Journal of Classical Analysis
Volume 21, Number 2 (2023), 187–191 doi:10.7153/jca-2023-21-11

ON GENERALIZED RIESZ SUMMABILITY OF FACTORED FOURIER SERIES

DEEPAK ACHARYA, SUBRATA KUMAR SAHU*, PURNA CHANDRA NAYAK AND UMAKANTA MISRA

Abstract. In the present article, we have established a result on $\varphi-|\bar{N},p_n;\delta,\mu|_k$ summability of general summability factor of Fourier series, generalizing a result on $|\bar{N},p_n|_k$ summability.

1. Introduction

Let $\sum a_n$ be a given infinite series with sequence of its partial sums (s_n) and (p_n) be a sequence of positive numbers such that

$$P_n=\sum_{\nu=0}^{n} p_\nu,\ (P_{-1}=p_{-1}=0,\ i\geqslant 1).$$

Then the sequence-to-sequence transformation

$$\sigma_n=\frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_\nu,\ (P_n\neq 0),$$

defines the sequence (σ_n) of the $(\bar{N},p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) [1]. The series $\sum a_n$ is said to be summable $|\bar{N},p_n|_k$, $k\geqslant 1$, if [2]

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|\triangle\sigma_{n-1}|^k<\infty.$$

Let (φ_n) be any sequence of positive real numbers. Then the series $\sum a_n$ is said to be $\varphi-|\bar{N},p_n|_k$, $k\geqslant 1$, summable if [9]

$$\sum_{n=1}^{\infty}\varphi_n^{k-1}|\sigma_n-\sigma_{n-1}|^k<\infty.$$

and for $\delta\geqslant 0$, it is said to be summable $\varphi-|\bar{N},p_n;\delta|_k$, if [9]

$$\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}|\sigma_n-\sigma_{n-1}|^k<\infty.$$

Mathematics subject classification (2020): 40D15, 42A24, 42A16.

Keywords and phrases: Summability factors, absolute summability, infinite series, Fourier series, Hölder inequality, Minkowski inequality.

* Corresponding author.

Further, for $\mu \geqslant 1$, it is said to be $\varphi - |\bar{N}, p_n; \delta, \mu|_k$ summable, if [9]

$$\sum_{n=1}^{\infty} \varphi_n^{\mu(\delta k+k-1)} |\sigma_n - \sigma_{n-1}|^k < \infty.$$

Clearly, by taking $\delta = 0$, $\mu = 1$ and $\varphi_n = \dfrac{P_n}{p_n}$, the summability method $\varphi - |\bar{N}, p_n; \delta, \mu|_k$ reduces to $|\bar{N}, p_n|_k$ summability.

Let f be a periodic function with period 2π and integrable-L over $(-\pi, \pi)$. Assuming that the constant term in the Fourier series of the function f to be zero, let

$$f(t) = \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} C_n(t).$$

Dealing with $|\bar{N}, p_n|_k$ summability factors and $\varphi - |\bar{N}, p_n; \delta|_k$ summability factors of Fourier series, many results have been done by different authors (see [3, 4, 5, 6, 7, 8, 10, 11]). Among those, Bor [4] has proved the following theorem:

THEOREM 1.1. *If* (λ_n) *is a non-negative and non-increasing sequence such that* $\sum p_n \lambda_n < \infty$, *where* (p_n) *is a sequence of positive numbers such that* $P_n \to \infty$ *as* $n \to \infty$ *and* $\sum_{v=1}^{n} P_v C_v(t) = O(P_n)$, *then the series* $\sum C_n(t) P_n \lambda_n$ *is summable* $|\bar{N}, p_n|_k$, $k \geqslant 1$.

Subsequently, dealing with $\varphi - |\bar{N}, p_n; \delta|_k, k \geqslant 1$, summability, Yildiz [11] has established the following theorem:

THEOREM 1.2. *Let* (p_n) *and* (λ_n) *be sequences satisfying the conditions of Theorem* 1.1 *and let* (φ_n) *be a sequence of positive real numbers such that*

(i) $\varphi_n p_n = O(P_n)$,

(ii) $\displaystyle\sum_{n=v+1}^{\infty} \varphi_n^{\delta k-1} (P_{n-1})^{-1} = O\left(\varphi_v^{\delta k} (P_v)^{-1}\right)$,

(iii) $\displaystyle\sum_{n=1}^{m} \varphi_n^{\delta k} p_n \lambda_n = O(1)$, *as* $m \to \infty$,

(iv) $\displaystyle\sum_{n=1}^{m} \varphi_n^{\delta k} P_n \triangle \lambda_n = O(1)$, *as* $m \to \infty$.

