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EXTENDED GENERALIZED VOIGT—TYPE FUNCTIONS AND RELATED BOUNDS

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Abstract. The principal aim of this paper is to introduce extended generalized Voigt–type function which contains the classical Voigt functions $K(x,y)$ and $L(x,y)$ as their particular cases. Functional bounding inequalities, monotonicity properties, log–convexity properties and Turán–type inequality results are presented for the investigated extended generalized Voigt–type function $\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y,z)$.

1. Introduction

The classical Voigt functions $K(x,y)$ and $L(x,y)$ occur frequently in a variety of problems of physics such as in astrophysical spectroscopy, emission, absorption and transfer of radiation in heated atmosphere, and plasma dispersion, and also in the theory of neutron reactions [25]. For various other investigations involving the Voigt functions, the interested reader may be referred to several recent papers on the subject, see, among others [5, 15, 20, 23] and the references cited therein. The functions $K(x,y), L(x,y)$, which are due essentially to Reiche [19] are defined by:

$$K(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-yt-\frac{t^2}{4}} \cos(xt)\,\mathrm{d}t, \tag{1}$$

and

$$L(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-yt-\frac{t^2}{4}} \sin(xt)\,\mathrm{d}t, \tag{2}$$

where in both cases $x \in \mathbb{R}$, $y \in \mathbb{R}_+$ [1].

The power series definition of the Bessel function of the first kind $J_\nu(z)$ of the order ν reads [26]

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}, \qquad z \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ \nu\in\mathbb{C},$$

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[1] Here, and in what follows, denote $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}_0^-$ and $\mathbb{N}_0$ the sets of complex, real, positive real, non-positive and non-negative integers, respectively.

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and the fact that [13, Eq. (10.39.2)]

$$J_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\sin(z), \quad \text{and} \quad J_{-\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\cos(z),$$

enable Srivastava and Miller [22] to define a generalization of $K(x,y)$ and $L(x,y)$ in (1) and (2) as

$$V_{\mu,\nu}(x,y)=\sqrt{\frac{x}{2}}\int_0^\infty t^\mu \, \mathrm{e}^{-yt-\frac{t^2}{4}} J_\nu(xt)\,\mathrm{d}t, \tag{3}$$

$$\left(\mu,\, x,\, y\in\mathbb{R}^+;\, \Re(\mu+\nu)>-1\right),$$

so that

$$K(x,y)=V_{\frac{1}{2},-\frac{1}{2}}(x,y), \quad \text{and} \quad L(x,y)=V_{\frac{1}{2},\frac{1}{2}}(x,y).$$

Recently, Pathan and Shahwan [16] (see also [11]) introduced a generalized Voigt function in the following form:

$$\Omega_{\mu,\alpha,\beta,\nu}(x,y)=\sqrt{\frac{x}{2}}\int_0^\infty t^\mu \, \mathrm{e}^{-yt-\frac{t^2}{4}}\, {}_1F_2\left[\alpha;\beta,1+\nu;-\frac{x^2t^2}{4}\right]\mathrm{d}t, \tag{4}$$

$$\left(\mu,\, x,\, y\in\mathbb{R}^+;\, \Re(\mu+\nu)>-1\right).$$

Clearly, by (3) and (4), we have the relation

$$\Omega_{\mu,\alpha,\alpha,\nu}(x,y)=\Gamma(\nu+1)\left(\frac{2}{x}\right)^\nu V_{\mu-\nu,\nu}(x,y).$$

Extending the generalized Voigt function (4) by introducing two more parameters in numerator and denominator as

$$\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y,z)=\sqrt{\frac{x}{2}}\int_0^\infty t^\mu \, \mathrm{e}^{-yt-zt^2}\, {}_2F_3\left[\alpha,\alpha';\beta,\beta',1+\nu;-\frac{x^2t^2}{4}\right]\mathrm{d}t, \tag{5}$$

$$\left(\mu,\, x,\, y,\, z\in\mathbb{R}^+;\, \Re(\mu+\nu)>-1\right),$$

for $\alpha'=\beta'$, the result in (4) of Pathan and Shahwan [16] follows.

In the present article, we first extend the generalized Voigt function (4) by introducing two more parameters. Then we derive the bounds for

$${}_2F_3\left[\alpha,\alpha';\beta,\beta',1+\nu;-\frac{x^2t^2}{4}\right]$$

by using the bound established by Parmar [14] for

$${}_1F_2\left[\alpha;\beta,1+\nu;-\frac{x^2t^2}{4}\right].$$

By virtue of the derived bounds for ${}_2F_3$, we establish the bounds for the extended generalized Voigt function $\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y,z)$ using its integral representation and estimating the ${}_1F_2$ function in the integrand in terms of the confluent Fox-Wright function ${}_1\Psi_0$. Monotonicity properties, log–convexity properties and Turán–type inequality results are presented for the investigated extended Voigt–type function.

