Journal of
Classical
Analysis
Volume 21, Number 2 (2023), 153–171 doi:10.7153/jca-2023-21-09

UNIQUENESS RESULTS ON L–FUNCTIONS AND CERTAIN DIFFERENCE–DIFFERENTIAL POLYNOMIALS

PREETHAM N. RAJ AND HARINA P. WAGHAMORE*

Abstract. Using the notion of weighted sharing, we study the value distribution of a L-function and an arbitrary meromorphic function when certain type of difference-differential polynomials generated by them share a non-zero small function or a non-zero rational function and obtain some uniqueness results which generalises and improves the recent results due to Hao and Chen [3], Mandal and Datta [11].

1. Introduction and main results

Throughout the paper, $\mathbb{C}$ denotes the complex plane and $\mathbb{N}$ denotes the set of natural numbers. Now, towards the end of twentieth century, a new class of Dirichlet series called the Selberg class was introduced by Atle Selberg, which has now become an important field of research in the analytic number theory. A L-function $\mathscr{L}$ means a Selberg class function with the Riemann Zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ as the prototype and the Selberg class S of L-function is defined as the set of all Dirichlet series $\mathscr{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ of a complex variable s that satisfy the following axioms (see [13]):

(i) Ramanujan hypothesis: $a(n) \ll n^{\varepsilon}$ for each $\varepsilon > 0$;

(ii) Analytic continuation: There is a non-negative integer k, such that $(s-1)^k \mathscr{L}(s)$ is an entire function of finite order;

(iii) Functional equation: $\mathscr{L}$ satisfies a functional equation of the type

$$\Lambda_{\mathscr{L}}(s) = \omega \overline{\Lambda_{\mathscr{L}}(1-\overline{s})},$$

where

$$\Lambda_{\mathscr{L}}(s) = \mathscr{L}(s) Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q, λ_j and complex numbers ν_j, ω, with $\Re(\nu_j) \geqslant 0$ and $|\omega| = 1$, where $\Re$ represents the real part and since the argument of the Gamma function should be positive, therefore $\Re(\lambda_j s + \nu_j) > 0$;

Mathematics subject classification (2020): 30D35.
Keywords and phrases: Meromorphic functions, L-functions, difference-differential polynomial, weighted sharing, uniqueness.

* Corresponding author.

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(iv) Euler product hypothesis: $\mathscr{L}$ can be written over primes as

$$\mathscr{L}(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right)$$

with suitable co-efficients $b(p^k)$ such that $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$, where the product is taken over all prime numbers p.

From Ramanujan Hypothesis it can be implied that the Dirichlet series $\mathscr{L}$ converges absolutely in the half-plane $\Re(s) > 1$ and then is extended meromorphically. The degree $d_{\mathscr{L}}$ of an L-function $\mathscr{L}$ is defined to be

$$d_{\mathscr{L}} = 2\sum_{j=1}^{K} \lambda_j,$$

where λ_j and K are respectively the positive real number and the positive integer as in axiom *(iii)* above.

The Nevanlinna value distribution theory is an important area of research which has seen extensive work. It primarily focuses on the analysis of the distribution of solutions to the equation $f(z) = a$, where f is an entire or meromorphic function in $\mathbb{C}$, $z \in \mathbb{C}$ and $a \in \mathbb{C} \cup \{\infty\}$. One can refer to Hayman [4], Yi and Yang [15], Yang [16] for the standard definitions and notations of Nevanlinna theory.

In general, for a meromorphic function $f(z)$, the quantity $m(r,f)$ denotes the proximity function of $f(z)$, while $N(r,f)$ denotes the counting function of poles of $f(z)$ whose multiplicities are taken into account (respectively $\overline{N}(r,f)$ denotes the reduced counting function when multiplicities are ignored). The quantity $N(r,a;f)$ (notation interchangable with $N\left(r,\frac{1}{f-a}\right)$) denotes the counting function of a points of $f(z)$ whose multiplicities are taken into account (respectively $\overline{N}(r,a;f)$ denotes the reduced counting function when multiplicities are ignored). The Nevanlinna characteristic function of a meromorphic function f plays a very important role in the value distribution theory and it is denoted by $T(r,f)$. We have $T(r,f) = m(r,f) + N(r,f)$, which clearly shows that $T(r,f)$ is non-negative.

The value distribution of an L-function $\mathscr{L}$, is concerened with the distribution of the zeros of $\mathscr{L}$ and more generally, with the roots of the equation $\mathscr{L}(s) = c$ for some $c \in \mathbb{C} \cup \{\infty\}$. Since L-functions are analytically continued as meromorphic functions, it is possible to study the value distribution and uniqueness outcomes between the L-functions and any arbitrary meromorphic functions.

To make our paper self sufficient, we state the following standard definitions of Nevanlinna theory and it is important to note that all the definitions discussed also applies to the L-function.

DEFINITION 1.1. [4] The order $\rho(f)$ and hyper-order $\rho_2(f)$ of a meromorphic function $f(z)$ are defined as,

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{\log r}$$

and

$$\rho_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ represents the Nevanlinna characteristic function of the meromorphic function f.

DEFINITION 1.2. [4] A meromorphic function $a(z)$ is said to be a small function of f, if $T(r,a) = S(r,f)$, where $S(r,f)$ denotes any quantity which satisfies $S(r,f) = o(T(r,f))$ as $r \to +\infty$ possibly outside a set I with finite linear measure.

DEFINITION 1.3. [4] Let $a \in \mathbb{C}$ and $k \in \mathbb{N}$. Then for a meromorphic function $f(z)$, we denote by $N_{(k}(r,a;f)$ the counting function for zeros of $f-a$ with multiplicities atleast k, and by $\overline{N}_{(k}(r,a;f)$ the one for which multiplicity is not counted. Similarly, we denote by $N_{k)}(r,a;f)$ the counting function for zeros of $f-a$ with multiplicities atmost k, and by $\overline{N}_{k)}(r,a;f)$ the one for which multiplicity is not counted. Then

$$N_k(r,a;f) = \overline{N}_{(1}(r,a;f) + \overline{N}_{(2}(r,a;f) + \cdots + \overline{N}_{(k}(r,a;f).$$

DEFINITION 1.4. [4] Let $f(z)$ and $g(z)$ be two meromorphic functions in the complex plane $\mathbb{C}$. If $f(z)-a$ and $g(z)-a$ assume the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicity), and if we do not consider the multiplicity, then we say that $f(z)$ and $g(z)$ share the value a IM (ignoring multiplicity), where a is a complex number.

DEFINITION 1.5. [5] Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a,f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a,f) = E_k(a,g)$, then we say that f and g share the value a with the weight k, which is written as f, g share (a,k), which implies that z_0 is a zero of $f(z)-a$ with multiplicity $m(\leqslant k)$ if and only if it is a zero of $g(z)-a$ with multiplicity $m(\leqslant k)$ and z_0 is a zero of $f(z)-a$ with multiplicity $m(>k)$ if and only if it is a zero of $g(z)-a$ with multiplicity $n(>k)$, where m is not necessarily equal to n.

Hence, it is clear that, if f, g share (a,k) then f, g share (a,p) for all non-negative integers p, where $0 \leqslant p \leqslant k$. We say that f, g share the value a IM or CM, if and only if f, g share $(a,0)$ or (a,∞) respectively.

DEFINITION 1.6. [5] Let f and g share the value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities are different from multiplicities of the corresponding a-points of g, where each a-point is counted only once. Clearly, $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$.

