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COMPLETE ASYMPTOTIC EXPANSION OF THE COMPOUND MEANS WITH APPLICATIONS

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Abstract. We present a complete asymptotic expansion of the compound mean $M \otimes N$ of two symmetric homogeneous means M and N and derive an efficient algorithm for computing coefficients in this expansion. This new approach is applied to obtain a simple formula for computing coefficients in the expansion of the arithmetic-geometric mean. We also give some other applications to the compounds of the classical means.

1. Introduction and motivation

Arithmetic-geometric mean is a famous mean obtained by a limiting iterative process of arithmetic and geometric means in a following way:

$$\begin{aligned} &A_0 = s, \quad G_0 = t, \quad s,t \in \mathbb{R}_+, \\ &A_n = \frac{A_{n-1}+G_{n-1}}{2}, \qquad G_n = \sqrt{A_{n-1}G_{n-1}}, \qquad n \geqslant 1. \end{aligned}$$

Both of these sequences converge to a same limit which is called arithmetic-geometric mean of the numbers s and t. This mean was studied by Gauss a long time ago, but it is still an interesting research subject of many mathematicians. Because of its fast convergence properties, this method is used to construct efficient algorithms for computing classical constants, elementary transcedental funtions and elliptic integrals ([2, 3]).

This process can be generalized to arbitrary bivariate means M and N. By bivariate mean we consider a symmetrical function $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\min(s,t) \leqslant F(s,t) \leqslant \max(s,t)$. Then, we define an iterative algorithm:

$$\begin{aligned} &M_0(s,t) = s, && N_0(s,t) = t, \\ &M_n(s,t) = M(M_{n-1},N_{n-1}), && N_n(s,t) = N(M_{n-1},N_{n-1}), \quad n \geqslant 1. \end{aligned} \tag{1.1}$$

If both of these sequences converge to the same limit, this common value is called the compound mean of s and t and is denoted by $M \otimes N(s,t)$. More details about this concept, existence and some properties of compound means can be found in [4, Ch.VI.3] and a comprehensive study on arithmetic-geometric mean and other bivariate

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means can be found in [16]. Recently, compound means were used in root finding method [15] and for solving automorphism equations for random means [14].

The problem with compound means is that they rarely have closed explicit forms and there is no an easy way to study them. Therefore it is useful to find another way of representing and analysing these means.

The asymptotic expansion of the mean $M(s,t)$ is a representation of mean in a form

$$M(x+s,x+t) = x\sum_{n=0}^{\infty} c_n(s,t)x^{-n}, \quad x\to\infty, \tag{1.2}$$

where $c_n(s,t)$ are polynomials of the degree n in variables s and t. If we consider homogeneous means, that is $M(\lambda s,\lambda t) = \lambda M(s,t)$, then $c_n(s,t)$ are homogeneous polynomials. The technique of developing asymptotic expansions of means is presented in a series of recently published papers [9, 10, 11]. This new method is successfully used in the comparison of classical means and establishing various relations between means, see cited papers for details. Also note that for expansions we use the equality sign $=$ instead of $\sim$ since they are usually convergent for x large enough.

In [5], authors derived the asymptotic expansion of the arithmetic-geometric mean and studied convergence and stationary properties of the coefficients in this expansion. In [1, 6], authors studied and presented the asymptotic expansion of the compound of the two power means and most recently, in [7], this was generalized to the compound of arbitrary two means. Authors presented an algorithm for computing coefficients in asymptotic expansions of composition of two means and they proved fast convergence and stationary properties of this iterative process. In a recent paper [13], author also studied the asymptotic behaviour of the compound means and gave some related numerical results.

First, we will present an efficient recursive formula for calculating coefficients in the asymptotic expansion of arbitrary compound mean which was not observed by authors in [7]. Then, we will derive a simple algorithm for computing coefficients in the expansion of the arithmetic-geometric mean and connect it with elliptic integral. Finally, we will also give some other examples of the compounds of some classical means.

In the sequel, the following fundamental lemma for transformation of power of asymptotic series will be crucial. The coefficients of the new series depend on the power r and initial sequence $\mathbf{a} = (a_n)_{n\in\mathbb{N}_0}$, it will be denoted here as $P[n,r,\mathbf{a}]$, see [12] for details.