2. Bounds for generalized Voigt function $\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y,z)$

We recall the bounds for J_ν, which ones we will employ on the positive real half–axis. Firstly, we mention von Lommel's results [7], [8, pp. 548–549] (see also [26, p. 406])

$$|J_\nu(t)| \leqslant 1, \qquad |J_{\nu+1}(t)| \leqslant \frac{1}{\sqrt{2}}, \qquad \nu \in \mathbb{R}^+, t \in \mathbb{R}, \tag{6}$$

and the bound by Minakshisundaram and Szász [9, p. 37]

$$|J_\nu(t)| \leqslant \frac{1}{\Gamma(\nu+1)} \left(\frac{|t|}{2}\right)^\nu, \qquad t \in \mathbb{R}. \tag{7}$$

Another bounds were derived by Landau [6], who gave in a sense best possible bounds for J_ν with respect to ν and t:

$$|J_\nu(t)| \leqslant b_L \nu^{-1/3}, \qquad b_L = \sqrt[3]{2} \sup_{t \geqslant 0} \mathrm{Ai}(t), \tag{8}$$

$$|J_\nu(t)| \leqslant c_L |t|^{-1/3}, \qquad c_L = \sup_{t \geqslant 0} t^{1/3} J_0(t), \tag{9}$$

where $\mathrm{Ai}(\cdot)$ stands for the Airy function

$$\mathrm{Ai}(x) = \frac{\pi}{2}\sqrt{\frac{x}{3}} \left(J_{-\frac{1}{3}}\left\{2(x/3)^{3/2}\right\} + J_{\frac{1}{3}}\left\{2(x/3)^{3/2}\right\}\right).$$

Olenko established the upper bound [12, Theorem 1]

$$\sup_{t \geqslant 0} \sqrt{t}|J_\nu(t)| \leqslant b_L \sqrt{\nu^{1/3} + \frac{\alpha_1}{\nu^{1/3}} + \frac{3\alpha_1^2}{10\nu}} = d_O, \qquad \nu \in \mathbb{R}^+, \tag{10}$$

where α_1 is the smallest positive zero of the Airy–function Ai and b_L is the Landau's constant from above. Further considerable upper bounds are listed, for example, in [2, 3, 17, 18, 24].

The generalized hypergeometric function with p numerator and q denominator parameters is defined by the power series

$${}_pF_q(\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z) = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \cdots (\alpha_p)_m}{(\beta_1)_m \cdots (\beta_q)_m} \frac{z^m}{m!}, \tag{11}$$

where, as usual, we make use of the Pochhammer symbol (or raising factorial)

$$(\tau)_0 = 1; \quad (\tau)_m = \tau(\tau+1)\cdots(\tau+m-1) = \frac{\Gamma(\tau+m)}{\Gamma(\tau)}, \qquad m \in \mathbb{N},$$

and $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-, j = \overline{1,s}$. The series (11) converges for all $z \in \mathbb{C}$ if $p \leqslant q$. It is divergent for all $z \neq 0$ when $p > q+1$, unless at least one numerator parameter is a negative integer in which case (11) is a polynomial. Finally, if $p = q+1$, the series converges in the unit circle $|z| < 1$, and also for $|z| = 1$ when $\Re\left(\sum\beta_j - \sum\alpha_j\right) > 0$ (consult [1]).

The Fox–Wright function extends the generalized hypergeometric function ${}_pF_q[z]$ which power series form reads [18, 21]:

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1,A_1),\cdots,(\alpha_p,A_p)\\(\beta_1,B_1),\cdots,(\beta_q,B_q)\end{matrix}\bigg|z\right] = \sum_{m=0}^{\infty}\frac{\prod\limits_{j=1}^{p}\Gamma(\alpha_j+mA_j)}{\prod\limits_{j=1}^{q}\Gamma(\beta_j+mB_j)}\frac{z^m}{m!}, \tag{12} $$

where $A_j > 0,\ j = 1,\cdots,p;\ B_j > 0,\ j = 1,\cdots,q$. The convergence conditions for the series at the right-hand side of (12) follow from the known asymptotic of the Euler Gamma–function. The defining series in (12) converges in the whole complex z-plane when

$$ \Delta = 1 + \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > 0. $$

If $\Delta = 0$, then the series in (12) converges for $|z| < \rho$, and $|z| = \rho$ under the condition $\Re(\theta) > \frac{1}{2}$ where

$$ \rho = \prod_{j=1}^{p} A_j^{-A_j} \cdot \prod_{j=1}^{q} B_j^{B_j}, \qquad \theta = \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j + \frac{p-q}{2}. $$