DEFINITION 1.7. Let $f(z)$ be a non-constant meromorphic function, then difference operators of $f(z)$ are defined as follows

$$\Delta_c f(z) = f(z+c) - f(z),$$

$$\Delta_c^u f(z) = \Delta_c^{u-1}(\Delta_c f(z)) = \sum_{r=0}^{u}(-1)^r \binom{u}{r} f(z+(u-r)c),$$

where c is a non-zero complex number, $u(\geqslant 2)$ is a positive integer.

In 2003, the following question was posed by Yang [7].

QUESTION. Let f be a meromorphic function in the complex plane and a,b,c are three distinct values, where $c \neq 0,\infty$. If f and the Riemann zeta function ζ share a,b CM and c IM, will then $f \equiv \zeta$?

In continuation with this, in 2010, Li [6] proved the following result.

THEOREM A. [6] *Let a and b be two distinct finite values and f be a meromorphic function in the complex plane with finitely many poles. If f and a non-constant L-function $\mathscr{L}$ share a* CM *and b* IM, *then $\mathscr{L} \equiv f$.*

In 2017, by considering the differential monomial, Liu et al., [8] proved the following uniqueness theorems.

THEOREM B. [8] *Let f be a non-constant meromorphic function, let $\mathscr{L}$ be a L-function, and let $n,k \in \mathbb{N}$ with $n > 3k+6$. If $(f^n)^{(k)}$ and $(\mathscr{L}^n)^{(k)}$ share* 1 CM, *then $f = t\mathscr{L}$ for a constant t satisfying $t^n = 1$.*

THEOREM C. [8] *Let f be a non-constant meromorphic function, let $\mathscr{L}$ be a L-function, and let $n,k \in \mathbb{N}$ with $n > 3k+6$. If $(f^n)^{(k)}(z) - z$ and $(\mathscr{L}^n)^{(k)}(z) - z$ share* 0 CM, *then $f = t\mathscr{L}$ for a constant t satisfying $t^n = 1$.*

In 2018, by considering differential polynomials instead of differential monomials, Hao and Chen [3] obtained the following uniqueness results.

THEOREM D. [3] *Let f be a non-constant meromorphic function, let $\mathscr{L}$ be a L-function and let $m,n,k \in \mathbb{N}$. Suppose that $[f^n(f-1)^m]^{(k)}$ and $[\mathscr{L}^n(\mathscr{L}-1)^m]^{(k)}$ share* 1 CM. *If $n > 3k+m+6$ and $k \geqslant 2$, then $f = \mathscr{L}$ or $[f^n(f-1)^m] \equiv [\mathscr{L}^n(\mathscr{L}-1)^m]$.*

THEOREM E. [3] *Let f be a non-constant meromorphic function, let $\mathscr{L}$ be an L-function and let $m,n,k \in \mathbb{N}$. Suppose that $[f^n(f-1)^m]^{(k)}$ and $[\mathscr{L}^n(\mathscr{L}-1)^m]^{(k)}$ share* 1 IM. *If $n > 7k+4m+11$ and $k \geqslant 2$, then $f = \mathscr{L}$ or $[f^n(f-1)^m] \equiv [\mathscr{L}^n(\mathscr{L}-1)^m]$.*

In 2021, by considering small function & rational function sharing and by considering difference-differential polynomials instead of differential polynomials in Theorems D & E, Mandal and Datta [11] obtained the following results.

THEOREM F. [11] *Let $\mathscr{L}$ be a non-constant L-function and f be a transcendental meromorphic function. Let k,n,η,μ_j $(j=1,2,\ldots,\eta)$, $\lambda = \sum_{j=1}^{\eta}\mu_j$ be positive integers such that $n > \lambda + \eta(2k+4)+4$ and $\omega_j \in \mathbb{C}\setminus\{0\}$ $(j=1,2,\ldots,\eta)$ be distinct constants. Also, let $\rho_2(\mathscr{L}) < 1$, $\rho_2(f) < 1$, with $[\mathscr{L}^n(z)\prod_{j=1}^{\eta}\mathscr{L}(z+\omega_j)^{\mu_j}]^{(k)}$ and $[f^n(z)\prod_{j=1}^{\eta}f(z+\omega_j)^{\mu_j}]^{(k)}$ share $(\rho(z),l)$ and f, $\mathscr{L}$ share $(\infty,0)$, where $\rho(z)$*

is a small function of f *and* $\mathscr{L}$. *If* $l=0$ *and* $n>\lambda+(\eta+1)(5k+7)$ *or* $l=1$ *and* $n>\lambda+\frac{3}{2}(\eta+1)(2k+3)$, *then one of the following holds:*

(i) $[\mathscr{L}^n(z)\prod_{j=1}^{\eta}\mathscr{L}(z+\omega_j)^{\mu_j}]^{(k)}\equiv[f^n(z)\prod_{j=1}^{\eta}f(z+\omega_j)^{\mu_j}]^{(k)}$,

(ii) $[\mathscr{L}^n(z)\prod_{j=1}^{\eta}\mathscr{L}(z+\omega_j)^{\mu_j}]^{(k)}\cdot[f^n(z)\prod_{j=1}^{\eta}f(z+\omega_j)^{\mu_j}]^{(k)}\equiv\rho(z)^2$.

THEOREM G. [11] *Let* $\mathscr{L}$ *be a non-constant L-function and f be a transcendental meromorphic function. Let* k,n,η,μ_j $(j=1,2,\ldots,\eta)$, $\lambda=\sum_{j=1}^{\eta}\mu_j$ *be positive integers such that* $n>\lambda+\eta(2k+4)+4$ *and* $\omega_j\in\mathbb{C}-\{0\}$ $(j=1,2,\ldots,\eta)$ *be distinct constants. Also let* $\rho_2(\mathscr{L})<1$, $\rho_2(f)<1$, $[\mathscr{L}^n(z)\prod_{j=1}^{\eta}\mathscr{L}(z+\omega_j)^{\mu_j}]^{(k)}$ *and* $[f^n(z)\prod_{j=1}^{\eta}f(z+\omega_j)^{\mu_j}]^{(k)}$ *share* $(R(z),l)$ *and* f, $\mathscr{L}$ *share* $(\infty,0)$, *where* $R(z)$ *is a rational function. If* $l=0$ *and* $n>\lambda+(\eta+1)(5k+7)$ *or* $l=1$ *and* $n>\lambda+\frac{3}{2}(\eta+1)(2k+3)$, *then one of the following holds:*

(i) $[\mathscr{L}^n(z)\prod_{j=1}^{\eta}\mathscr{L}(z+\omega_j)^{\mu_j}]^{(k)}\equiv[f^n(z)\prod_{j=1}^{\eta}f(z+\omega_j)^{\mu_j}]^{(k)}$,

(ii) $[\mathscr{L}^n(z)\prod_{j=1}^{\eta}\mathscr{L}(z+\omega_j)^{\mu_j}]^{(k)}\cdot[f^n(z)\prod_{j=1}^{\eta}f(z+\omega_j)^{\mu_j}]^{(k)}\equiv R(z)^2$.

Now, it will be interesting to study the above Theorems D, E, F & G by considering a more general form of difference-differential polynomial and also when the functions share ∞ with weight $w\in[0,\infty]$, instead of considering only ∞ IM sharing.

The main motivation to this paper is the fact that the L-function $\mathscr{L}$ has only one possible pole at $s=1$ in $\mathbb{C}$. By utilising this fact, we obtain the Lemmas 2.14, 2.15, 2.16 and Lemma 2.17 which are the heart of this paper.