LEMMA 1.1. *Let $a_0 \neq 0$ and $g(x)$ be a function with asymptotic expansion (as $x\to\infty$):*

$$g(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}.$$

Then for all real r it holds

$$[g(x)]^r \sim \sum_{n=0}^{\infty} P[n,r,\mathbf{a}]x^{-n},$$

where $P[0,r,\mathbf{a}] = a_0^r$ *and*

$$P[n,r,\mathbf{a}] = \frac{1}{na_0}\sum_{k=1}^{n}[k(1+r)-n]a_kP[n-k,r,\mathbf{a}], \quad n \geqslant 1.$$

2. Asymptotic expansion of the compound mean

It is shown in [9, 10] that simpler form of the coefficients in the asymptotic expansion of the mean (1.2) is obtained through variables α and β where

$$s = \alpha - \beta, \qquad t = \alpha + \beta,$$

and if $\alpha = 0$, we will have

$$M(x-\beta, x+\beta) = x\sum_{n=0}^{\infty} c_n(-\beta,\beta)x^{-n}.$$

Since c_n are homogeneous and symmetric polynomials of degree n, it follows

$$M(x-\beta, x+\beta) = x\sum_{n=0}^{\infty}\gamma_n\beta^{2n}x^{-2n}, \tag{2.1}$$

for some constants $(\gamma_n)_{n\in\mathbb{N}_0}$. The asymptotic expansion (2.1), which can be seen as the asymptotic expansion in one variable since

$$M(x-\beta, x+\beta) = xM\left(1-\tfrac{\beta}{x}, 1+\tfrac{\beta}{x}\right),$$

is sufficient to obtain the complete two variable asymptotic expansion (1.2) when $\alpha \neq 0$.

LEMMA 2.1. *Let M be symmetric homogeneous means with asymptotic expansion* (2.1)*. Then the coefficients $c_m(s,t)$, $t \neq \pm s$, in the asymptotic expansion* (1.2) *are given by the formula*

$$c_m(s,t) = (-1)^m\left(\frac{s+t}{2}\right)^m\sum_{n=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m-2}{m-2n}\gamma_n\left(\frac{t-s}{t+s}\right)^{2n}, \quad m \geqslant 0.$$

Proof. By simple computation and rearrangement of sums we obtain the following:

$$\begin{aligned}
M(x+s,x+t) &= M\left(x+\frac{(s+t)}{2}-\frac{(t-s)}{2}, x+\frac{(s+t)}{2}+\frac{(t-s)}{2}\right)\\
&= M(x+\alpha-\beta, x+\alpha+\beta) = \sum_{n=0}^{\infty}\gamma_n\beta^{2n}(x+\alpha)^{-2n+1}
\end{aligned}$$

$$= \sum_{m=0}^{\infty} \left[(-1)^m \alpha^m \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-2}{m-2n} \gamma_n \beta^{2n} \alpha^{-2n} \right] x^{-m+1}$$

$$= \sum_{m=0}^{\infty} \left[(-1)^m \left(\frac{s+t}{2} \right)^m \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-2}{m-2n} \gamma_n \left(\frac{t-s}{t+s} \right)^{2n} \right] x^{-m+1}. \quad \square$$

Let M and N be two arbitrary symmetric and homogeneous means with the asymptotic expansions

$$M(x-t, x+t) = \sum_{k=0}^{\infty} a_k t^{2k} x^{-2k+1}, \tag{2.2}$$

and

$$N(x-t, x+t) = \sum_{k=0}^{\infty} b_k t^{2k} x^{-2k+1}. \tag{2.3}$$

In [7], authors obtained the asymptotic expansion of the composition of means $H = F(M,N)$, where F is arbitrary mean with coefficients $(f_n)_{n\in\mathbb{N}_0}$. They proved that coefficients $(h_n)_{n\in\mathbb{N}_0}$ can be calculated by recursive algorithm

$$h_n = \sum_{k=0}^{\lfloor \frac{n}{2z} \rfloor} \gamma_k \sum_{j=0}^{n-2zk} P[j, 2k, \mathbf{d}] P[n-2zk-j, -2k+1, \mathbf{c}], \tag{2.4}$$

where sequences $\mathbf{c} = (c_n)_{n\in\mathbb{N}_0}$ and $\mathbf{d} = (d_n)_{n\in\mathbb{N}_0}$ are defined by

$$c_n = \frac{1}{2}(a_n + b_n), \qquad d_n = \frac{1}{2}(b_{n+z} - a_{n+z}). \tag{2.5}$$

Here z denotes the smallest number $m \geqslant 0$ such that $a_m - b_m \neq 0$, but notice that $z \geqslant 1$ since it holds $a_0 = b_0 = 1$.

