CHARACTERIZATIONS OF CERTAIN SEQUENCES OF $q$–POLYNOMIALS

P. NJIONOU SADJANG

Abstract. We provide a new characterization for those sequences of quasi-orthogonal polynomials which form also $q$-Appell sets.

1. Introduction

Throughout this paper, we use the following standard notations

$\mathbb{N} := \{1, 2, 3, \ldots\}$, \quad $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}$.

Let $P_n(x), \ n = 0, 1, 2, \ldots$ be a polynomial set, i.e. a sequence of polynomials with $P_n(x)$ of exact degree $n$. Assume further that

$$\frac{dP_n(x)}{dx} = P'_n(x) = nP_{n-1}(x) \quad \text{for} \quad n = 0, 1, 2, \ldots.$$ 

Such polynomial sets are called Appell sets and received considerable attention since P. Appell [2] introduced them in 1880.

Let $q$ be an arbitrary real number (with $q \not= 0, 1$) and define the $q$-derivative [6] of a function $f(x)$ by means of

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad \text{if} \ x \not= 0$$

and $D_q f(0) = f'(0)$ if $f$ is differentiable at $x = 0$, which furnishes a generalization of the differential operator $\frac{d}{dx}$.

A basic ($q$-)analogue of Appell sequences was first introduced by Sharma and Chak [9] as those polynomial sets \( \{P_n(x)\}_{n=0}^{\infty} \) which satisfy

$$D_q P_n(x) = [n]_q P_{n-1}(x), \quad n = 1, 2, 3, \ldots$$

where \([n]_q = (1-q^n)/(1-q)\). They called them $q$-harmonic. Later, Al-Salam [1] studied these families and referred to them as $q$-Appell sets in analogy with ordinary Appell sets. Note that when $q \to 1$, (2) reduces to

$$\frac{dP_n(x)}{dx} = nP_{n-1}(x), \quad n = 1, 2, 3, \ldots$$

Mathematics subject classification (2020): 33C65, 33C45, 33D05, 33D45, 11B68.

Keywords and phrases: Appell polynomial set, $q$-Bernoulli polynomials, $q$-Euler polynomials, orthogonal polynomials, quasi-orthogonal polynomials, $q$-difference equation.
so that we may think of \( q \)-Appell sets as a generalization of Appell sets. We will call these polynomial sets \( q \)-Appell sets of type I.

A sequence of polynomials \( \{Q_n\} \), \( n = 0, 1, 2, \ldots \) \( \deg Q_n(x) = n \) is said to be quasi-orthogonal if there is an interval \( (a, b) \) and a non-decreasing function \( \alpha(x) \) such that

\[
\int_a^b x^m Q_n(x) d\alpha(x) \begin{cases} 
= 0 & \text{for } 0 \leq m \leq n - 2 \\
\neq 0 & \text{for } 0 \leq m = n - 1 \\
\neq 0 & \text{for } 0 = m = n.
\end{cases}
\]

We say that two polynomial sets are related if one set is quasi-orthogonal with respect to the interval and the distribution of the orthogonality of the other set. Riesz [8] and Chihara [3] have shown that a necessary and sufficient condition for the quasi-orthogonality of the \( \{Q_n(x)\} \) is that there exist non-zero constants, \( \{a_n\}_{n=0}^\infty \) and \( \{b_n\}_{n=1}^\infty \), such that

\[
Q_n(x) = a_n P_n(x) + b_n P_{n-1}(x), \\
Q_0(x) = a_0 P_0(x)
\]

where the \( \{P_n(x)\}_{n=0}^\infty \) are the related orthogonal polynomials.

In 1967, Al-Salam has given in a very short paper [1] a characterization of those sequences of orthogonal polynomials \( \{P_n(x)\} \) which are also \( q \)-Appell sets. More precisely, He gave a characterization of those sequences of orthogonal polynomials for which \( D_q P_n(x) = [n]_q P_{n-1}(x) \) for \( n = 1, 2, 3, \ldots \).

The purpose of this paper is to study those classes of polynomial sets \( \{P_n(x)\} \) that are at the same time quasi-orthogonal sets and \( q \)-Appell sets of type I. Extension will be done to those polynomials \( \{P_n(x)\} \) that satisfy

\[
D_q P_n(x) = [n]_q P_{n-1}(qx).
\]

The later polynomials will be called \( q \)-Appell polynomials of type II and appear already in [5] where some of their properties are given.