We obtain the following main theorems.

THEOREM 1.1. *Let* f *be a non-constant meromorphic function with* $\rho_2(f)<1$ *and* $\mathscr{L}$ *be a L-function with* $\rho_2(\mathscr{L})<1$. *Let* $F=[f^n(z)(f(z)-1)^m\Delta_c^u f(z)]^{(k)}$ *and* $G=[\mathscr{L}^n(z)(\mathscr{L}(z)-1)^m\Delta_c^u\mathscr{L}(z)]^{(k)}$, *where* n,m,u,k *are positive integers with* $u>1$ *and c is a non-zero complex constant such that* $\Delta_c^u f\not\equiv 0$ *and* $\Delta_c^u\mathscr{L}\not\equiv 0$. *Suppose* F *and* G *share* $(a(z),l)$ *and* (∞,w), *where* $0\leqslant l<\infty$, $0\leqslant w\leqslant\infty$, $a(z)$ *is a small function of* f *and* $\mathscr{L}$, *and if one of the following conditions holds:*

(i) $l\geqslant 2$, $w=\infty$ *and* $n>2k+m+2u+6$;

(ii) $l\geqslant 2$, $0\leqslant w<\infty$ *and* $n>2k+m+u(3-\chi)+6$;

(iii) $l=1$, $w=0$ *and* $n>\frac{5}{2}k+\frac{3}{2}m+u\left(\frac{7}{2}-\chi\right)+7$;

(iv) $l=0$, $w=0$ *and* $n>5k+4m+u(6-\chi)+12$;

where $\chi=\limsup_{r\to\infty}\frac{N(r,\mathscr{L})}{T(r,f)+T(r,\mathscr{L})}$ *and* $\chi\in[0,1]$. *Then one of the following two conclusions holds:*

1. $F\equiv G$,
2. $F\cdot G\equiv a(z)^2$.

THEOREM 1.2. *Let f be a non-constant meromorphic function with $\rho_2(f) < 1$ and $\mathscr{L}$ be a L-function with $\rho_2(\mathscr{L}) < 1$. Let $F = [f^n(z)(f(z)-1)^m \Delta_c^u f(z)]^{(k)}$ and $G = [\mathscr{L}^n(z)(\mathscr{L}(z)-1)^m \Delta_c^u \mathscr{L}(z)]^{(k)}$, where n,m,u,k are positive integers with $u > 1$ and c is a non-zero complex constant such that $\Delta_c^u f \not\equiv 0$ and $\Delta_c^u \mathscr{L} \not\equiv 0$. Suppose F and G share $(R(z),l)$ and (∞,w), where $0 \leqslant l < \infty$, $0 \leqslant w \leqslant \infty$, $R(z)$ is a rational function, and if one of the following conditions holds:*

(i) $l \geqslant 2$, $w = \infty$ *and* $n > 2k+m+2u+6$;

(ii) $l \geqslant 2$, $0 \leqslant w < \infty$ *and* $n > 2k+m+u(3-\chi)+6$;

(iii) $l = 1$, $w = 0$ *and* $n > \frac{5}{2}k+\frac{3}{2}m+u\left(\frac{7}{2}-\chi\right)+7$;

(iv) $l = 0$, $w = 0$ *and* $n > 5k+4m+u(6-\chi)+12$;

where $\chi = \limsup_{r\to\infty} \frac{N(r,\mathscr{L})}{T(r,f)+T(r,\mathscr{L})}$ *and* $\chi \in [0,1]$. *Then one of the conclusions of* Theorem 1.1 *holds.*

THEOREM 1.3. *Let f be a non-constant meromorphic function with $\rho_2(f) < 1$ and $\mathscr{L}$ be a L-function with $\rho_2(\mathscr{L}) < 1$. Let $\Phi = [f^n(z)(f(z)-1)^m \prod_{j=1}^{\eta} f(z+c_j)^{\mu_j}]^{(k)}$ and $\Psi = [\mathscr{L}^n(z)(\mathscr{L}(z)-1)^m \prod_{j=1}^{\eta} \mathscr{L}(z+c_j)^{\mu_j}]^{(k)}$, where n,m,k,η,μ_j $(j = 1,2,\ldots,\eta)$ and $\lambda = \sum_{j=1}^{\eta} \mu_j$ are positive integers and c_j $(j = 1,2\ldots,\eta)$ be complex constants. Suppose Φ and Ψ share $(a(z),l)$ and (∞,w), where $0 \leqslant l < \infty$, $0 \leqslant w \leqslant \infty$, $a(z)$ is a small function of f and $\mathscr{L}$, and if one of the following conditions holds:*

(i) $l \geqslant 2$, $w = \infty$ *and* $n > 2k+m+2\lambda+4$;

(ii) $l \geqslant 2$, $0 \leqslant w < \infty$ *and* $n > 2k+m+\lambda(3-\chi)+4$;

(iii) $l = 1$, $w = 0$ *and* $n > \frac{5}{2}k+\frac{3}{2}m+\lambda\left(\frac{7}{2}-\chi\right)+\frac{7}{2}$;

(iv) $l = 0$, $w = 0$ *and* $n > 5k+4m+\lambda(6-\chi)+7$;

where $\chi = \limsup_{r\to\infty} \frac{N(r,\mathscr{L})}{T(r,f)+T(r,\mathscr{L})}$ *and* $\chi \in [0,1]$. *Then one of the following two conclusions holds:*

1. $\Phi \equiv \Psi$,
2. $\Phi \cdot \Psi \equiv a(z)^2$.

THEOREM 1.4. *Let f be a non-constant meromorphic function with $\rho_2(f) < 1$ and $\mathscr{L}$ be a L-function with $\rho_2(\mathscr{L}) < 1$. Let $\Phi = [f^n(z)(f(z)-1)^m \prod_{j=1}^{\eta} f(z+c_j)^{\mu_j}]^{(k)}$ and $\Psi = [\mathscr{L}^n(z)(\mathscr{L}(z)-1)^m \prod_{j=1}^{\eta} \mathscr{L}(z+c_j)^{\mu_j}]^{(k)}$, where n,m,k,η,μ_j $(j = 1,2,\ldots,\eta)$ and $\lambda = \sum_{j=1}^{\eta} \mu_j$ are positive integers and c_j $(j = 1,2\ldots,\eta)$ be complex constants. Suppose Φ and Ψ share $(R(z),l)$ and (∞,w), where $0 \leqslant l < \infty$, $0 \leqslant w \leqslant \infty$, $R(z)$ is a rational function, and if one of the following conditions holds:*

(i) $l \geqslant 2$, $w = \infty$ *and* $n > 2k+m+2\lambda+4$;

(ii) $l \geqslant 2,\ 0 \leqslant w < \infty$ *and* $n > 2k + m + \lambda(3-\chi) + 4$*;*

(iii) $l = 1,\ w = 0$ *and* $n > \frac{5}{2}k + \frac{3}{2}m + \lambda\left(\frac{7}{2} - \chi\right) + \frac{7}{2}$*;*

(iv) $l = 0,\ w = 0$ *and* $n > 5k + 4m + \lambda(6-\chi) + 7$*;*

where $\chi = \limsup_{r\to\infty} \frac{N(r,\mathscr{L})}{T(r,f)+T(r,\mathscr{L})}$ *and* $\chi \in [0,1]$*. Then one of the conclusions of Theorem 1.3 holds.*

REMARK 1.1. In Theorem 1.1, if we replace n by $n-1$ and take $u = 0$, then $\Delta_c^u f(z) = f(z)$, $\Delta_c^u \mathscr{L}(z) = \mathscr{L}(z)$ and the equations F and G becomes, $F = [f^n(z)(f(z) - 1)^m]^{(k)}$ and $G = [\mathscr{L}^n(z)(\mathscr{L}(z)-1)^m]^{(k)}$. When we consider the value sharing $l = 0$ and $w = 0$, the condition for n reduces to $n-1 > 5k+4m+12$. If we take $k = 2$ then, we get $n > 4m+23$, which is definitely an improvement of Theorem E, which has $n > 4m+25$. Thus, Theorem 1.1 generalises as well as improves Theorem E.