In the special case $A_r = B_s = 1;\ r = 1,\cdots,p;\ s = 1,\cdots,q$, the Fox–Wright function ${}_p\Psi_q[z]$ reduces (up to the multiplicative constant) to the generalized hypergeometric function

$$ {}_p\Psi_q\left[\begin{matrix}(a_1,1),\cdots,(a_p,1)\\(b_1,1),\cdots,(b_q,1)\end{matrix}\bigg|z\right] = \frac{\Gamma(a_1)\cdots\Gamma(a_p)}{\Gamma(b_1)\cdots\Gamma(b_q)}\ {}_pF_q\left[\begin{matrix}\alpha_1,\cdots,\alpha_p\\\beta_1,\cdots,\beta_q\end{matrix}\bigg|z\right]. $$

We also will need in the exposition the integral formula [4, p. 200, Eq. 13.1(95)]

$$ \begin{aligned} &{}_{p+1}F_{q+1}[\rho,\alpha_1,\cdots,\alpha_p;\rho+\sigma,\beta_1,\cdots,\beta_q;z] \\ &\qquad = \frac{\Gamma(\rho+\sigma)}{\Gamma(\rho)\Gamma(\sigma)}\int_0^1 u^{\rho-1}(1-u)^{\sigma-1}\ {}_pF_q[\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;zu]\,\mathrm{d}u, \end{aligned} \tag{13} $$

valid under

$$ p \leqq q+1;\ \min\{\Re(\rho),\Re(\sigma)\} > 0;\ |z| < 1 \text{ when } p = q+1. $$

For $p = q-1 = 0, \rho = \alpha, \rho+\sigma = \beta, \beta_1 = 1+\nu$ and $z = -\frac{x^2t^2}{4}$, (13) becomes

$$ \begin{aligned} {}_1F_2\left[\alpha;\beta,\nu+1;-\frac{x^2t^2}{4}\right] = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\beta-\alpha-1} \\ \cdot\ {}_0F_1\left(-;\nu+1;-\frac{ux^2t^2}{4}\right)\mathrm{d}t. \end{aligned} \tag{14} $$

Applying the relation [13, 26]

$$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)}\, {}_0F_1\left(-\,;\nu+1;-\frac{1}{4}z^2\right), \qquad \nu \in \mathbb{C}\setminus\mathbb{Z}^-, \tag{15}$$

in (14), we obtain the integral

$$\begin{aligned} {}_1F_2\left[\alpha;\beta,1+\nu;-\frac{x^2t^2}{4}\right] &= \frac{2^\nu\Gamma(\beta)\Gamma(\nu+1)}{(xt)^\nu\Gamma(\alpha)\Gamma(\beta-\alpha)} \\ &\qquad \cdot \int_0^1 u^{\alpha-\frac{\nu}{2}-1}(1-u)^{\beta-\alpha-1}\; J_\nu(xt\sqrt{u})\,\mathrm{d}u. \end{aligned} \tag{16}$$

Recently, Parmar [14] obtained sharp bounding inequalities for the generalized Voigt function $\Omega_{\mu,\alpha,\beta,\nu}(x,y)$ by making use of the integral representation (16) and applying several known upper bounds for the first-kind of the Bessel function $J_\nu(x)$ for the ${}_1F_2$ with appealing the bounds of Lommel's, Minakshisundaram and Szász, Landau and Olenko, respectively given in (6), (7), (8), (9) and (10). These bound are given in Lemma below.

LEMMA 1. [14] *Let* $\alpha, \beta, x, t, \nu \in \mathbb{R}^+$ *such, that* $2\min\{\alpha,\beta\} > \nu$. *Then we have*

$$\left|{}_1F_2\left[\alpha;\beta,1+\nu;-\frac{x^2t^2}{4}\right]\right| \leqq \frac{2^{\nu-\frac{1}{2}}\,\Gamma(\nu+1)\Gamma(\beta)\,\Gamma\left(\alpha-\frac{\nu}{2}\right)}{|xt|^\nu\Gamma(\alpha)\Gamma\left(\beta-\frac{\nu}{2}\right)}. \tag{17}$$

In the same parameter range

$$\left|{}_1F_2\left[\alpha;\beta,1+\nu;-\frac{x^2t^2}{4}\right]\right| \leqq \frac{b_L\,2^\nu\,\Gamma(\beta)\,\Gamma\left(\alpha-\frac{\nu}{2}\right)\,\Gamma(\nu+1)}{\sqrt[3]{\nu}\,|xt|^\nu\Gamma(\alpha)\Gamma\left(\beta-\frac{\nu}{2}\right)}. \tag{18}$$