2. Preliminaries results and definitions

Let us introduce the so-called \( q \)-Pochhammer symbol

\[
(x;q)_n = \begin{cases} 
(1-x)(1-xq)\ldots(1-xq^{n-1}) & n = 1, 2, \ldots \\
1 & n = 0.
\end{cases}
\]

For a non-negative integer \( n \), the \( q \)-factorial is defined by

\[
[n]_q! = \prod_{k=0}^n [k]_q \quad \text{for } n \geq 1, \quad \text{and} \quad [0]_q! = 1.
\]

The \( q \)-binomial coefficients are defined by

\[
\begin{aligned}
\binom{n}{k}_q &= \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \quad (0 \leq k \leq n).
\end{aligned}
\]
We will use the following two $q$-analogues of the exponential function $e^x$ (see for example [6, 7] and the references therein)

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!},$$

and

$$E_q(x) = \sum_{k=0}^{\infty} \frac{q^{\frac{k}{2}}}{[k]_q!} x^k.$$  \(4\)

These two functions are related by the equation (see [6])

$$e_q(x) E_q(-x) = 1.$$ \(6\)

The basic hypergeometric or $q$-hypergeometric function $_r\phi_s$ is defined by the series

$$_r\phi_s\left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array} \mid q; z \right) := \sum_{k=0}^{\infty} \frac{(a_1, \cdots, a_r)_k}{(b_1, \cdots, b_s)_k} \left((-1)^k q^{\frac{k}{2}}\right)^{1+s-r} \frac{z^k}{(q;q)_k},$$

where

$$(a_1, \cdots, a_r)_k := (a_1; q)_k \cdots (a_r; q)_k.$$

The Al Salam-Carlitz I polynomials [7, p. 534] have the $q$-hypergeometric representation

$$U_n^{(a)}(x; q) = (-a)^n q^\left(\frac{n}{2}\right)_2 \phi_1\left(\begin{array}{c} q^{-n}, x^{-1} \\ 0 \end{array} \mid q; \frac{qx}{a} \right).$$

The Al-Salam Carlitz I polynomials fulfil the three-term recurrence relation

$$x U_n^{(a)}(x; q) = U_{n+1}^{(a)}(x; q) + (a + 1) q^n U_n^{(a)}(x; q) - a q^{n-1} (1 - q^n) U_{n-1}^{(a)}(x; q),$$

and the $q$-derivative rule

$$D_q U_n^{(a)}(x; q) = [n]_q U_{n-1}^{(a)}(x; q).$$

It is therefore clear that the Al-Salam Carlitz I polynomials form a $q$-Appell set.

The Al-Salam-Carlitz II polynomials [7, p. 537] have the $q$-hypergeometric representation

$$V_n^{(a)}(x; q) = (-a)^n q^\left(\frac{n}{2}\right)_2 \phi_0\left(\begin{array}{c} q^{-n}, x \\ q^n \end{array} \mid q; \frac{q^n}{a} \right).$$

Note that the Al Salam-Carlitz I polynomials and the Al Salam-Carlitz II polynomials are related in the following way:

$$U_n^{(a)}(x, q^{-1}) = V_n^{(a)}(x; q).$$
The Al-Salam Carlitz II polynomials fulfil the three-term recurrence relation
\[ xV_n^{(a)}(x; q) = V_{n+1}^{(a)}(x; q) + (a + 1)q^{-n}V_n^{(a)}(x; q) + aq^{2n+1}(1 - q^n)V_{n-1}^{(a)}(x; q), \]
and the \( q \)-derivative rule
\[ D_q V_n^{(a)}(x; q) = q^{-n-1}[n]_q V_n^{(a)}(qx; q). \]

Let us introduce the modified Al-Salam Carlitz II polynomials \( \gamma_n^{(a)}(x; q) \) by the relation
\[ \gamma_n^{(a)}(x; q) = q^{\binom{n}{2}} V_n^{(a)}(x; q). \] (7)

Then we have the following proposition.

**Proposition 1.** The polynomial sequence \( \{\gamma_n^{(a)}(x; q)\}_{n=0}^{\infty} \) is a \( q \)-Appell polynomial set of type II.

**Proposition 2.** (See [4, Theorem 1]) For \( \{Q_n(x)\} \) to be a set of polynomials quasi-orthogonal with respect to an interval \((a, b)\) and a distribution \( d\alpha(x) \), it is necessary and sufficient that there exist a set of nonzero constants \( \{T_k\}_{k=0}^{\infty} \) and a set of polynomials \( \{P_n(x)\} \) orthogonal with respect to \((a, b)\) and \( d\alpha(x) \) such that
\[ P_n(x) = \sum_{k=0}^{n} T_k Q_k(x), \quad n \geq 0. \] (8)

**Proposition 3.** (See [4, Theorem 2]) A necessary and sufficient condition that the set \( \{Q_n(x)\}_{n=0}^{\infty} \) where each \( Q_n(x) \) is a polynomial of degree precisely \( n \), be quasi-orthogonal is that it satisfies
\[ Q_{n+1}(x) = (x + b_n)Q_n(x) - c_n Q_{n-1}(x) + d_n \sum_{k=0}^{n-2} T_k Q_k(x), \]
for all \( n \), with \( d_0 = d_1 = 0 \).