REMARK 1.2. In Theorem 1.3, if we take $m = 0$, $\eta = 1$ and $\mu_j = 1$ $(\forall j = 1\cdots\eta)$, then we get $\lambda = 1$, and hence the condition for n reduces to $n > \frac{5k}{2} + 7 - \chi$ (when $l = 1$, $w = 0$) and $n > 5k + 13 - \chi$ (when $l = 0, w = 0$) respectively, which are definitely improvements when compared to $n > 6k+10$ (when $l = 1$, $w = 0$) and $n > 10k + 15$ (when $l = 0$, $w = 0$) respectively, in Theorem F. Thus, Theorem 1.3, generalises as well as improves Theorem F.

Similarly, Theorem 1.4, generalises and improves Theorem G.

2. Lemmas

In this section we provide all the necessary lemmas required to prove the theorems. Let us define,

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.1}$$

LEMMA 2.1. [4] *Let* $f(z)$ *be a meromorphic function and* $a \in \mathbb{C}$*. Then*

$$T\left(r, \frac{1}{f}\right) = T(r,f) + O(1),$$
$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1).$$

LEMMA 2.2. [15] *Let* $f(z)$ *be a non-constant meromorphic function and let* $a_0(z)$, $a_1(z), \ldots, a_n(z) (\not\equiv 0)$ *be small functions with respect to* f*. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r,f) + S(r,f).$$

LEMMA 2.3. [17] *Let $f(z)$ be a non-constant meromorphic function with $\rho_2(f) < 1$ and let $c \in \mathbb{C} \setminus \{0\}$. Then*

$$\begin{aligned} T(r, f(z+c)) &= T(r,f) + S(r,f), \\ N(r, f(z+c)) &= N(r,f) + S(r,f), \\ N\left(r, \frac{1}{f(z+c)}\right) &= N\left(r, \frac{1}{f(z)}\right) + S(r,f). \end{aligned}$$

LEMMA 2.4. [2] *Let $f(z)$ be a meromorphic function with $\rho_2(f) < 1$. Then for $\varepsilon > 0$ we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\rho_2-\varepsilon}}\right) = S(r,f),$$

for all r outside of a set of finite logarithmic measure.

LEMMA 2.5. [9] *Let $f(z)$ be a non-constant meromorphic function and $p, k \in \mathbb{N}$. Then*

$$\begin{aligned} T(r, f^{(k)}) &\leqslant T(r,f) + k\overline{N}(r,f) + S(r,f), \\ N_p\left(r, \frac{1}{f^{(k)}}\right) &\leqslant T(r, f^{(k)}) - T(r,f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r,f), \\ N_p\left(r, \frac{1}{f^{(k)}}\right) &\leqslant N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f). \end{aligned}$$

LEMMA 2.6. [16] *Let $f(z) = \frac{a_o + a_1 z + \ldots + a_n z^n}{b_0 + b_1 z + \ldots + b_m z^m}$ be a non-constant rational function defined in the complex plane $\mathbb{C}$, where $a_0, a_1, \ldots, a_n (\neq 0)$ and $b_0, b_1, \ldots, b_m (\neq 0)$ are complex constants. Then $T(r,f) = t \log r + O(1)$, where $t = \max(n,m)$.*

LEMMA 2.7. [1] *Let F, G be two non-constant meromorphic function. If F, G share $(1,2)$ and (∞, k), where $0 \leqslant k \leqslant \infty$ and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r,F) \leqslant\ & N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}_*(r, \infty; F, G) \\ & + S(r,F) + S(r,G), \end{aligned}$$

where $\overline{N}_*(r,\infty;F,G)$ denotes the reduced counting function of those poles of F whose multiplicities differ from the multiplicities of the corresponding poles of G. Similar result holds for $T(r,G)$.

LEMMA 2.8. [12] *Let F, G be two non-constant meromorphic functions sharing $(1,1)$ and $(\infty,0)$. If $H \not\equiv 0$, then*

$$\begin{aligned} T(r,F) \leqslant\ & N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \frac{3}{2}\overline{N}(r,F) + \overline{N}(r,G) + \overline{N}_*(r, \infty; F, G) \\ & + \frac{1}{2}\overline{N}\left(r, \frac{1}{F}\right) + S(r,F) + S(r,G). \end{aligned}$$

Similar result holds for $T(r,G)$.

LEMMA 2.9. [12] *Let F, G be two non-constant meromorphic functions sharing $(1,0)$ and $(\infty,0)$. If $H \not\equiv 0$, then*

$$T(r,F) \leqslant N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+3\overline{N}(r,F)+2\overline{N}(r,G)+\overline{N}_*(r,\infty;F,G)$$
$$+2\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,F)+S(r,G).$$

Similar result holds for $T(r,G)$.

LEMMA 2.10. [14] *Let $\mathscr{L}$ be a L-function with degree d. Then*

$$T(r,L)=\frac{d}{\pi}r\log r+O(r).$$

LEMMA 2.11. [10] *Let $\mathscr{L}$ be a L-function. Then $N(r,\infty;\mathscr{L})=N(r,\mathscr{L})=S(r,\mathscr{L})$ $=O(\log r)$.*

LEMMA 2.12. [11] *Let f be a non-constant meromorphic function and $\mathscr{L}$ be a L-function. If f and $\mathscr{L}$ share ∞ IM, then $\overline{N}(r,f)=\overline{N}(r,\mathscr{L})=S(r,\mathscr{L})=O(\log r)$.*

LEMMA 2.13. *Let f be a non-constant meromorphic function and $\mathscr{L}$ be a L-function. If f and $\mathscr{L}$ share ∞ CM, then $N(r,f)=N(r,\mathscr{L})=S(r,\mathscr{L})=O(\log r)$.*

Proof. Proof follows easily from Lemma 2.12 □

LEMMA 2.14. *Let f be a non-constant meromorphic function with $\rho_2(f)<1$ and $\mathscr{L}$ be a L-function with $\rho_2(\mathscr{L})<1$. Let $F_1=[f^n(z)(f(z)-1)^m\Delta_c^u f(z)]$, where n,m,u,k are positive integers with $u>1$ and c is a non-zero complex constant such that $\Delta_c^u f\not\equiv 0$. Then*

(i). If f and $\mathscr{L}$ share ∞ CM, we have

$$(n+m)T(r,f)\leqslant T(r,F_1)+S(r,f).$$

(ii). If f and $\mathscr{L}$ share ∞ IM or (∞,w), where $0\leqslant w<\infty$, we have

$$(n+m-u)T(r,f)\leqslant T(r,F_1)+S(r,f).$$

Proof. Since f is a meromorphic function, from Lemmas 2.1, 2.2, and 2.3, we have

$$\begin{aligned}(n+m+1)T(r,f)&=T(r,f^{n+m+1})+S(r,f)\\&\leqslant T(r,f^n(f-1)^m f)+S(r,f)\leqslant T\left(r,\frac{f(z).F_1}{\Delta_c^u f}\right)+S(r,f)\\&\leqslant T(r,F_1)+T\left(r,\frac{\Delta_c^u f}{f(z)}\right)+S(r,f)\\&\leqslant T(r,F_1)+m\left(r,\frac{\Delta_c^u f}{f(z)}\right)+N\left(r,\frac{\Delta_c^u f}{f(z)}\right)+S(r,f).\end{aligned} \tag{2.2}$$