We will now show that this algorithm can be also applied for calculating the compound of means $M \otimes N$. We state the following theorem.

THEOREM 2.2. *Let M and N be symmetric homogeneous means whose asymptotic expansions are given by* (2.2) *and* (2.3)*, and let their compound $M \otimes N$ have the asymptotic expansion*

$$M \otimes N(x-t, x+t) = \sum_{n=0}^{\infty} \gamma_n t^{2n} x^{-2n+1}. \tag{2.6}$$

Then the coefficients (γ_n) satisfy the recursive relation

$$\begin{aligned} \gamma_0 &= 1, \\ \gamma_n &= \sum_{k=0}^{\lfloor \frac{n}{2z} \rfloor} \gamma_k \sum_{j=0}^{n-2zk} P[j, 2k, \mathbf{d}] P[n-2zk-j, -2k+1, \mathbf{c}], \quad n \geqslant 1, \end{aligned} \tag{2.7}$$

where the sequences $\mathbf{c} = (c_n)_{n\in\mathbb{N}_0}$ and $\mathbf{d} = (d_n)_{n\in\mathbb{N}_0}$ are defined by (2.5).

Proof. By definition of the compound mean and its construction, it can easily be seen that the compound mean $M \otimes N$ satisfies relation

$$M \otimes N(M,N) = M \otimes N, \tag{2.8}$$

which is also known as the Gauss functional equation. Now we can apply result (2.4) for the composition of the means. By comparing the asymptotic expansions of the function on the right and left-hand side in (2.8), if follows that coefficients γ_n in the asymptotic expansion (2.6) are given by recursive relation (2.7) which proves the theorem. □

Let us show the first few coefficients in the expansion of the arbitrary compound mean. For a $z = 1$, which is usually the case for most of the means, we obtain the following coefficients:

$$\begin{aligned}
\gamma_0 &= 1,\\
\gamma_1 &= \frac{1}{2}(a_1+b_1),\\
\gamma_2 &= \frac{1}{2}(a_2+b_2) + \frac{1}{8}(a_1-b_1)^2(a_1+b_1),\\
\gamma_3 &= \frac{1}{2}(a_3+b_3) - \frac{1}{16}(a_1-b_1)(a_1^2-b_1^2-4(a_2-b_2))(a_1+b_1),\\
&\vdots
\end{aligned}$$

Regarding the general choice of variables, from asymptotic expansion (2.6) we can easily derive expansion of the type (1.2) by applying Lemma 2.1, where coefficients (γ_n) are defined in (2.7).

COROLLARY 2.3. *The asymptotic expansion of a compound mean $M \otimes N$ reads as:*

$$\begin{aligned}
&M \otimes N(x+s,x+t)\\
&\quad = x + \frac{s+t}{2} + \frac{(s-t)^2}{8}(a_1+b_1)x^{-1} - \frac{(s-t)^2(s+t)}{16}(a_1+b_1)x^{-2}\\
&\qquad + \frac{(s-t)^2}{128}\Big((s^2+t^2)\left(a_1^3+b_1^3-a_1b_1(a_1+b_1)+4(a_1+b_1)+4(a_2+b_2)\right)\\
&\qquad - 2st\left(a_1^3+b_1^3-a_1b_1(a_1+b_1)-4(a_1+b_1)+4(a_2+b_2)\right)\Big)x^{-3} + \mathcal{O}(x^{-4}).
\end{aligned}$$

From the proof of the Lemma 2.1 we can also deduce the more convenient expansion through the variables α and β:

$$\begin{aligned}
M \otimes N(x+\alpha-\beta, x+\alpha+\beta) &= x + \alpha + \frac{\beta^2}{2}(a_1+b_1)x^{-1} - \frac{\alpha\beta^2}{2}(a_1+b_1)x^{-2}\\
&+ \beta^2\left(\frac{\beta^2}{2}(a_2+b_2) + \frac{\alpha^2}{2}(a_1+b_1) + \frac{\beta^2}{8}(a_1+b_1)(a_1-b_1)^2\right)x^{-3} + \mathcal{O}(x^{-4}).
\end{aligned}$$

3. Asymptotic expansion of the arithmetic-geometric mean and related means

Asymptotic expansion of the arithmetic-geometric mean was derived in [5]. Authors obtained the following:

$$A\otimes G(x-t,x+t)=x-\frac{1}{4}t^2x^{-1}-\frac{5}{64}t^4x^{-3}-\frac{11}{256}t^6x^{-5}-\frac{469}{16384}t^8x^{-7}-\dots$$

Coefficients in this expansion were obtained by a tedious iterative procedure until their stationarity was achieved and there was not any direct algorithm or formula describing the sequence of coefficients in this expansion.