**Proposition 4.** (See [1, Theorem 4.1]) If \( \{Q_n(x)\}_{n=0}^{\infty} \) is a \( q \)-Appell set which are also orthogonal, then there exists a non zero constant \( b \) such that
\[ Q_n(x) = b^n U_n^{(a/b)} \left( \frac{x}{b} \right), \]
for all \( n \geq 0 \).
3. Some notes on $q$-Appell polynomials of type II

As mentioned earlier in the manuscript, $q$-Appell polynomials of type II are those polynomial sets $\{P_n\}$ satisfying the relation

$$D_q P_n(x) = [n]_q P_{n-1}(qx).$$

Let us recall that the following Cauchy product for infinite series applies

$$\left( \sum_{n=0}^{\infty} A_n \right) \left( \sum_{n=0}^{\infty} B_n \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} A_k B_{n-k} \right).$$

(9)

In particular, if $A_n = \frac{a_n x^n}{[n]_q !}$ and $B_n = \frac{b_n x^n}{[n]_q !}$, then we have

$$\left( \sum_{n=0}^{\infty} \frac{a_n x^n}{[n]_q !} \right) \left( \sum_{n=0}^{\infty} \frac{b_n x^n}{[n]_q !} \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{[n]_q}{[k]_q} a_k b_{n-k} \right) \frac{x^n}{[n]_q !}.$$

(10)

3.1. Four equivalent statements

In this section, we give several characterizations of $q$-Appell sets of type II.

**Theorem 1.** Let $\{f_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials. Then the following are all equivalent:

1. $\{f_n(x)\}_{n=0}^{\infty}$ is a $q$-Appell set of type II.

2. There exists a sequence $(a_k)_{k \geq 0}$; independent of $n$; $a_0 \neq 0$; such that

$$f_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{(n-k)/2} a_k x^{n-k}.$$

3. $\{f_n(x)\}_{n=0}^{\infty}$ is generated by

$$A(t)E_q(xt) = \sum_{n=0}^{\infty} f_n(x) \frac{t^n}{[n]_q !},$$

where

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_q !},$$

(11)

is called the determining function for $\{f_n(x)\}_{n=0}^{\infty}$.

4. There exists a sequence $(a_k)_{k \geq 0}$; independent of $n$; $a_0 \neq 0$; such that

$$f_n(x) = \left( \sum_{k=0}^{\infty} a_k q^{(n-k)/2} D_q^k \right) x^n.$$
Proof. First, we prove that \((1) \implies (2) \implies (3) \implies (1)\).

(1) \implies (2). Since \(\{f_n(x)\}_{n=0}^{\infty}\) is a polynomial set, it is possible to write

\[
f_n(x) = \sum_{k=0}^{n} a_{n,k} \binom{n}{k}_q q^{(n-k)} x^{n-k}, \quad n = 1, 2, \ldots, \tag{12}
\]

where the coefficients \(a_{n,k}\) depend on \(n\) and \(k\) and \(a_{n,0} \neq 0\). We need to prove that these coefficients are independent of \(n\). By applying the operator \(D_q\) to each member of (12) and taking into account that \(\{f_n(x)\}_{n=0}^{\infty}\) is a \(q\)-Appell polynomial set of type II, we obtain

\[
f_{n-1}(qx) = \sum_{k=0}^{n-1} a_{n,k} \binom{n-1}{k}_q q^{(n-1-k)} (qx)^{n-1-k}, \quad n = 1, 2, \ldots, \tag{13}
\]

since \(D_q x^0 = 0\). Shifting index \(n \to n + 1\) in (13) and making the substitution \(x \to xq^{-1}\), we get

\[
f_n(x) = \sum_{k=0}^{n} a_{n+1,k} \binom{n}{k}_q q^{(n-k)} x^{n-k}, \quad n = 0, 1, \ldots, \tag{14}
\]

Comparing (12) and (14), we have \(a_{n+1,k} = a_{n,k}\) for all \(k\) and \(n\), which means that \(a_{n,k} = a_k\) is independent of \(n\).

(2) \implies (3). From (2), and the identity (10), we have

\[
\sum_{n=0}^{\infty} f_n(x) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k}_q a_k x^{n-k} \right) \frac{t^n}{[n]_q!} = \left( \sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_q!} \right) \left( \sum_{n=0}^{\infty} \frac{q^{(2)}(1)}{[n]_q!} (xt)^n \right) = A(t)E_q(xt).
\]

(3) \implies (1). Assume that \(\{f_n(x)\}_{n=0}^{\infty}\) is generated by

\[
A(t)E_q(xt) = \sum_{n=0}^{\infty} f_n(x) \frac{t^n}{[n]_q!}.
\]

Then, applying the operator \(D_q\) to each side of this equation,

\[
tA(t)E_q(qxt) = \sum_{n=0}^{\infty} D_q f_n(x) \frac{t^n}{[n]_q!}.
\]

Moreover, we have

\[
tA(t)E_q(qxt) = \sum_{n=0}^{\infty} f_n(qx) \frac{t^{n+1}}{[n]_q!} = \sum_{n=0}^{\infty} [n]_q f_{n-1}(qx) \frac{t^n}{[n]_q!}.
\]

By comparing the coefficients of \(t^n\), we obtain (1).