If f and $\mathscr{L}$ share ∞ CM, then from (2.2), Lemma 2.4 and Lemma 2.13, we have

$$(n+m+1)T(r,f) \leqslant T(r,F_1) + N\left(r,\frac{1}{f}\right) + N(r,\Delta_c^u f) + S(r,f)$$

Thus,

$$(n+m)T(r,f) \leqslant T(r,F_1) + S(r,f).$$

If f and $\mathscr{L}$ share ∞ IM or (∞,w), where $0 \leqslant w < \infty$, then from (2.2), Lemma 2.4 and Lemma 2.12, we have

$$\begin{aligned}
(n+m+1)T(r,f) &\leqslant T(r,F_1) + N\left(r,\frac{\sum_{r=0}^{u}(-1)^r\binom{u}{r}f(z+(u-r)c)}{f(z)}\right) + S(r,f) \\
&\leqslant T(r,F_1) + N\left(r,\frac{\sum_{r=0}^{u-1}(-1)^r\binom{u}{r}f(z+(u-r)c)}{f(z)}\right) \\
&\quad + N\left(r,(-1)^u\frac{f(z)}{f(z)}\right) + S(r,f) \\
&\leqslant T(r,F_1) + N\left(r,\frac{1}{f(z)}\right) + \sum_{r=0}^{u-1} N(r,f(z+(u-r)c)) + S(r,f) \\
&\leqslant T(r,F_1) + T(r,f) + uN(r,f) + S(r,f) \\
&\leqslant T(r,F_1) + (u+1)T(r,f) + S(r,f).
\end{aligned}$$

Therefore, we have $(n+m-u)T(r,f) \leqslant T(r,F_1) + S(r,f)$.

This completes the proof of Lemma 2.14. □

LEMMA 2.15. *If $\mathscr{L}$ is a L-function and $G_1 = [\mathscr{L}^n(\mathscr{L}-1)^m\Delta_c^u\mathscr{L}]$, where n,m,u,k are positive integers with $u > 1$ and c is a non-zero complex constant such that $\Delta_c^u\mathscr{L} \not\equiv 0$. Then*

$$(n+m)T(r,\mathscr{L}) \leqslant T(r,G_1) + S(r,\mathscr{L}).$$

Proof. The proof follows easily from the proof of Lemma 2.14, using Lemma 2.10 and Lemma 2.11. □

LEMMA 2.16. *Let f be a non-constant meromorphic function with $\rho_2(f) < 1$ and $\mathscr{L}$ be a L-function with $\rho_2(\mathscr{L}) < 1$. Let $\Phi_1 = f^n(f-1)^m\prod_{j=1}^{\eta} f(z+c_j)^{\mu_j}$ and $\lambda = \sum_{j=1}^{\eta}\mu_j$, where n,m,η,μ_j $(j=1,2,\ldots,\eta)$ are positive integers and c_j $(j=1,2,\ldots,\eta)$ are complex constants. Then*

(i). If f and $\mathscr{L}$ share ∞ CM*, we have*

$$(n+m)T(r,f) \leqslant T(r,\Phi_1) + S(r,f).$$

(ii). If f and $\mathscr{L}$ share ∞ IM *or (∞,w), where $0 \leqslant w < \infty$, we have*

$$(n+m-\lambda)T(r,f) \leqslant T(r,\Phi_1) + S(r,f).$$

Proof. Since f is a meromorphic function, from Lemmas 2.1, 2.2 and 2.3, we have

$$\begin{aligned}(n+m+\lambda)T(r,f) &= T(r,f^{n+m+\lambda})+S(r,f)\\ &\leqslant T(r,f^n(f-1)^m f^\lambda)+S(r,f)\\ &\leqslant T\left(r,\frac{\Phi_1 f^\lambda}{\prod_{j=1}^{\eta} f(z+c_j)^{\mu_j}}\right)+S(r,f)\\ &\leqslant T(r,\Phi_1)+T\left(r,\frac{\prod_{j=1}^{\eta} f(z+c_j)^{\mu_j}}{f^\lambda}\right)+S(r,f)\\ &\leqslant T(r,\Phi_1)+m\left(r,\frac{\prod_{j=1}^{\eta} f(z+c_j)^{\mu_j}}{f^\lambda}\right)+N\left(r,\frac{\prod_{j=1}^{\eta} f(z+c_j)^{\mu_j}}{f^\lambda}\right)\\ &\quad +S(r,f)\\ (n+m+\lambda)T(r,f) &\leqslant T(r,\Phi_1)+\sum_{j=1}^{\eta} m\left(r,\frac{f(z+c_j)^{\mu_j}}{f^{\mu_j}}\right)+\sum_{j=1}^{\eta} N\left(r,\frac{f(z+c_j)^{\mu_j}}{f^{\mu_j}}\right)\\ &\quad +S(r,f). \end{aligned} \tag{2.3}$$

If f and $\mathscr{L}$ share ∞ CM, then from (2.3), Lemma 2.4 and Lemma 2.13, we have

$$(n+m+\lambda)T(r,f) \leqslant T(r,\Phi_1)+\lambda T(r,f)+S(r,f)$$

Thus,

$$(n+m)T(r,f) \leqslant T(r,\Phi_1)+S(r,f).$$

If f and $\mathscr{L}$ share ∞ IM or (∞,w), where $0 \leqslant w < \infty$, then from (2.3), Lemma 2.4 and Lemma 2.12, we have

$$(n+m+\lambda)T(r,f) \leqslant T(r,\Phi_1)+2\lambda T(r,f)+S(r,f)$$

Thus,

$$(n+m-\lambda)T(r,f) \leqslant T(r,\Phi_1)+S(r,f).$$

This completes the proof of Lemma 2.16. □

LEMMA 2.17. *If $\mathscr{L}$ is a L-function and $\Psi_1 = \mathscr{L}^n(\mathscr{L}-1)^m \prod_{j=1}^{\eta}\mathscr{L}(z+c_j)^{\mu_j}$ and $\lambda = \sum_{j=1}^{\eta}\mu_j$, where n,m,η,μ_j $(j=1,2,\ldots,\eta)$ are positive integers and c_j are complex constants. Then*

$$(n+m)T(r,\mathscr{L}) \leqslant T(r,\Psi_1)+S(r,\mathscr{L}).$$

Proof. The proof follows easily from the proof of Lemma 2.16, using Lemma 2.10 and Lemma 2.11. □

3. Proof of Theorems

3.1. Proof of Theorem 1.1

Proof. Let

$$F_1 = f^n(z)(f(z)-1)^m \Delta_c^u f(z) \quad \text{and} \quad G_1 = \mathscr{L}^n(z)(\mathscr{L}(z)-1)^m \Delta_c^u \mathscr{L}(z);$$

$$F = [f^n(z)(f(z)-1)^m \Delta_c^u f(z)]^{(k)} \quad \text{and} \quad G = [\mathscr{L}^n(z)(\mathscr{L}(z)-1)^m \Delta_c^u \mathscr{L}(z)]^{(k)};$$

$$\mathscr{F} = \frac{[f^n(z)(f(z)-1)^m \Delta_c^u f(z)]^{(k)}}{a(z)} \quad \text{and} \quad \mathscr{G} = \frac{[\mathscr{L}^n(z)(\mathscr{L}(z)-1)^m \Delta_c^u \mathscr{L}(z)]^{(k)}}{a(z)}.$$

Clearly $\mathscr{F}$ and $\mathscr{G}$ share $(1,l)$ and (∞,w) except for zeros and poles of $a(z)$.