Applying Theorem 2.2, we can now obtain these coefficients directly by algorithm (2.7). This is much easier than before, but still in this algorithm we have to apply Lemma 1.1 twice. We will now show that coefficients in the expansion of the arithmetic-geometric mean satisfy simple recursive formula and can be calculated very efficiently.

First let us recall asymptotic expansions of the arithmetic and geometric mean derived in [10]:

$$\begin{aligned}&A(x-t,x+t)=x,\\&G(x-t,x+t)=x-\frac{1}{2}t^2x^{-1}-\frac{1}{8}t^4x^{-3}-\frac{1}{16}t^6x^{-5}-\dots.\end{aligned}$$

Following remark will be useful in the sequel.

REMARK 3.1. The asymptotic expansion of the geometric mean

$$G(x-t,x+t)=\sum_{n=0}^{\infty}g_nt^{2n}x^{-2n+1}$$

has the following coefficients

$$g_0=1,\qquad g_n=-\frac{1}{2^{2n-1}}C_{n-1},\quad n\geqslant 1,\tag{3.1}$$

where C_n denotes the n-th Catalan number. Namely, in the paper [10], authors showed that $g_n=-\frac{(2n-3)!!}{2^nn!}$, for $n\geqslant 1$ (note that $(-1)!!=1$). By simple computation this can be written through Catalan numbers $C_n=\frac{1}{n+1}\binom{2n}{n}$ and the (3.1) holds.

By using properties of the Catalan numbers, we will derive following algorithm for calculating coefficients in the expansion of $A\otimes G$.

THEOREM 3.2. *The coefficients (γ_m) in the asymptotic expansion of compound mean $A\otimes G$ are given by*

$$\begin{aligned}&\gamma_0=1,\\&\gamma_m=\frac{1}{2^{2m}}\sum_{n=0}^{\lfloor\frac{m}{2}\rfloor}\gamma_n\sum_{k=0}^{m-2n}\frac{m-k}{m+k}\binom{k+m}{m}\binom{m-k-2}{m-k-2n},\quad m\geqslant 1.\end{aligned}\tag{3.2}$$

Proof. If we directly apply Theorem 2.2 and calculate coefficients by (2.7), note that sequences (c_n) and (d_n) coincide in most terms. Therefore it is better to get one step back and group these coefficients in a more efficient way.

Proof of (2.4) is based on the following technique:

$$F(M,N) = \frac{N+M}{2} + f_1\left(\frac{N-M}{2}\right)^2\left(\frac{N+M}{2}\right)^{-1} + f_2\left(\frac{N-M}{2}\right)^4\left(\frac{N+M}{2}\right)^{-3} + \dots$$

where the composition $F(M,N)$ was written as the asymptotic expansion in terms of the asymptotic sequence

$$F_n = \left(\frac{N-M}{2}\right)^{2n}\left(\frac{N+M}{2}\right)^{-2n+1} + \mathscr{O}(x^{-2n(z+1)+1}), \quad n \geqslant 1.$$

For details see [7].

In our case, functional equation (2.8) reads as

$$A \otimes G(x-t, x+t) = A \otimes G(G(x-t, x+t), A(x-t, x+t)).$$

On the left-hand side, we have the asymptotic expansion of the $A \otimes G$ and on the right-hand side we have same expansion but with the variables $\frac{1}{2}(A-G)$ and $\frac{1}{2}(A+G)$. Hence, we have the following calculations:

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_n t^{2n} x^{-2n+1} \\
&= \sum_{n=0}^{\infty} c_n \left(-\frac{1}{2}\sum_{k=1}^{\infty} g_k t^{2k} x^{-2k+1}\right)^{2n}\left(x + \frac{1}{2}\sum_{k=1}^{\infty} g_k t^{2k} x^{-2k+1}\right)^{-2n+1} \\
&= \sum_{n=0}^{\infty} c_n \left(\frac{1}{4}t^2 x^{-1}\sum_{k=0}^{\infty} C_k \frac{t^{2k}}{2^{2k}} x^{-2k}\right)^{2n} \\
&\qquad \times \sum_{j=0}^{\infty}\binom{-2n+1}{j} x^{-2n+1-j}\left(-\frac{1}{4}t^2 x^{-1}\sum_{k=0}^{\infty} C_k \frac{t^{2k}}{2^{2k}} x^{-2k}\right)^{j} \\
&= \sum_{n=0}^{\infty} c_n \sum_{j=0}^{\infty}\binom{-2n+1}{j} x^{-2n+1-j}\left(\frac{1}{4}t^2 x^{-1}\right)^{2n+j}(-1)^j\left(\sum_{k=0}^{\infty} C_k \frac{t^{2k}}{2^{2k}} x^{-2k}\right)^{2n+j} \\
&= \sum_{n=0}^{\infty} c_n \sum_{j=0}^{\infty}\binom{-2n+1}{j}\left(\frac{1}{4}t^2 x^{-2}\right)^{2n+j} x(-1)^j \sum_{k=0}^{\infty} C(k, 2n+j)\frac{t^{2k}}{2^{2k}} x^{-2k},
\end{aligned}$$

where $C(k,j)$ equals δ_{0k} for $j=0$ and it denotes k-fold Catalan product

$$C(k,j) = \sum_{\substack{i_1+\dots+i_j=k \\ i_n \geqslant 0}} C_{i_1}\cdots C_{i_j}$$

for $j \neq 0$. After substitution $\frac{t^2}{4}x^{-2} = y^{-1}$ we obtain

$$\sum_{n=0}^{\infty} c_n 2^{2n} y^{-n} = \sum_{n=0}^{\infty} c_n \sum_{j=0}^{\infty} \binom{-2n+1}{j} (-1)^j y^{-2n-j} \sum_{k=0}^{\infty} C(k, 2n+j) y^{-k},$$

and further

$$\begin{aligned}
\sum_{m=0}^{\infty} c_m 2^{2m} y^{-m} &= \sum_{m=0}^{\infty} \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} c_n \sum_{k=0}^{m-2n} \binom{-2n+1}{-2n+m-k} (-1)^{m-k} C(k, m-k) y^{-m} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} c_n \sum_{k=0}^{m-2n} \binom{m-k-2}{m-k-2n} C(k, m-k) y^{-m}.
\end{aligned}$$

Catalan's k-fold convolution formula (see [8]) states that

$$C(k, j) = \frac{j}{2k+j} \binom{2k+j}{k+j}.$$

Now it follows that $c_0 = 1$ and for $m \geqslant 1$ we have

$$\begin{aligned}
c_m &= \frac{1}{2^{2m}} \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} c_n \sum_{k=0}^{m-2n} \binom{m-k-2}{m-k-2n} C(k, m-k) \\
&= \frac{1}{2^{2m}} \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} c_n \sum_{k=0}^{m-2n} \binom{m-k-2}{m-k-2n} \frac{m-k}{m+k} \binom{k+m}{m},
\end{aligned}$$

which completes the proof. $\square$

REMARK 3.3. It is well-known that arithmetic-geometric mean has an integral representation in terms of the elliptic integral of the first kind, see [16]. Namely, it holds

$$A \otimes G(s, t) = \left(\frac{2}{\pi} \int_0^{\pi/2} \frac{1}{\sqrt{s^2 \cos^2 \theta + t^2 \sin^2 \theta}} d\theta \right)^{-1}.$$

Therefore, we can apply Lemma 1.1 for $r = -1$ to the coefficients of the $A \otimes G$ and easily obtain asymptotic expansion of the elliptic integral, which has already been done in [5].