Next, \((2) \iff (4)\) is obvious. This ends the proof of the theorem. □
3.2. Algebraic structure

We denote a given polynomial set \( \{ f_n(x) \}_{n=0}^\infty \) by a single symbol \( f \) and refer to \( f_n(x) \) as the \( n \)-th component of \( f \). We define (see [2, 10]) on the set \( \mathcal{P} \) of all polynomial sets the following operation \( + \). This operation is given by the rule that \( f + g \) is the polynomial set whose \( n \)-th component is \( f_n(x) + g_n(x) \) provided that the degree of \( f_n(x) + g_n(x) \) is exactly \( n \). We also define the operation \( * \) (which appears here for the first time) such that if \( f \) and \( g \) are two sets whose \( n \)-th components are, respectively,

\[
f_n(x) = \sum_{k=0}^{n} \alpha(n,k)x^k, \quad g_n(x) = \sum_{k=0}^{n} \beta(n,k)x^k,
\]

then \( f * g \) is the polynomial set whose \( n \)-th component is

\[
(f * g)_n(x) = \sum_{k=0}^{n} \alpha(n,k)q^{-\binom{k}{2}}g_k(x).
\]

If \( \lambda \) is a real or complex number, then \( \lambda f \) is defined as the polynomial set whose \( n \)-th component is \( \lambda f_n(x) \). We obviously have

\[
f + g = g + f \quad \text{for all} \quad f, g \in \mathcal{P},
\]

\[
\lambda f * g = (f * \lambda g) = \lambda (f * g).
\]

Clearly, the operation \( * \) is not commutative on \( \mathcal{P} \). One commutative subclass is the set \( \mathcal{A} \) of all Appell polynomials (see [2]).

In what follows, \( \mathcal{A}(q) \) denotes the class of all \( q \)-Appell sets of type II.

In \( \mathcal{A}(q) \) the identity element (with respect to \( * \)) is the \( q \)-Appell set of type II \( \mathcal{I} = \left\{ q^{\binom{n}{2}}x^n \right\} \). Note that \( \mathcal{I} \) has the determining function \( A(t) = 1 \). This is due to the identity (5). Next we state the following Lemma.

**Lemma 1.** Let \( f, g, h \in \mathcal{A}(q) \) with the determining functions \( A(t) \), \( B(t) \) and \( C(t) \) respectively. Then

1. \( f + g \in \mathcal{A}(q) \) if \( A(0) + B(0) \neq 0 \),
2. \( f + g \) belongs to the determining function \( A(t) + B(t) \),
3. \( f + (g + h) = (f + g) + h \).

Next we state and prove the following theorem.

**Theorem 2.** If \( f, g, h \in \mathcal{A}(q) \) with the determining functions \( A(t) \), \( B(t) \) and \( C(t) \) respectively, then

1. \( f * g \in \mathcal{A}(q) \),
2. \( f * g = g * f \),
3. \( f \ast g \) belongs to the determining function \( A(t)B(t) \),

4. \( f \ast (g \ast h) = (f \ast g) \ast h \).

**Proof.** It is enough to prove the first part of the theorem. The rest follows directly. According to Theorem 1, we may put

\[
f_{n}(x) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \\ q \end{array} \right] q^{\frac{n-k}{2}} a_{k}x^{n-k} = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \\ q \end{array} \right] q^{\frac{k}{2}} a_{n-k}x^{k}
\]

so that

\[
A(t) = \sum_{n=0}^{\infty} a_{n} \frac{t^{n}}{[n]_{q}!}.
\]

Hence

\[
\sum_{n=0}^{\infty} (f \ast g)_{n}(x) \frac{t^{n}}{[n]_{q}!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \\ q \end{array} \right] a_{n-k}g_{k}(x) \right) \frac{t^{n}}{[n]_{q}!} = \left( \sum_{n=0}^{\infty} a_{n} \frac{t^{n}}{[n]_{q}!} \right) \left( \sum_{n=0}^{\infty} g_{n}(x) \frac{t^{n}}{[n]_{q}!} \right) = A(t)B(t)E_{q}(xt).
\]

This ends the proof of the theorem. \( \square \)

**Corollary 1.** Let \( f \in \mathcal{A}(q) \) then there is a set \( g \in \mathcal{A}(q) \) such that

\[
f \ast g = g \ast f = I.
\]

Indeed \( g \) belongs to the determining function \((A(t))^{-1}\) where \( A(t) \) is the determining function for \( f \).

In view of Corollary 1 we shall denote this element \( g \) by \( f^{-1} \). We are further motivated by Theorem 2 and its corollary to define \( f^{0} = I, f^{n} = f \ast (f^{n-1}) \) where \( n \) is a non-negative integer, and \( f^{-n} = f^{-1} \ast (f^{-n+1}) \). We note that we have proved that the system \((\mathcal{A}(q), \ast)\) is a commutative group. In particular this leads to the fact that if

\[
f \ast g = h
\]

and if any two of the elements \( f, g, h \) are \( q \)-Appell of type II then the third is also \( q \)-Appell of type II.