Next, for $\min\{\alpha,\beta\} > \nu$, *there follows*

$$\left|{}_1F_2\left[\alpha;\beta,1+\nu;-\frac{x^2t^2}{4}\right]\right| \leqq \frac{2^{\nu-1}\,\Gamma(\beta)\,\Gamma(\alpha-\nu)}{\Gamma(\alpha)\Gamma(\beta-\nu)}. \tag{19}$$

and

$$\left|{}_1F_2\left[\alpha;\beta,1+\nu;-\frac{x^2t^2}{4}\right]\right| \leqslant \begin{cases} \dfrac{c_L\,2^\nu\,\Gamma(\beta)\,\Gamma\left(\alpha-\frac{\nu}{2}-\frac{1}{6}\right)\Gamma(\nu+1)}{|xt|^{\nu+\frac{1}{3}}\,\Gamma(\alpha)\Gamma\left(\beta-\frac{\nu}{2}-\frac{1}{6}\right)}, \\[2ex] \dfrac{d_O\,2^\nu\,\Gamma(\beta)\,\Gamma\left(\alpha-\frac{\nu}{2}-\frac{1}{4}\right)\Gamma(\nu+1)}{|xt|^{\nu+\frac{1}{2}}\,\Gamma(\alpha)\Gamma\left(\beta-\frac{\nu}{2}-\frac{1}{4}\right)}, \end{cases} \tag{20}$$

here the bound above holds if $6\min\{\alpha,\beta\} > 3\nu+1$, *whilst the expression below appears for* $4\min\{\alpha,\beta\} > 2\nu+1$.

Now by using the integral representation (13), if we specify $p=q-1=1, \rho=\alpha, \rho+\sigma=\beta,\ \alpha_1=\alpha',\ \beta_1=\beta',\ \beta_2=1+\nu$, and $z=-\frac{x^2t^2}{4}$, we conclude

$$ {}_2F_3\left[\alpha,\alpha';\beta,\beta',1+\nu;-\frac{x^2t^2}{4}\right]=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\beta-\alpha-1} \cdot {}_1F_2\left(\alpha';\beta',\nu+1;-\frac{ux^2t^2}{4}\right)\mathrm{d}t. \tag{21} $$

In the sequel we present our results which consist of functional and uniform upper bounds for the modulus of generalized hypergeometric functions ${}_1F_2$, ${}_2F_3$ under specific combinations of upper and lower parameters. The main derivation tools are the auxiliary Lemma 1 and the integral representation (16).

THEOREM 1. *Let* $\alpha,\beta,\alpha',\beta',\ x,\ t,\ \nu\in\mathbb{R}^+$. *When* $2\min\{\alpha,\beta,\alpha',\beta'\}>\nu$ *we have*

$$ \left|{}_2F_3\left[\alpha,\alpha';\beta,\beta',1+\nu;-\frac{x^2t^2}{4}\right]\right|\leqq\frac{2^{\nu-\frac12}\,\Gamma(\nu+1)\Gamma(\beta)\Gamma(\beta')\Gamma\left(\alpha-\frac{\nu}{2}\right)\Gamma\left(\alpha'-\frac{\nu}{2}\right)}{|xt|^{\nu}\Gamma(\alpha)\Gamma\left(\beta-\frac{\nu}{2}\right)\Gamma(\alpha')\Gamma\left(\beta'-\frac{\nu}{2}\right)}. \tag{22} $$

Moreover,

$$ \left|{}_2F_3\left[\alpha,\alpha';\beta,\beta',1+\nu;-\frac{x^2t^2}{4}\right]\right|\leqq\frac{b_L\,2^{\nu}\,\Gamma(\nu+1)\,\Gamma(\beta)\Gamma(\beta')\Gamma\left(\alpha-\frac{\nu}{2}\right)\Gamma\left(\alpha'-\frac{\nu}{2}\right)}{\sqrt[3]{\nu}\,|xt|^{\nu}\Gamma(\alpha)\Gamma(\alpha')\Gamma\left(\beta-\frac{\nu}{2}\right)\Gamma\left(\beta'-\frac{\nu}{2}\right)}. \tag{23} $$