We shall define Ω as:

$$\Omega = \left(\frac{\mathscr{F}''}{\mathscr{F}'} - \frac{2\mathscr{F}'}{\mathscr{F}-1}\right) - \left(\frac{\mathscr{G}''}{\mathscr{G}'} - \frac{2\mathscr{G}'}{\mathscr{G}-1}\right). \tag{3.1}$$

From Lemma 2.5, we have

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right) &\leqslant N_2\left(r,\frac{1}{F}\right) + S(r,F) \\ &\leqslant T(r,F) - T(r,F_1) + N_{k+2}\left(r,\frac{1}{F_1}\right) + S(r,F_1). \end{aligned} \tag{3.2}$$

Similarly from Lemma 2.5, we have

$$\begin{aligned} N_2\left(r,\frac{1}{G}\right) &\leqslant N_2\left(r,\frac{1}{G}\right) + S(r,G) \\ &\leqslant T(r,F) - T(r,G_1) + N_{k+2}\left(r,\frac{1}{G_1}\right) + S(r,G_1). \end{aligned} \tag{3.3}$$

Now, we consider the following two main cases:

Case 1. Suppose $\Omega \not\equiv 0$, then in this case we consider the following 4 subcases.

Subcase 1.1. Suppose $l \geqslant 2$ and $w = \infty$, then from Lemma 2.7, we have

$$\begin{aligned} T(r,\mathscr{F}) \leqslant{} & N_2\left(r,\frac{1}{\mathscr{F}}\right) + N_2\left(r,\frac{1}{\mathscr{G}}\right) + \overline{N}(r,\mathscr{F}) + \overline{N}(r,\mathscr{G}) \\ & + \overline{N}_*(r,\infty;\mathscr{F},\mathscr{G}) + S(r,\mathscr{F}) + S(r,\mathscr{G}). \end{aligned} \tag{3.4}$$

Since $a(z)$ is a small function of f and $\mathscr{L}$, we can write

$$\begin{aligned} T(r,F) \leqslant{} & N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + \overline{N}(r,F) + \overline{N}(r,G) \\ & + \overline{N}_*(r,\infty;F,G) + S(r,F) + S(r,G). \end{aligned} \tag{3.5}$$

Now, by using (3.2) in (3.5) and from Lemma 2.5, we attain

$$T(r,F) \leqslant T(r,F) - T(r,F_1) + N_{k+2}\left(r,\frac{1}{F_1}\right) + N_{k+2}\left(r,\frac{1}{G_1}\right) + k\overline{N}(r,G_1)$$
$$+ \overline{N}(r,F_1) + \overline{N}(r,G_1) + \overline{N}_*(r,\infty;F_1,G_1) + S(r,F_1) + S(r,G_1). \tag{3.6}$$

From (3.6), by using Lemmas 2.11, 2.12, 2.13 and by *(i)* of Lemma 2.14, we can deduce that

$$(n+m)T(r,f) \leqslant (k+2)\overline{N}\left(r,\frac{1}{f}\right) + mN\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{\Delta_c^u f}\right) + mN\left(r,\frac{1}{\mathscr{L}}\right)$$
$$+ (k+2)\overline{N}\left(r,\frac{1}{\mathscr{L}}\right) + N\left(r,\frac{1}{\Delta_c^u \mathscr{L}}\right) + S(r,f) + S(r,\mathscr{L})$$
$$\leqslant (k+m+u+3)\{T(r,f) + T(r,\mathscr{L})\} + S(r,f) + S(r,\mathscr{L}). \tag{3.7}$$

Similarly, by using Lemma 2.15, we obtain

$$(n+m)T(r,\mathscr{L}) \leqslant (k+m+u+3)\{T(r,f) + T(r,\mathscr{L})\} + S(r,f) + S(r,\mathscr{L}). \tag{3.8}$$

Now, by combining (3.7) and (3.8), we get

$$(n+m)\{T(r,f)+T(r,\mathscr{L})\} \leqslant 2(k+m+u+3)\{T(r,f)+T(r,\mathscr{L})\} + S(r,f) + S(r,\mathscr{L}).$$

This is a contradiction to $n > 2k+m+2u+6$.

Subcase 1.2. Suppose $l \geqslant 2$ and $0 \leqslant w < \infty$.

Then from Lemma 2.11, 2.12, 2.13 and by using *(ii)* of Lemma 2.14 in (3.6), we deduce

$$(n+m-u)T(r,f) \leqslant (k+2)\overline{N}\left(r,\frac{1}{f}\right) + mN\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{\Delta_c^u f}\right) + mN\left(r,\frac{1}{\mathscr{L}}\right)$$
$$+ (k+2)\overline{N}\left(r,\frac{1}{\mathscr{L}}\right) N\left(r,\frac{1}{\Delta_c^u \mathscr{L}}\right) + S(r,f) + S(r,\mathscr{L}).$$

Thus, we have

$$(n+m-u)T(r,f) \leqslant (k+m+u+3)T(r,f) + (k+m+u+3)T(r,\mathscr{L}) + S(r,f) + S(r,\mathscr{L}). \tag{3.9}$$

Similarly, by using Lemma 2.15, we arrive at

$$(n+m)T(r,\mathscr{L}) \leqslant (k+m+u+3)T(r,\mathscr{L}) + (k+m+u+3)T(r,f) + S(r,f) + S(r,\mathscr{L}). \tag{3.10}$$

Now, by adding (3.9) and (3.10), we obtain

$$(n+m)\{T(r,f)+T(r,\mathscr{L})\} - uT(r,f) \leqslant 2(k+m+u+3)T(r,f)$$
$$+ 2(k+m+u+3)T(r,\mathscr{L}) + S(r,f) + S(r,\mathscr{L}),$$
$$(n+m-u)\{T(r,f)+T(r,\mathscr{L})\} \leqslant (2k+2m+2u+6)\{T(r,f)+T(r,\mathscr{L})\}$$
$$- uN(r,\mathscr{L}) + S(r,f) + S(r,\mathscr{L}).$$

This is a contradiction to $n > 2k+m+u(3-\chi)+6$, where $\chi = \limsup_{r\to\infty} \frac{N(r,\mathscr{L})}{T(r,f)+T(r,\mathscr{L})}$ and $\chi \in [0,1]$.