**Proposition 5.** If \( f \) is a \( q \)-Appell set of type II with the determining function \( A(t) \), if we put

\[
A^{-1}(t) = \sum_{n=0}^{\infty} b_{n} \frac{t^{n}}{[n]_{q}!},
\]

therefore

\[
x^{n} = q^{-\binom{n}{2}} \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \\ q \end{array} \right] b_{k}f_{n-k}(x).
\]
Proof. Since \( f \) is a \( q \)-Appell set of type II, we have

\[
\sum_{n=0}^{\infty} q^{\binom{n}{2}} x^n \frac{t^n}{[n]_q!} = (A(t))^{-1} A(t) E_q(x t)
\]

\[
= \left( \sum_{n=0}^{\infty} b_n \frac{t^n}{[n]_q!} \right) \left( \sum_{n=0}^{\infty} f_n(x) \frac{t^n}{[n]_q!} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{n!}{k!} q b_{n-k} f_k(x) \right) \frac{t^n}{[n]_q!}.
\]

The result follows by comparing the coefficients of \( t^n \). \( \square \)

4. Characterization results

4.1. Quasi-orthogonal \( q \)-Appell polynomials of type I

In this section, we characterize quasi-orthogonal polynomial sets that are also \( q \)-Appell set of type I.

**Theorem 3.** If \( \{Q_n(x)\}_{n=0}^{\infty} \) is a \( q \)-Appell set which are quasi-orthogonal. Then, there exist three real numbers \( b, c \) and \( \lambda \), such that

\[
Q_{n+1}(x) = (x + bq^n)Q_n(x) - c q^n [n]_q Q_{n-1}(x) + d_n \sum_{k=0}^{n-2} \frac{\lambda^k}{[k]_q} Q_k(x).
\]  
(15)

**Proof.** Assume that \( \{Q_n(x)\}_{n=0}^{\infty} \) is a \( q \)-Appell set which are quasi-orthogonal and \( \{P_n(x)\}_{n=0}^{\infty} \) the related orthogonal family. From Proposition 3, there exist three sequences \( \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty} \) and \( \{d_n\}_{n=0}^{\infty} \) with \( d_0 = d_1 = 0 \) such that

\[
Q_{n+1}(x) = (x + b_n)Q_n(x) - c_n Q_{n-1}(x) + d_n \sum_{k=0}^{n-2} T_k Q_k(x).
\]  
(16)

If we \( q \)-differentiate (16) and using the fact that \( \{Q_n(x)\}_{n=0}^{\infty} \) is a \( q \)-Appell, we get after some simplifications

\[
Q_n(x) = \left( x + \frac{b_n}{q} \right) Q_{n-1}(x) - \frac{c_n}{q} \frac{[n-1]_q}{[n]_q} Q_{n-2}(x) + \frac{d_n}{q[n]_q} \sum_{k=0}^{n-3} [k+1]_q T_{k+1} Q_k(x).
\]  
(17)

Next, if we replace \( n \) by \( n - 1 \) in (16), we obtain

\[
Q_n(x) = (x + b_{n-1})Q_n(x) - c_{n-1} Q_{n-2}(x) + d_{n-1} \sum_{k=0}^{n-3} T_k Q_k(x).
\]  
(18)

If we compare (17) and (18), we see that we should have

\[
b_n = q b_{n-1}, \quad c_n = q \frac{[n]_q}{[n-1]_q} c_{n-1},
\]  
(19)
and
\[ d_n[k + 1]qT_{k+1} = q[n]q d_{n-1} T_k, \quad k = 0, 1, \ldots n - 3. \quad (20) \]

Equation (19) gives
\[ b_n = q^n b_0, \quad \text{and} \quad c_n = q^{n-1}[n]q c_1. \]

Next, (20) gives for \( k = 0 \) and \( k = n - 3 \) the relations
\[ d_n = \frac{q[n]q}{T_1} d_{n-1} \quad \text{and} \quad d_n = \frac{q[n]q T_{n-1}}{[n-2]q T_{n-2}} d_{n-1}. \quad (21) \]

If, for a given \( k \geq 2 \), \( d_k = 0 \), it follows from (21) that \( d_k = 0 \) for all \( k \). In this case (16) becomes a three-term recurrence relation
\[ Q_{n+1}(x) = (x + b_n)Q_n(x) - c_n Q_{n-1}(x). \quad (22) \]

In this case, from Proposition 4, it is seen that \( \{Q_n(x)\}_{n=0}^{\infty} \) is essentially the sequence of Al-Salam Carlitz I polynomials. Thus, in this case, \( \{Q_n(x)\}_{n=0}^{\infty} \) is not a sequence of quasi-orthogonal polynomials. Thus, we must have \( d_k \neq 0 \) for \( k \geq 2 \).

Again, using (21), we have for all \( n \geq 0 \)
\[ T_n = \frac{T_{n-1}}{[n]q} \quad \text{for} \quad n \geq 0. \]

This last relation gives
\[ T_n = \frac{T_1^n}{[n]q^n}. \]

Setting \( b_0 = b, c_1 = c \) and \( T_1 = \lambda \), this ends the proof or the theorem. \( \square \)

**Theorem 4.** Let \( \{Q_n(x)\}_{n=0}^{\infty} \) be a monic polynomial set with \( Q_0(x) = 1 \). The following assertions are equivalent:

1. \( \{Q_n(x)\}_{n=0}^{\infty} \) is quasi-orthogonal and is a \( q \)-Appell set, \( n \geq 1 \).