Next, for $\min\{\alpha,\beta,\alpha',\beta'\}>\nu$, *it is*

$$ \left|{}_2F_3\left[\alpha,\alpha';\beta,\beta',1+\nu;-\frac{x^2t^2}{4}\right]\right|\leqq\frac{2^{\nu-1}\,\Gamma(\beta')\,\Gamma(\alpha'-\nu)}{\Gamma(\alpha')\Gamma(\beta'-\nu)}, \tag{24} $$

and

$$ \left|{}_2F_3\left[\alpha,\alpha';\beta,\beta',1+\nu;-\frac{x^2t^2}{4}\right]\right|\leqq\begin{cases}\dfrac{c_L\,2^{\nu}\,\Gamma(\nu+1)\,\Gamma(\beta)\,\Gamma(\beta')\,\Gamma\left(\alpha-\frac{\nu}{2}-\frac16\right)\Gamma\left(\alpha'-\frac{\nu}{2}-\frac16\right)}{|xt|^{\nu+\frac13}\,\Gamma(\alpha)\Gamma(\alpha')\Gamma\left(\beta-\frac{\nu}{2}-\frac16\right)\Gamma\left(\beta'-\frac{\nu}{2}-\frac16\right)},\\[2ex] \dfrac{d_O\,2^{\nu}\,\Gamma(\nu+1)\,\Gamma(\beta)\Gamma(\beta')\,\Gamma\left(\alpha-\frac{\nu}{2}-\frac14\right)\Gamma\left(\alpha'-\frac{\nu}{2}-\frac14\right)}{|xt|^{\nu+\frac12}\,\Gamma(\alpha)\Gamma(\alpha')\Gamma\left(\beta-\frac{\nu}{2}-\frac14\right)\Gamma\left(\beta'-\frac{\nu}{2}-\frac14\right)}.\end{cases} \tag{25} $$

Here the bound above holds if $6\min\{\alpha,\alpha',\beta,\beta'\}>3\nu+1$, *while the expression below appears for* $4\min\{\alpha,\alpha',\beta,\beta'\}>2\nu+1$.

Proof. Considering the integral representation (21), we observe that

$$ \left|{}_2F_3\left[\alpha,\alpha';\beta,\beta',1+\nu;-\frac{x^2t^2}{4}\right]\right|\leqq\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\beta-\alpha-1} \cdot\left|{}_1F_2\left[\alpha;\beta,1+\nu;-\frac{ux^2t^2}{4}\right]\right|\mathrm{d}u. \tag{26} $$

By appealing appropriately the functional bound (17) for $|{}_1F_2|$ in the integrand of (26) and then using the integral definition of Beta function, *viz.*

$$\mathrm{B}(t,s) = \int_0^1 u^{t-1}(1-u)^{s-1}\,\mathrm{d}u, \qquad \min\{t,s\} > 0,$$

we get (22). In a similar manner (23) is established by virtue of the second result (18).
The bounds (19) and (20) give the inequalities (24) and (25), respectively. □

Our next goal is to give a functional bound for the function $\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y,z)$ by exploring its integral representation (4) with the help of Theorem 1.

THEOREM 2. *Let* $\alpha,\beta,\alpha',\beta',x,y,z,\mu,\nu > 0$. *For all* $\min\{\alpha,\beta,\alpha',\beta'\} > \nu$ *it is*

$$\left|\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y,z)\right| \leqq \frac{2^{\nu-2}\Gamma(\nu+1)\Gamma(\beta)\Gamma\left(\alpha-\frac{\nu}{2}\right)\Gamma(\beta')\Gamma\left(\alpha'-\frac{\nu}{2}\right)}{|x|^{\nu-\frac{1}{2}}\, z^{\frac{\mu-\nu+1}{2}}\Gamma(\alpha)\Gamma\left(\beta-\frac{\nu}{2}\right)\Gamma(\alpha')\Gamma\left(\beta'-\frac{\nu}{2}\right)} \cdot {}_1\Psi_0\left[\begin{matrix}\left(\frac{\mu-\nu+1}{2},\frac{1}{2}\right)\\ -\end{matrix}\,\middle|\, -\frac{y}{\sqrt{z}}\right]. \tag{27}$$

In the same parameter range holds

$$\left|\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y,z)\right| \leqq \frac{b_L 2^{\nu-\frac{3}{2}}\Gamma(\beta)\Gamma(\alpha-\frac{\nu}{2})\Gamma(\beta')\Gamma(\alpha'-\frac{\nu}{2})\Gamma(\nu+1)}{\sqrt[3]{\nu}\,|x|^{\nu-\frac{1}{2}}\, z^{\frac{\mu-\nu+1}{2}}\Gamma(\alpha)\Gamma(\beta-\frac{\nu}{2})\Gamma(\alpha')\Gamma(\beta'-\frac{\nu}{2})} \cdot {}_1\Psi_0\left[\begin{matrix}\left(\frac{\mu-\nu+1}{2},\frac{1}{2}\right)\\ -\end{matrix}\,\middle|\, -\frac{y}{\sqrt{z}}\right]. \tag{28}$$