Then the series $\sum C_n(t) P_n \lambda_n$ *is summable* $\varphi - |\bar{N}, p_n; \delta|_k$, $k \geqslant 1$, $0 \leqslant \delta k < 1$.

We require the following lemma to prove our main theorem.

LEMMA 1.3. ([4]) *If* (λ_n) *is a non-negative and non-increasing sequence such that* $\sum p_n \lambda_n$ *is convergent, where* (p_n) *is a sequence of positive numbers such that* $P_n \to \infty$ *as* $n \to \infty$, *then* $p_n \lambda_n = O(1)$ *and* $\sum P_n \triangle \lambda_n < \infty$.

2. Main results

However, generalizing Theorem 1.1 and Theorem 1.2, we establish the following theorem.

THEOREM 2.1. *Let (p_n) and (λ_n) be sequences satisfying the conditions of Theorem 1.1 and let (φ_n) be a sequence of positive real numbers such that*

(i) $\varphi_n p_n = O(P_n)$,

(ii) $\sum_{n=v+1}^{\infty} \varphi_n^{\mu(\delta k+k-1)-k} (P_{n-1})^{-1} = O\Big(\varphi_v^{\mu(\delta k+k-1)-k+1} (P_v)^{-1}\Big)$,

(iii) $\sum_{n=1}^{m} \varphi_n^{\mu(\delta k+k-1)-k+1} p_n \lambda_n = O(1)$, *as* $m \to \infty$, *and*

(iv) $\sum_{n=1}^{m} \varphi_n^{\mu(\delta k+k-1)-k+1} P_n \triangle\lambda_n = O(1)$, *as* $m \to \infty$.

Then the series $\sum C_n(t) P_n \lambda_n$ *is summable* $\varphi - |\bar{N}, p_n; \delta, \mu|_k$, $k \geqslant 1$, $0 \leqslant \delta k < 1$, $\mu \geqslant 1$.

Proof. Let $I_n(t)$ be the sequence of $(\bar{N}, p_n)$ means of the series $\sum C_n(t) P_n \lambda_n$. Then, by definition, we have

$$I_n(t) = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{i=0}^{v} C_i(t) P_i \lambda_i = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{n-1}) C_v(t) P_v \lambda_v.$$

Then, for $n \geqslant 1$, we have

$$I_n(t) - I_{n-1}(t) = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} C_v(t) P_v \lambda_v.$$

Using Abel's transformation, we have

$$\begin{aligned} I_n(t) - I_{n-1}(t) &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \triangle(P_{v-1}\lambda_v) \sum_{r=1}^{v} C_r(t) P_r + \frac{p_n}{P_n} \lambda_n \sum_{r=1}^{n} C_r(t) P_r \\ &= O(1) \Big\{ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} (P_v \triangle\lambda_v - p_v \lambda_v) P_v \Big\} + O(1) p_n \lambda_n \\ &= O(1) \Big\{ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} (P_v P_v \triangle\lambda_v - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} (P_v p_v \lambda_v) + p_n \lambda_n \Big\} \\ &= O(1) \Big\{ I_{n,1} + I_{n,2} + I_{n,3} \Big\}. \end{aligned}$$

Using Minkowski's inequality, in order to establish the theorem, it is sufficient to show that $\sum_{n=1}^{\infty} \varphi_n^{\mu(\delta k+k-1)}|I_{n,r}|^k < \infty$, for $r = 1,2,3$. Now we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\mu(\delta k+k-1)}|I_{n,1}|^k &= \sum_{n=1}^{m+1} \varphi_n^{\mu(\delta k+k-1)}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_vP_v\triangle\lambda_v\right|^k \\
&\leqslant \sum_{n=2}^{m+1} \varphi_n^{\mu(\delta k+k-1)-k}\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1} P_vP_v\triangle\lambda_v\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_vP_v\triangle\lambda_v\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1} \varphi_n^{\mu(\delta k+k-1)-k}\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1} P_vP_v\triangle\lambda_v\right\} \\
&= O(1)\sum_{v=1}^{m} P_vP_v\triangle\lambda_v \sum_{n=v+1}^{m+1} \varphi_n^{\mu(\delta k+k-1)-k}\frac{1}{P_{n-1}} \\
&= O(1)\sum_{v=1}^{m} \varphi_v^{\mu(\delta k+k-1)-k+1}P_v\triangle\lambda_v = O(1) \text{ as } m\to\infty.
\end{aligned}$$