2. There exists three constants \( \alpha, \beta \) and \( \lambda \) (\( \beta, \lambda \neq 0 \)) such that
\[ Q_n(x) = \beta^n U_n^{(\alpha/\beta)} \left( \frac{x}{\beta}; q \right) - \frac{\beta^n[n]q}{\lambda} U_{n-1}^{(\alpha/\beta)} \left( \frac{x}{\beta}; q \right), \quad n \geq 1, \]

where \( U_n^{(\alpha)}(x; q) \) are the Al-Salam Carlitz I polynomials.

**Proof.** Suppose first that \( \{Q_n(x)\}_{n=0}^{\infty} \) is quasi-orthogonal and is a \( q \)-Appell set, \( n \geq 1 \). Then, by Theorem 3, the \( Q_n \)'s satisfy a recurrence relation of the form (15). Let us define the polynomial set \( \{P_n(x)\}_{n=0}^{\infty} \) by
\[ P_n(x) = \frac{[n]q!}{\lambda^n} \sum_{k=0}^{n} \frac{\lambda^k}{[k]q} Q_k(x). \quad (23) \]

It is not difficult to see that
\[ D_q P_n(x) = \frac{[n]q!}{\lambda^n} \sum_{k=1}^{n} \frac{\lambda^k}{[k]q} [k]q Q_{k-1}(x) \]
\[ = [n]q \frac{[n-1]q!}{\lambda^{n-1}} \sum_{k=0}^{n-1} \frac{\lambda^k}{[k]q} Q_k(x) \]
\[ = [n]q P_{n-1}(x). \]
Hence, \( \{P_n(x)\}_{n=0}^{\infty} \) is a \( q \)-Appell set. Moreover, \( \{P_n(x)\}_{n=0}^{\infty} \) are the orthogonal set (see Proposition 2) related to \( \{Q_n(x)\}_{n=0}^{\infty} \). By Proposition 4, there exist \( \alpha \) and \( \beta \) such that
\[
P_n(x) = \beta^n U_n^{(\alpha/\beta)} \left( \frac{x}{\beta}; q \right).
\]

Next, from (23), it follows that
\[
Q_n(x) = \frac{[n]!}{\lambda^n} (P_n(x) - P_{n-1}(x)).
\]

The first implication of the theorem follows.

Conversely, assume that there exists three constants \( \alpha \), \( \beta \) and \( \lambda \) \((\beta, \lambda \neq 0)\) such that
\[
Q_n(x) = \beta^n U_n^{(\alpha/\beta)} \left( \frac{x}{\beta}; q \right) - \frac{\beta^n [n]!}{\lambda} U_{n-1}^{(\alpha/\beta)} \left( \frac{x}{\beta}; q \right), \quad n \geq 1.
\]

It can be seen that \( \{Q_n(x)\}_{n=0}^{\infty} \) is quasi orthogonal set. It remains to prove that \( \{Q_n(x)\}_{n=0}^{\infty} \) is \( q \)-Appell. Using the fact that \( D_q[f(ax)] = a[D_qf](ax) \). We have
\[
D_q U_n^{(\alpha/\beta)} \left( \frac{x}{\beta}; q \right) = \frac{1}{\beta} U_{n-1}^{(\alpha/\beta)} \left( \frac{x}{\beta}; q \right).
\]
It follows that \( D_q Q_n(x) = [n]_q Q_{n-1}(x) \). This ends the proof of the theorem. \( \square \)

### 4.2. Orthogonal \( q \)-Appell polynomials of type II

In this section we determine those real sets in \( \mathcal{A}(q) \) which are also orthogonal. It is well known [11] that a set of real orthogonal polynomials satisfies a recurrence relation of the form
\[
P_{n+1}(x) = (A_n x + B_n) P_n(x) + C_n P_{n-1}(x), \quad n \geq 1, \tag{24}
\]

with
\[
P_0(x) = 1, \quad P_1(x) = A_0 x + B_0.
\]

Here \( A_n, B_n \) and \( C_n \) are real constants which do not depend on \( n \).

If we \( q \)-differentiate (24) and assume that the polynomial set \( \{P_n(x)\} \) is \( q \)-Appell of type II, we get:
\[
[n + 1]_q P_n(qx) = [n]_q (A_n x + B_n) P_{n-1}(qx) + A_n P_n(qx) + [n - 1]_q C_n P_{n-2}(qx). \tag{25}
\]

Substituting \( n \) by \( n + 1 \) and \( x \) by \( xq^{-1} \) in (25), it follows that
\[
P_{n+1}(x) = \left( \frac{[n + 1]_q A_{n+1}}{[n + 2]_q - A_{n+1}} x + \frac{[n + 1]_q B_{n+1}}{[n + 2]_q - A_{n+1}} \right) P_n(x) + \frac{[n]_q C_{n+1}}{[n + 2]_q - A_{n+1}} P_{n-1}(x). \tag{26}
\]