But the complete elliptic integral of the first kind is usually defined through one parameter $m \in [0, 1]$ in a way

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}}$$

and it has a famous series representation:

$$K(m) = \frac{\pi}{2} \sum_{k=0}^{\infty} \left[\frac{(2k-1)!!}{(2k)!!} \right]^2 m^k = \frac{\pi}{2} \left(1 + \frac{1}{4} m + \frac{9}{64} m^2 + \frac{25}{256} m^3 + \dots \right)$$

Since corresponding relation with arithmetic-geometric mean is:

$$A \otimes G(1-\sqrt{m}, 1+\sqrt{m}) = \frac{\pi}{2}\big(K(m)\big)^{-1},$$

and if we apply asymptotic expansion of $A \otimes G$ with $x = 1$ and $t = \sqrt{m}$, we have

$$\sum_{k=0}^{\infty} \gamma_k m^k = \left(\sum_{k=0}^{\infty} \left[\frac{(2k-1)!!}{(2k)!!} \right]^2 m^k \right)^{-1}. \tag{3.3}$$

Therefore, we can obtain coefficients (γ_n) directly from the known series of the elliptic integral together with the Lemma 1.1 for $r = -1$. By this discussion, we have proved the following result.

The coefficients (γ_n) in the asymptotic expansion of compound mean $A \otimes G$ are given by

$$\gamma_0 = 1, \qquad \gamma_n = -\sum_{k=1}^{n} \left[\frac{(2k-1)!!}{(2k)!!} \right]^2 \gamma_{n-k}, \quad n \geqslant 1. \tag{3.4}$$

REMARK 3.4. In fact, by Proposition 3.1, it follows that (3.4) can be written through the Catalan numbers and the coefficients (g_n) of the geometric mean:

$$\gamma_n = -\sum_{k=1}^{n} [(2k-1)g_n]^2 \gamma_{n-k}, \quad n \geqslant 1.$$

While both algorithms (3.2) and (3.4) define the same sequence, recursion (3.4) has simpler coefficients, but recursion (3.2) uses half less terms of (γ_n) in its calculation.

We can also obtain the asymptotic expansion of $A \otimes G$ in two variables by applying Corollary 2.3. Here are the first few coefficients:

$$\begin{aligned} A \otimes G(x+s, x+t) = x + \frac{s+t}{2} - \frac{(s-t)^2}{16} x^{-1} + \frac{(s-t)^2(s+t)}{32} x^{-2} \\ - \frac{(s-t)^2}{1024}\big(21(s^2+t^2) + 22st\big) x^{-3} + \mathscr{O}(x^{-4}), \end{aligned}$$

or in the more convenient form through the variables α and β:

$$\begin{aligned} A \otimes G(x+\alpha-\beta, x+\alpha+\beta) = x + \alpha - \frac{\beta^2}{4} x^{-1} + \frac{\alpha\beta^2}{4} x^{-2} \\ - \frac{\beta^2}{64}\big(16\alpha^2 + 5\beta^2\big) x^{-3} + \mathscr{O}(x^{-4}). \end{aligned}$$

Next, we will derive the coefficients in the compound of the arithmetic and harmonic mean.

THEOREM 3.5. *The coefficients* (h_n) *in the asymptotic expansion of* $A \otimes H$ *are given by*

$$h_0 = 1, \qquad h_n = \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-2}{n-2k} h_k, \quad n \geqslant 1. \tag{3.5}$$

Proof. We will directly apply Theorem 2.2. Since the coefficients of the arithmetic and harmonic mean are given by

$$\begin{aligned} &a_0 = 1,\ a_n = 0,\ n \geqslant 1, \\ &b_0 = 1,\ b_1 = -1,\ b_n = 0,\ n \geqslant 2, \end{aligned}$$

it follows $z = 1$ and

$$\begin{aligned} &c_0 = 1,\ c_1 = -\frac{1}{2},\ c_n = 0,\ n \geqslant 2, \\ &d_0 = \frac{1}{2},\ d_n = 0,\ n \geqslant 1. \end{aligned}$$

Now we have

$$\begin{aligned} &P[j, 2k, \mathbf{d}] = \begin{cases} \frac{1}{2^{2k}}\ , & j = 0, \\ 0\ , & j \neq 0, \end{cases} \\ &P[m, -2k+1, \mathbf{c}] = \binom{-2k+1}{m} \frac{(-1)^m}{2^m}, \end{aligned}$$

and by (2.7) it follows

$$\begin{aligned} &\sum_{j=0}^{n-2zk} P[j, 2k, \mathbf{d}] P[n-2zk-j, -2k+1, \mathbf{c}] \\ &\qquad = P[0, 2k, \mathbf{d}] P[n-2k, -2k+1, \mathbf{c}] \\ &\qquad = \frac{1}{2^{2k}} \binom{-2k+1}{n-2k} \frac{(-1)^n}{2^{n-2k}} = \frac{1}{2^n} \binom{n-2}{n-2k}, \end{aligned}$$

which proves the theorem. $\square$

REMARK 3.6. Since $A \otimes H$ is equal to the geometric mean, that is $h_n = g_n$, it follows that $g_n = -\frac{C_{n-1}}{2^{2n-1}}$ satisfy relation (3.5) which is a valid identity for the Catalan numbers.