Subcase 1.3. Suppose $l=1$ and $w=0$, then from Lemma 2.8, we have

$$T(r,\mathscr{F}) \leqslant N_2\left(r,\frac{1}{\mathscr{F}}\right)+N_2\left(r,\frac{1}{\mathscr{G}}\right)+\frac{3}{2}\overline{N}(r,\mathscr{F})+\overline{N}(r,\mathscr{G})+\overline{N}_*(r,\infty;\mathscr{F},\mathscr{G})$$
$$+\frac{1}{2}\overline{N}\left(r,\frac{1}{\mathscr{F}}\right)+S(r,\mathscr{F})+S(r,\mathscr{G}). \tag{3.11}$$

Since $a(z)$ is a small function of f and $\mathscr{L}$, we can write

$$T(r,F) \leqslant N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\frac{3}{2}\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_*(r,\infty;F,G)$$
$$+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+S(r,F)+S(r,G). \tag{3.12}$$

Now, by using (3.2) in (3.12) and from Lemma 2.5, we have

$$T(r,F) \leqslant T(r,F)-T(r,F_1)+N_{k+2}\left(r,\frac{1}{F_1}\right)+N_{k+2}\left(r,\frac{1}{G_1}\right)+k\overline{N}(r,G_1)$$
$$+\frac{3}{2}\overline{N}(r,F_1)+\overline{N}(r,G_1)+\overline{N}_*(r,\infty;F_1,G_1)+\frac{1}{2}N_{k+1}\left(r,\frac{1}{F_1}\right)$$
$$+\frac{k}{2}\overline{N}(r,F_1)+S(r,F_1)+S(r,G_1). \tag{3.13}$$

Now, from Lemmas 2.11, 2.12, 2.13 and by using *(ii)* of Lemma 2.14 in (3.13), we can deduce that

$$(n+m-u)T(r,f) \leqslant (k+2)\overline{N}\left(r,\frac{1}{f}\right)+mN\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{\Delta_c^u f}\right)+mN\left(r,\frac{1}{\mathscr{L}}\right)$$
$$+(k+2)\overline{N}\left(r,\frac{1}{\mathscr{L}}\right)+N\left(r,\frac{1}{\Delta_c^u \mathscr{L}}\right)+\frac{1}{2}(k+1)\overline{N}\left(r,\frac{1}{f}\right)$$
$$+\frac{m}{2}N\left(r,\frac{1}{f}\right)+\frac{1}{2}N\left(r,\frac{1}{\Delta_c^u f}\right)+S(r,f)+S(r,\mathscr{L}).$$

Thus, we can write

$$(n+m-u)T(r,f) \leqslant \left(\frac{3k}{2}+\frac{3m}{2}+\frac{3u}{2}+4\right)T(r,f)$$
$$+(k+m+u+3)T(r,\mathscr{L})+S(r,f)+S(r,\mathscr{L}). \tag{3.14}$$

Similarly, by using Lemma 2.15, we can arrive at

$$(n+m)T(r,\mathscr{L}) \leqslant \left(\frac{3k}{2}+\frac{3m}{2}+\frac{3u}{2}+4\right)T(r,\mathscr{L})$$
$$+(k+m+u+3)T(r,f)+S(r,f)+S(r,\mathscr{L}). \tag{3.15}$$

Now, by combining (3.14) and (3.15), and then by simplifying, we attain

$$(n+m-u)\{T(r,f)+T(r,\mathscr{L})\} \leqslant \left(\frac{5k}{2}+\frac{5m}{2}+\frac{5u}{2}+7\right)\{T(r,f)+T(r,\mathscr{L})\} \\ -uN(r,\mathscr{L})+S(r,f)+S(r,\mathscr{L}),$$

which is a contradiction to

$$n>\frac{5k}{2}+\frac{3m}{2}+u\left(\frac{7}{2}-\chi\right)+7,$$

where $\chi=\limsup_{r\to\infty}\frac{N(r,\mathscr{L})}{T(r,f)+T(r,\mathscr{L})}$ and $\chi\in[0,1]$.

Subcase 1.4. Suppose $l=0$ and $w=0$, then from Lemma 2.9, we have

$$T(r,\mathscr{F}) \leqslant N_2\left(r,\frac{1}{\mathscr{F}}\right)+N_2\left(r,\frac{1}{\mathscr{G}}\right)+3\overline{N}(r,\mathscr{F})+2\overline{N}(r,\mathscr{G})+\overline{N}_*(r,\infty;\mathscr{F},\mathscr{G}) \\ +2\overline{N}\left(r,\frac{1}{\mathscr{F}}\right)+\overline{N}\left(r,\frac{1}{\mathscr{F}}\right)+S(r,\mathscr{F})+S(r,\mathscr{G}). \tag{3.16}$$

Since $a(z)$ is a small function of f and $\mathscr{L}$, we can write

$$T(r,F) \leqslant N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+3\overline{N}(r,F)+2\overline{N}(r,G)+\overline{N}_*(r,\infty;F,G) \\ +2\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,F)+S(r,G). \tag{3.17}$$

Now, by using (3.2) in (3.17) and from Lemma 2.5, we deduce

$$T(r,F) \leqslant T(r,F)-T(r,F_1)+N_{k+2}\left(r,\frac{1}{F_1}\right)+N_{k+2}\left(r,\frac{1}{G_1}\right)+k\overline{N}(r,G_1) \\ +3\overline{N}(r,F_1)+2\overline{N}(r,G_1)+\overline{N}_*(r,\infty;F_1,G_1)+2N_{k+1}\left(r,\frac{1}{F_1}\right) \\ +2k\overline{N}(r,F_1)+N_{k+1}\left(r,\frac{1}{G_1}\right)+k\overline{N}(r,G_1)+S(r,F_1)+S(r,G_1). \tag{3.18}$$

Now, from Lemmas 2.11, 2.12, 2.13 and by using *(ii)* of Lemma 2.14 in (3.18), we attain

$$(n+m-u)T(r,f) \leqslant (k+2)\overline{N}\left(r,\frac{1}{f}\right)+mN\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{\Delta_c^u f}\right)+mN\left(r,\frac{1}{\mathscr{L}}\right) \\ +(k+2)\overline{N}\left(r,\frac{1}{\mathscr{L}}\right)+N\left(r,\frac{1}{\Delta_c^u\mathscr{L}}\right)+2(k+1)\overline{N}\left(r,\frac{1}{f}\right) \\ +2mN\left(r,\frac{1}{f}\right)+2N\left(r,\frac{1}{\Delta_c^u f}\right)+(k+1)\overline{N}\left(r,\frac{1}{\mathscr{L}}\right) \\ +mN\left(r,\frac{1}{\mathscr{L}}\right)+N\left(r,\frac{1}{\Delta_c^u\mathscr{L}}\right)+S(r,f)+S(r,\mathscr{L}).$$

Thus, we can write

$$(n+m-u)T(r,f) \leqslant (3k+3m+3u+7)T(r,f) + (2k+2m+2u+5)T(r,\mathscr{L}) + S(r,f) + S(r,\mathscr{L}). \tag{3.19}$$

Similarly, by using Lemma 2.15, we can deduce that

$$(n+m)T(r,\mathscr{L}) \leqslant (3k+3m+3u+7)T(r,\mathscr{L}) + (2k+2m+2u+5)T(r,f) + S(r,f) + S(r,\mathscr{L}). \tag{3.20}$$

Now, by combining (3.19) and (3.20), and then by simplifying, we obtain

$$(n+m-u)\{T(r,f)+T(r,\mathscr{L})\} \leqslant (5k+5m+5u+12)\{T(r,f)+T(r,\mathscr{L})\} - uN(r,\mathscr{L}) + S(r,f) + S(r,\mathscr{L}),$$

which is a contradiction to

$$n > 5k+4m+u(6-\chi)+12,$$

where $\chi = \limsup_{r\to\infty} \frac{N(r,\mathscr{L})}{T(r,f)+T(r,\mathscr{L})}$ and $\chi \in [0,1]$.