Next, for $\min\{\alpha',\beta'\} > \nu$, *we have*

$$\left|\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y,z)\right| \leqq \frac{2^{\nu-\frac{5}{2}}\sqrt{x}\,\Gamma(\beta')\Gamma(\alpha'-\nu)}{z^{\frac{\mu+1}{2}}\Gamma(\alpha')\Gamma(\beta'-\nu)}\, {}_1\Psi_0\left[\begin{matrix}\left(\frac{\mu+1}{2},\frac{1}{2}\right)\\ -\end{matrix}\,\middle|\, -\frac{y}{\sqrt{z}}\right], \tag{29}$$

and

$$\left|\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y,z)\right| \leqslant \begin{cases} \dfrac{c_L\, 2^{\nu-\frac{3}{2}}\Gamma(\beta)\Gamma(\alpha-\frac{\nu}{2}-\frac{1}{6})\Gamma(\beta')\Gamma(\alpha'-\frac{\nu}{2}-\frac{1}{6})\Gamma(\nu+1)}{|x|^{\nu-\frac{1}{6}}\, z^{\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{3}}\Gamma(\alpha)\Gamma(\beta-\frac{\nu}{2}-\frac{1}{6})\Gamma(\alpha')\Gamma(\beta'-\frac{\nu}{2}-\frac{1}{6})} \\ \qquad \cdot {}_1\Psi_0\left[\begin{matrix}\left(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{3},\frac{1}{2}\right)\\ -\end{matrix}\,\middle|\, -\frac{y}{\sqrt{z}}\right], \\[2ex] \dfrac{d_O\, 2^{\nu-\frac{3}{2}}\Gamma(\beta)\Gamma(\alpha-\frac{\nu}{2}-\frac{1}{4})\Gamma(\beta')\Gamma(\alpha'-\frac{\nu}{2}-\frac{1}{4})\Gamma(\nu+1)}{2\,|x|^{\nu}\, z^{\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{4}}\Gamma(\alpha)\Gamma(\beta-\frac{\nu}{2}-\frac{1}{4})\Gamma(\alpha')\Gamma(\beta'-\frac{\nu}{2}-\frac{1}{4})} \\ \qquad \cdot {}_1\Psi_0\left[\begin{matrix}\left(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{4},\frac{1}{2}\right)\\ -\end{matrix}\,\middle|\, -\frac{y}{\sqrt{z}}\right], \end{cases} \tag{30}$$

where the bound above holds if $6\min\{\alpha,\alpha',\beta,\beta'\} > 3\nu+1$*, while the expression below appears for* $4\min\{\alpha\alpha',\beta,\beta'\} > 2\nu+1$*. Here* ${}_1\Psi_0$ *is the confluent Fox–Wright function for the case* $p=1$ *and* $q=0$ *in* (12).

Proof. Let us evaluate the integral similar to given in [24, p. 198, Equation (15)]

$$\mathscr{I}_\mu(y,z) = \int_0^\infty t^{\mu-1}\, e^{-yt-zt^2}\, dt,$$

by the series expansion of the exponential e^{-yt} in the kernel into its Taylor-Maclaurin series and legitimate integral–sum interchange and then using the familiar Gamma formula [13]

$$\Gamma(\eta)\xi^{-\eta} = \int_0^\infty e^{-\xi t} t^{\eta-1}\, dt, \qquad \Re(\xi)>0,\ \Re(\eta)>0,$$

we obtain

$$\mathscr{I}_\mu(y,z) = \frac{1}{2z^{\frac{\mu}{2}}}\sum_{m=0}^\infty \frac{\Gamma\left(\dfrac{\mu+m}{2}\right)}{m!}\left(-\frac{y}{\sqrt{z}}\right)^m = \frac{1}{2z^{\frac{\mu}{2}}}\,{}_1\Psi_0\left[\begin{matrix}(\frac{\mu}{2},\frac{1}{2})\\ -\end{matrix}\,\middle|\,-\frac{y}{\sqrt{z}}\right],$$

where we have used the defintion of Fox–Wright hypergeometric function for the case $p=1$ and $q=0$ in (12), say confluent Fox–Wright function ${}_1\Psi_0$.