Applying Theorem 2.2 we can also obtain the compound of the geometric-harmonic mean, but we will show here that we can obtain direct connection with the asymptotic expansion of the arithmetic-geometric mean and derive simple formula without any recursion.

THEOREM 3.7. *The coefficients (f_n) in the asymptotic expansion of $G\otimes H$ are given by*

$$f_0 = 1,$$
$$f_n = -(4n-1)\left[\frac{(2n-3)!!}{(2n)!!}\right]^2, \quad n \geqslant 1.$$

Proof. Note that if we start iterative procedure for arithmetic-geometric mean with $A_0 = \frac{1}{s}$, $B_0 = \frac{1}{t}$, we have $A_n = \frac{1}{H_n}$ and $B_n = \frac{1}{G_n}$, where H_n and G_n are the n-th terms in the iterative procedure with means G and H for initial points s and t, that is

$$G\otimes H(s,t) = (A\otimes G(\tfrac{1}{s},\tfrac{1}{t}))^{-1} = st(A\otimes G(s,t))^{-1}. \tag{3.6}$$

Therefore it holds

$$G\otimes H(x-t,x+t) = (x^2-t^2)(A\otimes G(x-t,x+t))^{-1}.$$

Now applying (3.4), more precisely (3.3) for the inverse of $A\otimes G$, we have

$$\begin{aligned}
&G\otimes H(x-t,x+t)\\
&\quad = x\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 t^{2n}x^{-2n} - t^2x^{-1}\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 t^{2n}x^{-2n}\\
&\quad = x + \sum_{n=1}^{\infty}\left(\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 - \left[\frac{(2n-3)!!}{(2n-2)!!}\right]^2\right) t^{2n}x^{-2n+1}\\
&\quad = x + \sum_{n=1}^{\infty}\left[\frac{(2n-3)!!}{(2n)!!}\right]^2 [(2n-1)^2-4n^2]t^{2n}x^{-2n+1},
\end{aligned}$$

and the proof follows. □

Same as before, we can obtain the asymptotic expansion of $G\otimes H$ in two variables by applying Corollary 2.3:

$$\begin{aligned}
G\otimes H(x+s,x+t) = x &+ \frac{s+t}{2} - \frac{3(s-t)^2}{16}x^{-1} + \frac{3(s-t)^2(s+t)}{64}x^{-2}\\
&- \frac{(s-t)^2}{1024}\big(55(s^2+t^2)+82st\big)x^{-3} + \mathscr{O}(x^{-4}),
\end{aligned}$$

or in simpler form through the variables α and β:

$$\begin{aligned}
G\otimes H(x+\alpha-\beta,x+\alpha+\beta) = x+\alpha &- \frac{3\beta^2}{4}x^{-1} + \frac{3\alpha\beta^2}{4}x^{-2}\\
&- \frac{\beta^2}{64}\big(48\alpha^2+7\beta^2\big)x^{-3} + \mathscr{O}(x^{-4}).
\end{aligned}$$

REMARK 3.8. Connection (3.6) between $G \otimes H$ and $A \otimes G$ from the previous theorem can be generalized in the following way. Suppose we have the coefficients (M_n) and (N_n) in the iterative procedure of two compoundable means M and N as defined in (1.1). If exists a suitable function, i.e. a continuous function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $n \in \mathbb{N}_0$ it holds

$$\begin{aligned} A_{n+1} &= \frac{1}{2}(f(M_n) + f(N_n)) = f(M_{n+1}) = f(M(M_n, N_n)), \\ B_{n+1} &= \sqrt{f(M_n) f(N_n)} = f(N_{n+1}) = f(N(M_n, N_n)), \end{aligned}$$

then we have

$$M \otimes N(s,t) = f^{-1}(A \otimes G(f(s), f(t)))$$

For example, for general power mean M_r compounded with the geometric mean G the function $f(x) = x^r$ and it holds:

$$M_r \otimes G(s,t) = (A \otimes G(s^r, t^r))^{\frac{1}{r}}.$$

This principle may be useful for obtaining asymptotic expansions of such compounds.

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