By comparing (24) and (26) we get
\[
\frac{[n + 1]_q A_{n+1}}{[n + 2]_q - A_{n+1}} = qA_n, \quad \frac{[n + 1]_q B_{n+1}}{[n + 2]_q - A_{n+1}} = B_n \quad \text{and} \quad \frac{[n]_q C_{n+1}}{[n + 2]_q - A_{n+1}} = C_n.
\]
so that

\[ A_n = q^n, \quad B_n = B_0 \quad \text{and} \quad C_n = C_1 (1 - q^n). \]

Hence, \( \{ P_n(x) \} \) is given by

\[
P_{n+1}(x) = (q^n x + B_0) P_n(x) + C_1 (1 - q^n) P_{n-1}(x),
\]

\[ P_0(x) = 1, \quad P_1(x) = x + B_0. \]

From the recurrence relation of the Al-Salam Carlitz II polynomials (see [7, p. 538]), one can see that the polynomial sequence \( \{ R_n(x) \} \) with

\[ R_n(x) = \beta^n q^{\binom{n}{2}} V_n^{\left( \frac{\alpha}{\beta} ; q \right)} \left( \frac{x}{\beta} ; q \right), \]

satisfies the recurrence relation

\[
x R_n(x) = R_{n+1}(x) + (q^n x - (\alpha + \beta)) R_n(x) - \alpha \beta (1 - q^n) R_{n-1}(x),
\]

with \( R_0(x) = 1 \) and \( R_1(x) = x - (\alpha + \beta) \). It is therefore clear that

\[
P_n(x) = \beta^n q^{\binom{n}{2}} V_n^{\left( \frac{\alpha}{\beta} ; q \right)} \left( \frac{x}{\beta} ; q \right),
\]

where \( \alpha + \beta = -B_0 \) and \( \alpha \beta = -C_1 \).

We thus have the following theorem.

**Theorem 5.** The set of q-Appell polynomials of type II which are also orthogonal is given (27) or (29).

### 4.3. Quasi-orthogonal q-Appell polynomials of type II

**Theorem 6.** If \( \{ Q_n(x) \}_{n=0}^{\infty} \) is a q-Appell set of type II of quasi-orthogonal polynomials, then there exist three reel numbers \( B_0, C_1 \) and \( \lambda \), such that

\[
Q_{n+1}(x) = (q^n x + B_0) Q_n(x) + C_1 (1 - q^n) Q_{n-1}(x) + \frac{[n]_q!}{\lambda^n} \sum_{k=0}^{n-2} \frac{\lambda^k}{[k]_q!} Q_k(x).
\]

**Proof.** Assume that \( \{ Q_n(x) \}_{n=0}^{\infty} \) is a q-Appell set which is quasi-orthogonal and \( \{ P_n(x) \}_{n=0}^{\infty} \) the related orthogonal family. From Proposition 3, there exist four sequences \( \{ A_n \}_{n=0}^{\infty}, \{ B_n \}_{n=0}^{\infty}, \{ C_n \}_{n=0}^{\infty} \) and \( \{ E_n \}_{n=0}^{\infty} \) with \( E_0 = E_1 = 0 \) such that

\[
Q_{n+1}(x) = (A_n x + B_n) Q_n(x) + C_n Q_{n-1}(x) + E_n \sum_{k=0}^{n-2} T_k Q_k(x).
\]
If we $q$-differentiate (31) and use the fact that $\{Q_n(x)\}_{n=0}^{\infty}$ is a $q$-Appell set of type II, we get after some simplifications

\[
Q_{n+1}(x) = \left(\frac{[n+1]_q q^{-1} A_{n+1}}{[n+2]_q - A_{n+1}} x + \frac{[n+1]_q B_{n+1}}{[n+2]_q - A_{n+1}}\right) Q_n(x) \\
+ \frac{[n]_q C_{n+1}}{[n+2]_q - A_{n+1}} Q_{n-1}(x) \\
+ \frac{E_{n+1}}{[n+2]_q - A_{n+1}} \sum_{k=0}^{n-2} [k+1]_q T_{k+1} Q_k(x).
\]  

(32)

By comparing (31) and (32) we get

\[
A_n = q^n, \quad B_n = B_0 \quad \text{and} \quad C_n = C_1 (1 - q^n),
\]

and

\[
E_n T_k = \frac{E_{n+1} [k+1]_q T_{k+1}}{[n+2]_q - A_{n+1}} = \frac{[k+1]_q T_{k+1}}{[n+1]_q} E_{n+1},
\]

For $k = 0$ and $k = n - 2$, we obtain the following

\[
E_{n+1} = \frac{[n+1]_q}{T_1} E_n, \quad T_n = \frac{E_{n+1}[n+2]_q}{E_{n+2}[n]_q} T_{n-1}.
\]

(33)

If, for a given $k \geq 2$, $E_k = 0$, it follows from (33) that $E_k = 0$ for all $k$. In this case (31) becomes a three-term recurrence relation

\[
Q_{n+1}(x) = (A_n x + B_n) Q_n(x) + C_n Q_{n-1}(x).
\]

(34)

In this case, from Theorem 5, it is seen that $\{Q_n(x)\}_{n=0}^{\infty}$ is essentially the sequence of Al-Salam Carlitz II polynomials. Thus, in this case, $\{Q_n(x)\}_{n=0}^{\infty}$ is not a sequence of quasi-orthogonal polynomials. Thus, we must have $E_k \neq 0$ for $k \geq 2$.