Applying the first inequality (22) to the modulus of the generalized hypergeometric function ${}_2F_3$ occurring in the integrand of (5), we easily conclude that

$$\left|\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y,z)\right| \leqq \frac{2^{\nu-\frac{1}{2}}\Gamma(\nu+1)\Gamma(\beta)\Gamma(\beta')\Gamma\left(\alpha-\frac{\nu}{2}\right)\Gamma\left(\alpha'-\frac{\nu}{2}\right)}{|x|^\nu\Gamma(\alpha)\Gamma\left(\beta-\frac{\nu}{2}\right)\Gamma(\alpha')\Gamma\left(\beta'-\frac{\nu}{2}\right)} \times \int_0^\infty t^{\mu-\nu}\, e^{-yt-zt^2}\, dt. \tag{31}$$

Now applying integral representation $\mathscr{I}_\mu(y,z)$ given above in (31), we arrive at (27). Similarly using the inequalities (23), (24) and (25) as above and simplifying, we get the results (28), (29) and (30), respectively. □

3. Complete monotonicity, log–convexity and Turán type inequalities

In this section, we prove the complete monotonicity and log–convexity properties, and certain Turán–type inequalities for the extended generalized Voigt–type function $\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y,z)$ in (5).

The function $h:(0,\infty)\to\mathbb{R}$ is completely monotonic (or totally monotonic), if h has derivatives of all (positive integer) orders and satisfies

$$(-1)^m h^{(m)}(x) \geqslant 0, \qquad m\in\mathbb{N}_0;\ x>0.$$

We say that a function $h:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ is said to be logrithimically convex, or simply log-convex, if its natural logarithm $\log h$ is convex, that is, for all $x,y\in[a,b]$ and $\lambda\in[0,1]$, we have

$$h(\lambda x+(1-\lambda)y)\leqslant\big[h(x)\big]^{\lambda}\,\big[h(y)\big]^{1-\lambda}.$$

Every completely monotonic function is log–convex, see [27, p. 167].

THEOREM 3. *Let $\nu>-1$, $x>0$ and $t\in(0,1)$. Then*

- *the function $y\mapsto\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y,z)$ is completely monotonic and log–convex on $(0,\infty)$,*
- *the function $z\mapsto\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y,z)$ is completely monotonic and log–convex on $(0,\infty)$,*
- *the function $\mu\mapsto\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y,z)$ is log–convex on $(0,\infty)$ for all $y,z>0$,*
- *the function $(y,z)\mapsto\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y,z)$ is log–convex on $(0,\infty)$.*

Proof. Differentiating $\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y,z)$ with respect to y we arrive at

$$\left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^{m}\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y,z)=\sqrt{\frac{x}{2}}\int_0^1 t^{\mu}\;t^{m}\,\mathrm{e}^{-yt-zt^2}\;{}_2F_3\left[\alpha,\alpha';\beta,\beta',1+\nu;-\frac{x^2t^2}{4}\right]\mathrm{d}t,$$

where $m\in\mathbb{N}_0$. Consequently

$$(-1)^m\left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^{m}\;\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y,z)\geqslant 0,$$

for all $t\in(0,1)$. So the function $y\mapsto\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y,z)$ is completely monotone in y.

By the complete monotone behavior of $y\mapsto\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y,z)$ it is also log–convex for $y>0$ for all $\mu>0,\,z>0$. Hence, for a suitably used $y_1,y_2>0$ we can write

$$\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,\lambda y_1+(1-\lambda)y_2,z)\leqq\big[\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y_1,z)\big]^{\lambda}\,\big[\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y_2,z)\big]^{1-\lambda}. \tag{32}$$

In a similar manner, we can prove the next assertions concerning the complete monotonicity and log–convexity of $\Omega_{\mu,\alpha,\beta,\nu}^{\alpha',\beta'}(x,y,z)$ in z.

For the next statement, we recall the Hölder–Rogers inequality for integrals [10, p. 54]

$$\int_a^b|f(t)g(t)|\,\mathrm{d}t=\left(\int_0^1|f(t)|^p\;\mathrm{d}t\right)^{1/p}\left(\int_0^1|g(t)|^q\,\mathrm{d}t\right)^{1/q}, \tag{33}$$

where $p>1$, $1/p+1/q=1$, and the real functions $f\in \mathrm{L}^p[a,b]$ and $g\in \mathrm{L}^p[a,b]$. Applying the integral representation (5), with the help of (33) we conclude