Case 2. Suppose $\Omega \equiv 0$. Then from (3.1)

$$\frac{\mathscr{F}''}{\mathscr{F}'} - \frac{2\mathscr{F}'}{\mathscr{F}-1} \equiv \frac{\mathscr{G}''}{\mathscr{G}'} - \frac{2\mathscr{G}'}{\mathscr{G}-1}.$$

By integrating the above equation twice, we get

$$\frac{1}{\mathscr{F}-1} = \frac{A}{\mathscr{G}-1} + B, \tag{3.21}$$

where $A(\neq 0)$ and B are constants. From (3.21), it is obvious that $\mathscr{F}$ and $\mathscr{G}$ share the values 1 and ∞ CM (and hence they share the values 1 and ∞ IM as well) and therefore $n > 2k+m+2u+6$. Now, we discuss the following three subcases separately.

Subcase 2.1. Suppose that $B \neq 0$ and $A = B$. Then from (3.21),we obtain that

$$\frac{1}{\mathscr{F}-1} = \frac{B\mathscr{G}}{\mathscr{G}-1}. \tag{3.22}$$

If $B = -1$, then from (3.22), we obtain

$$\mathscr{F} \cdot \mathscr{G} = 1,$$

i.e.,

$$[f^n(f-1)^m \Delta_c^u f]^{(k)} \cdot [\mathscr{L}^n(\mathscr{L}-1)^m \Delta_c^u \mathscr{L}]^{(k)} = a(z)^2,$$

which is one of the conclusions of Theorem 1.1.

If $B \neq -1$, then from (3.22), we have

$$\frac{1}{\mathscr{F}} = \frac{B\mathscr{G}}{(1+B)\mathscr{G}-1}$$

and hence, $\overline{N}\left(r, \frac{1}{\mathscr{G}-\frac{1}{1+B}}\right) = \overline{N}\left(r, \frac{1}{\mathscr{F}}\right)$. Now, from the Second Fundamental Theorem of Nevanlinna, we deduce that

$$\begin{aligned} T(r,G) &\leqslant T(r,\mathscr{G}) + S(r,\mathscr{G}) \\ &\leqslant \overline{N}\left(r,\frac{1}{\mathscr{G}}\right) + \overline{N}\left(r,\frac{1}{\mathscr{G}-\frac{1}{1+B}}\right) + \overline{N}(r,\mathscr{G}) + S(r,\mathscr{G}) \\ &\leqslant \overline{N}\left(r,\frac{1}{\mathscr{G}}\right) + \overline{N}\left(r,\frac{1}{\mathscr{F}}\right) + \overline{N}(r,\mathscr{G}) + S(r,\mathscr{G}) \\ &\leqslant \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,G) + S(r,G). \end{aligned} \tag{3.23}$$

From Lemma 2.5 and (3.23), we have

$$\begin{aligned} T(r,G) \leqslant T(r,G) - T(r,G_1) + N_{k+1}\left(r,\frac{1}{G_1}\right) + N_{k+1}\left(r,\frac{1}{F_1}\right) \\ + k\overline{N}(r,F_1) + \overline{N}(r,G_1) + S(r,G_1) + S(r,F_1). \end{aligned}$$

This implies,

$$\begin{aligned} (n+m)T(r,\mathscr{L}) &\leqslant (k+1)\overline{N}\left(r,\frac{1}{\mathscr{L}}\right) + mN\left(r,\frac{1}{\mathscr{L}}\right) + N\left(r,\frac{1}{\Delta_c^u\mathscr{L}}\right) + mN\left(r,\frac{1}{f}\right) \\ &\quad + (k+1)\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{\Delta_c^u f}\right) + S(r,f) + S(r,\mathscr{L}) \\ &\leqslant (k+m+u+2)\{T(r,f) + T(r,\mathscr{L})\} + S(r,f) + S(r,\mathscr{L}). \end{aligned}$$

Similarly, we obtain

$$(n+m)T(r,f) \leqslant (k+m+u+2)\{T(r,f)+T(r,\mathscr{L})\} + S(r,f) + S(r,\mathscr{L}).$$

Now, by combining the above two inequalities, we obtain

$$(n+m)\{T(r,f)+T(r,\mathscr{L})\} \leqslant (2k+2m+2u+4)\{T(r,f)+T(r,\mathscr{L})\} + S(r,f) + S(r,\mathscr{L}).$$

This is a contradiction to $n > 2k+m+2u+6$.

Subcase 2.2. Let $B \neq 0$ and $A \neq B$. Then from (3.21), we get

$$\mathscr{F} = \frac{(B+1)\mathscr{G}-(B-A+1)}{B\mathscr{G}+(A-B)}$$

and hence $\overline{N}\left(r,\frac{1}{\mathscr{G}-\left(\frac{B-A+1}{B+1}\right)}\right)=\overline{N}\left(r,\frac{1}{\mathscr{F}}\right)$. Proceeding in a similar manner to Subcase 2.1, we arrive at a contradiction.

Subcase 2.3. Let $B=0$ and $A\neq 0$. Then from (3.21), we get

$$\mathscr{F}=\frac{\mathscr{G}-(1-A)}{A} \qquad \text{and} \qquad \mathscr{G}=A\mathscr{F}-(A-1).$$

If $A\neq 1$, it follows that $\overline{N}\left(r,\frac{1}{\mathscr{F}-\frac{A-1}{A}}\right)=\overline{N}\left(r,\frac{1}{\mathscr{G}}\right)$ and $\overline{N}\left(r,\frac{1}{\mathscr{G}-(1-A)}\right)=\overline{N}\left(r,\frac{1}{\mathscr{F}}\right)$.

By using the similar argument as in Subcase 2.1, we arrive at a contradiction.

Thus $A=1$, which implies $\mathscr{F}=\mathscr{G}$ and therefore

$$[f^n(f-1)^m\Delta_c^u f]^{(k)}=[\mathscr{L}^n(\mathscr{L}-1)^m\Delta_c^u\mathscr{L}]^{(k)}.$$

This completes the proof of Theorem 1.1. □

3.2. Proof of Theorem 1.2 to 1.4

Since f and $\mathscr{L}$ are meromorphic functions and $R(z)$ is a rational function therefore $R(z)$ is a small function of f and $\mathscr{L}$. Thus, Theorem 1.2 can be proved in similar lines as Theorem1.1.

Theorems 1.3 and 1.4 can be proved by using Lemma 2.16 and Lemma 2.17 and by following similar argument as that of Theorem 1.1.

4. Conclusion

We have studied the value distribution of a L-function and an arbitrary meromorphic function using the concept of weighted sharing when certain type of difference-differential polynomials generated by them share a non-zero small function or a non-zero rational function. The fact that the L-function $\mathscr{L}$ has only one possible pole at $s=1$ in $\mathbb{C}$ is the central idea of this paper and our results greatly generalises and improves the earlier results due to Hao and Chen, Mandal and Datta. Also we can pose the following open questions.

OPEN QUESTION 1. Can the condition for n in Theorem 1.1. to 1.4. be still reduced?

OPEN QUESTION 2. What happens to the condition n if we use weakly weighted sharing or truncated weighted sharing or partial sharing?

Acknowledgements. Authors are very much thankful to the editor and refrees for their careful reading and valuable suggestions which helped to improve the manuscript significantly.

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(Received May 7, 2022)

Preetham N. Raj
Department of Mathematics, Jnanabharathi Campus
Bangalore University
Bangalore, India – 560 056
e-mail: preethamnraj@gmail.com

Harina P. Waghamore
Department of Mathematics, Jnanabharathi Campus
Bangalore University
Bangalore, India – 560 056
e-mail: harinapw@gmail.com