Now, when $k > 1$, applying Hölder's inequality with indices k and k' where $\frac{1}{k}+\frac{1}{k'}$, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\mu(\delta k+k-1)}|I_{n,2}|^k &= \sum_{n=2}^{m+1} \varphi_n^{\mu(\delta k+k-1)}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_vp_v\lambda_v\right|^k \\
&\leqslant \sum_{n=2}^{m+1} \varphi_n^{\mu(\delta k+k-1)-k}\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1} P_v^kp_v\triangle\lambda_v^k\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\right\}^{k-1} \\
&= O(1)\sum_{v=1}^{m} P_v^kp_v\lambda_v^k \sum_{n=v+1}^{m+1} \varphi_n^{\mu(\delta k+k-1)-k}\frac{1}{P_{n-1}} \\
&= O(1)\sum_{v=1}^{m} \varphi_v^{\mu(\delta k+k-1)-k+1}\frac{1}{P_v}(P_v\lambda_v)^kp_v \\
&= O(1)\sum_{v=1}^{m} \varphi_v^{\mu(\delta k+k-1)-k+1}p_v\lambda_v = O(1) \text{ as } m\to\infty.
\end{aligned}$$

by virtue of the hypothesis of Theorem 1.1 and Lemma 1.3. Finally, using the fact that $P_n\lambda_n = O(1)$, by Lemma 1.3, we obtain that

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{\mu(\delta k+k-1)}|I_{n,3}|^k &= \sum_{n=1}^{m} \varphi_n^{\mu(\delta k+k-1)}|p_n\lambda_n|^k \\
&\leqslant \sum_{n=1}^{m} \varphi_n^{\mu(\delta k+k-1)-k+1}\varphi_n^{k-1}(p_n\lambda_n)^{k-1}(p_n\lambda_n) \\
&= \sum_{n=1}^{m} \varphi_n^{\mu(\delta k+k-1)-k+1}(\varphi_np_n)^{k-1}(\lambda_n)^{k-1}(p_n\lambda_n)
\end{aligned}$$

$$= O(1)\sum_{n=1}^{m} \varphi_n^{\mu(\delta k+k-1)-k+1} P_n\lambda_n)^{k-1} p_n\lambda_n$$

$$= O(1)\sum_{n=1}^{m} \varphi_n^{\mu(\delta k+k-1)-k+1} p_n\lambda_n = O(1) \text{ as } m\to\infty.$$

by virtue of the hypotheses of Theorem 1.2. This completes the proof. □

REFERENCES

[1] G. H. HARDY, *Divergent Series*, Oxford University Press, Oxford, 1949.

[2] H. BOR, *On two summability methods*, Math. Proc. Cambridge Philos. Soc., **97** (1985), 147–149.

[3] H. BOR, *Multipliers for* $|\bar{N},p_n|_k$ *summability of Fourier series*, Bull. Inst. Math. Acad. Sin. (N.S.), **17**, (1989), 285–290.

[4] H. BOR, *On the absolute summability factors of Fourier series*, J. Comput. Anal. Appl., **8**, (2006), 223–227.

[5] H. BOR, *Some new results on infinite series and Fourier series*, Positivity, **19**, (2015), 467–473.

[6] H. BOR, *Some new results on absolute Riesz summability of infinite series and Fourier Series*, Positivity, **20**, (2016), 599–605.

[7] H. S. OZARSLAN, *On the local property of* $\varphi-|\bar{N},p_n;\delta|_k$ *summability of factored Fourier Series*, Bull. Inst. Math. Acad. Sin. (N.S.), **25**, (1997), 311–316.

[8] H. S. OZARSLAN, *A note on* $|\bar{N},p_n|_k$ *summability factors*, Int. J. Pure Appl. Math. **13**, (2004), 485–490.

[9] H. SEYHAN, *The absolute summability methods*, Ph.D. thesis, Ereiyes University, Kayseri, 1995.

[10] K. H. MISHRA, *Multipliers for* $|\bar{N},p_n|_k$ *summability of Fourier series*, Bull. Inst. Math. Acad. Sin. (N.S.), **14**, (1986), 431–438.

[11] S. YILDIZ, *On Riesz Summability factors of Fourier series*, Transaction of A. Razmadze Mathematical Institute, **171**, (2017), 328–33.

(Received June 6, 2023)

Deepak Acharya
Department of Mathematics
NIST Institute of Science and Technology (Autonomous)
Berhampur, Odisha – 761 008, India
e-mail: dpak.acharya888@gmail.com

Subrata Kumar Sahu
Department of Mathematics
NIST Institute of Science and Technology (Autonomous)
Berhampur, Odisha – 761 008, India
e-mail: subrataviit@gmail.com

Purna Chandra Nayak
Department of Mathematics
Bhadrak College (Autonomous)
Bhadrak, Odisha – 756 100, India
e-mail: pcnayak02@gmail.com

Umakanta Misra
Department of Mathematics
NIST Institute of Science and Technology (Autonomous)
Berhampur, Odisha – 761 008, India
e-mail: umakanta_misra@yahoo.com