$$\begin{aligned}
\Omega^{\alpha',\beta'}_{\lambda\mu_1+(1-\lambda)\mu_2,\alpha,\beta,\nu}(x,y,z) &= \sqrt{\frac{x}{2}}\int_0^1 \frac{t^{\lambda\mu_1+(1-\lambda)\mu_2}}{\mathrm{e}^{yt+zt^2}}\,{}_2F_3\left[\alpha,\alpha';\beta,\beta',1+\nu;-\frac{x^2t^2}{4}\right]\mathrm{d}t\\
&= \sqrt{\frac{x}{2}}\int_0^1 \left\{t^{\mu_1}\mathrm{e}^{-yt-zt^2}\,{}_2F_3\left[\alpha,\alpha';\beta,\beta',1+\nu;-\frac{x^2t^2}{4}\right]\right\}^{\lambda}\\
&\qquad\cdot\left\{t^{\mu_2}\mathrm{e}^{-yt-zt^2}\,{}_2F_3\left[\alpha,\alpha';\beta,\beta',1+\nu;-\frac{x^2t^2}{4}\right]\right\}^{1-\lambda}\mathrm{d}t\\
&\leqq \left\{\sqrt{\frac{x}{2}}\int_0^1 t^{\mu_1}\,\mathrm{e}^{-yt-zt^2}\,{}_2F_3\left[\alpha,\alpha';\beta,\beta',1+\nu;-\frac{x^2t^2}{4}\right]\mathrm{d}t\right\}^{\lambda}\\
&\qquad\cdot\left\{\sqrt{\frac{x}{2}}\int_0^1 t^{\mu_2}\,\mathrm{e}^{-yt-zt^2}\,{}_2F_3\left[\alpha,\alpha';\beta,\beta',1+\nu;-\frac{x^2t^2}{4}\right]\mathrm{d}t\right\}^{1-\lambda}.
\end{aligned}$$

This is equivalent to

$$\Omega^{\alpha',\beta'}_{\lambda\mu_1+(1-\lambda)\mu_2,\alpha,\beta,\nu}(x,y,z)\leqq\left[\Omega^{\alpha',\beta'}_{\mu_1,\alpha,\beta,\nu}(x,y,z)\right]^{\lambda}\left[\Omega^{\alpha',\beta'}_{\mu_2,\alpha,\beta,\nu}(x,y,z)\right]^{1-\lambda},$$

which proves the assertion that the function $\mu\mapsto\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y,z)$ is log–convex for all $\mu_1,\mu_2>0$, $\lambda\in[0,1]$ and $y,z>0$.

Moreover, the function $(y,z)\mapsto\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y,z)$ is log–convex for all $y_1,y_2,z_1,z_2>0$, $\mu>0$, $\lambda\in[0,1]$ and $x>0$ and hence equivalent to

$$\begin{aligned}
&\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,\lambda y_1+(1-\lambda)y_2,z_1+(1-\lambda)z_2)\\
&\qquad\leqslant\left[\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y_1,z_1)\right]^{\lambda}\left[\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y_2,z_2)\right]^{1-\lambda},
\end{aligned}$$

which finishes the proof of the theorem. □

THEOREM 4. *For the same parameter range as in* Theorem 3 *there holds the Turán inequality*

$$\{\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y,z)\}^2-\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y-1,z)\,\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y+1,z)\leqq 0.$$

Moreover, for the same parameter space we have the Turán inequalities

$$\begin{aligned}
&\{\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y,z)\}^2-\Omega^{\alpha',\beta'}_{\mu-1,\alpha,\beta,\nu}(x,y,z)\Omega^{\alpha',\beta'}_{\mu+1,\alpha,\beta,\nu}(x,y,z)\leqq 0,\\
&\{\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y,z)\}^2-\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y,z-1)\,\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y,z+1)\leqq 0.
\end{aligned}$$

Finally, we have

$$\{\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y,z)\}^2-\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y-1,z-1)\,\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y+1,z+1)\leqq 0.$$

Proof. Choosing $y_1 = y-1$, $y_2 = y+1$ and $\lambda = \frac{1}{2}$ in (32), we conclude the first Turán inequality in Theorem 4. Finally, for other assertion, specifying $z_1 = z-1$, $\mu_1 = \mu - 1$, $z_2 = z+1$, $\mu_2 = \mu+1$, $y_1 = y-1$, $z_1 = z-1$, $y_2 = y+1$, $z_2 = z+1$ and $\lambda = \frac{1}{2}$, respectively, we deduce the remaining Turán inequalities. □

4. Concluding remarks and observations

In present paper, we introduce the extended generalized Voigt–type function which contains the classical Voigt functions $K(x,y)$ and $L(x,y)$ as their particular cases. Then by virtue of the derived bounds for ${}_2F_3$, we established the bounds for the extended generalized Voigt–type function $\Omega^{\alpha',\beta'}_{\mu,\alpha,\beta,\nu}(x,y,z)$ using its integral representation and estimating the ${}_1F_2$ function in the integrand in terms of the confluent Fox–Wright function ${}_1\Psi_0$. Finally, we studied monotonicity properties, log–convexity properties and Turán–type inequality for the newly introduced extended generalized Voigt–type function.

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