Again, using (33), we have for all $n \geq 0$ the identities $E_n = \frac{[n]_q !}{T_1^n}$ and $\frac{T_{n-1}}{[n]_q T_n} = \frac{1}{T_1}$. This last relation gives $T_n = \frac{T_1^n}{[n]_q !}$. Setting $T_1 = \lambda$, this ends the proof of the theorem. □

**Theorem 7.** Let $\{Q_n(x)\}_{n=0}^{\infty}$ be a polynomial set. The following assertions are equivalent:

1. $\{Q_n(x)\}_{n=0}^{\infty}$ is quasi-orthogonal and is a $q$-Appell set of type II.
2. There exists three constants $\alpha$, $\beta$ and $\gamma$ ($\beta, \gamma \neq 0$) such that

\[
Q_n(x) = \beta^n q^{\binom{n}{2}} V_n^{(\beta)} \left(\frac{x}{\beta}; q\right) - \frac{\beta^{n-1} q^{\binom{n-1}{2}} [n]_q ! \lambda^n}{[\alpha]_q} V_{n-1}^{(\alpha)} \left(\frac{x}{\beta}; q\right), \quad (n \geq 1),
\]

where $V_n^{(\alpha)}(x; q)$ are the Al-Salam Carlitz II polynomials.
Proof. Suppose first that \( \{Q_n(x)\}_{n=0}^{\infty} \) is quasi-orthogonal and is a \( q \)-Appell set of type II. Then, by Theorem 6, the \( Q_n \)'s satisfy a recurrence relation of the form (30). Let us define the polynomial set \( \{P_n(x)\}_{n=0}^{\infty} \) by

\[
P_n(x) = \frac{[n]_q!}{\lambda^n} \sum_{k=0}^{n} \frac{\lambda^k}{[k]_q!} Q_k(x).
\] (35)

It is not difficult to see that

\[
D_q P_n(x) = \frac{[n]_q!}{\lambda^n} \sum_{k=1}^{n} \frac{\lambda^k}{[k]_q!} [k]_q Q_{k-1}(qx)
\]

\[
= [n]_q \frac{[n-1]_q!}{\lambda^{n-1}} \sum_{k=0}^{n-1} \frac{\lambda^k}{[k]_q!} Q_k(qx)
\]

\[
= [n]_q P_{n-1}(qx).
\]

Hence, \( \{P_n(x)\}_{n=0}^{\infty} \) is a \( q \)-Appell set of type II. Moreover, \( \{P_n(x)\}_{n=0}^{\infty} \) is the orthogonal set (see Proposition 2) related to \( \{Q_n(x)\}_{n=0}^{\infty} \). By Theorem 5, there exist \( \alpha \) and \( \beta \) such that

\[
P_n(x) = \beta^n q^{\binom{n}{2}} V_n^{(\frac{\alpha}{\beta})} \left( \frac{x}{\beta}; q \right).
\]

Next, from (35), it follows easily that

\[
Q_n(x) = P_n(x) - \frac{[n]_q!}{\lambda^n} P_{n-1}(x)
\]

\[
= \beta^n q^{\binom{n}{2}} V_n^{(\frac{\alpha}{\beta})} \left( \frac{x}{\beta}; q \right) - \beta^{n-1} q^{\binom{n-1}{2}} \frac{[n]_q!}{\lambda^n} V_{n-1}^{(\frac{\alpha}{\beta})} \left( \frac{x}{\beta}; q \right),
\]

The first implication of the theorem follows.

Conversely, assume that there exist three constants \( \alpha \), \( \beta \) and \( \gamma \) (\( \beta, \gamma \neq 0 \)) such that

\[
Q_n(x) = \beta^n q^{\binom{n}{2}} V_n^{(\frac{\alpha}{\beta})} \left( \frac{x}{\beta}; q \right) - \beta^{n-1} q^{\binom{n-1}{2}} \frac{[n]_q!}{\lambda^n} V_{n-1}^{(\frac{\alpha}{\beta})} \left( \frac{x}{\beta}; q \right), \quad (n \geq 1).
\]

It can be seen that \( \{Q_n(x)\}_{n=0}^{\infty} \) is a quasi-orthogonal set. It remains to prove that \( \{Q_n(x)\}_{n=0}^{\infty} \) is a \( q \)-Appell set. Using the fact that \( D_q [f(ax)] = a[D_q f(ax)] \), we get

\[
D_q V_n^{(\alpha/\beta)} \left( \frac{x}{\beta}; q \right) = \frac{[n]_q q^{-n+1}}{\beta} V_{n-1}^{(\alpha/\beta)} \left( \frac{qx}{\beta}; q \right).
\]

It follows that \( D_q Q_n(x) = [n]_q Q_{n-1}(qx) \). This ends the proof of the theorem. \( \square \)
REFERENCES


(Received August 11, 2023)

P. Njionou Sadjang
National Higher Polytechnic School
University of Douala
Cameroon
e-mail: pnjionou@yahoo.fr