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SERIES INVOLVING POLYGAMMA FUNCTIONS AND CERTAIN VARIANT EULER HARMONIC SUMS

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Abstract. In this paper, we begin by introducing integral formulas related to the psi functions. Following that, we delve into specific series that involve polygamma functions, leveraging Eulerian numbers to express them as finite series of double integrals. Subsequently, we employ the findings from the second part to study variant Euler harmonic sums. Finally, we offer closed-form evaluations for a number of distinct instances of these variant Euler harmonic sums.

1. Introduction and preliminaries

The generalized harmonic numbers $H_n^{(s)}(u)$ of order s are defined by

$$H_n^{(s)}(u) := \sum_{j=1}^{n} \frac{1}{(j+u)^s} \quad \left(s \in \mathbb{C},\, u \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1},\, n \in \mathbb{N}\right), \tag{1}$$

and $H_n^{(s)} := H_n^{(s)}(0)$ are the n-harmonic numbers of order s. The classical harmonic numbers $H_n := H_n^{(1)}$ are given by

$$H_n = \sum_{j=1}^{n} \frac{1}{j} = \gamma + \psi(n+1) \quad (n \in \mathbb{Z}_{\geqslant 0}) \quad \text{and} \quad H_0 := 0. \tag{2}$$

Here γ is the familiar Euler-Mascheroni constant (see, e.g., [23, Section 1.2]) and $\psi(z)$ denotes the digamma (or psi) function defined by

$$\psi(z) := \frac{d}{dz}(\log \Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \left(z \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}\right), \tag{3}$$

where $\Gamma(z)$ is the familiar Gamma function (see, e.g., [23, Section 1.1]). Among the many identities associated with the psi function, the following one is brought to attention (see, e.g., [23, Section 1.3]):

$$\psi(z+n) = \psi(z) + \sum_{k=1}^{n} \frac{1}{z+k-1} \quad (n \in \mathbb{N}). \tag{4}$$

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Here and elsewhere, an empty sum is assumed to be nil. Here and in the sequel, let $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Z}$ stand for the sets of complex numbers, real numbers, and integers, respectively. Also let $S_{>\ell}$, $S_{\geqslant\ell}$, $S_{<\ell}$, and $S_{\leqslant\ell}$ be the subsets of the set S ($\mathbb{R}$ or $\mathbb{Z}$) which are greater than, greater than or equal to, less than , and less than or equal to some $\ell\in\mathbb{R}$, respectively. Particularly, put $\mathbb{N}:=\mathbb{Z}_{\geqslant 1}$. The polygamma function $\psi^{(k)}(z)$ defined by

$$\psi^{(k)}(z):=\frac{d^k}{dz^k}\{\psi(z)\}=(-1)^{k+1}\,k!\sum_{r=0}^{\infty}\frac{1}{(r+z)^{k+1}}=(-1)^{k+1}\,k!\,\zeta(k+1,z) \tag{5}$$

$$(k\in\mathbb{N};\; z\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0})$$

has the recurrence

$$\psi^{(k)}(z+1)=\psi^{(k)}(z)+\frac{(-1)^k\,k!}{z^{k+1}}\quad (k\in\mathbb{Z}_{\geqslant 0};\; z\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}). \tag{6}$$

Here $\zeta(s,z)$ is the generalized (or Hurwitz) zeta function defined by

$$\zeta(s,z)=\sum_{m=0}^{\infty}\frac{1}{(m+z)^s}\quad (\Re(s)>1,\, z\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}). \tag{7}$$

The following identities are noteworthy:

$$\zeta(s,1)=\zeta(s)\quad\text{and}\quad \zeta(s,z)=\zeta(s,n+z)+\sum_{m=0}^{n-1}\frac{1}{(m+z)^s}\quad (n\in\mathbb{N}), \tag{8}$$

where $\zeta(s)$ is Riemann zeta function. Recall the following integral formula (see, e.g., [8, Entry 4.261-17]): For $\min\{\Re(\mu),\Re(\nu)\}>0$,

$$\begin{aligned}&\int_0^1 (\ln x)^2\, x^{\mu-1}\,(1-x)^{\nu-1}\,\mathrm{d}x\\ &\quad = B(\mu,\nu)\left\{[\psi(\mu)-\psi(\mu+\nu)]^2+\psi'(\mu)-\psi'(\mu+\nu)\right\},\end{aligned} \tag{9}$$

where $B(\mu,\nu)$ is the Beta function (see, e.g., [23, pp. 7–10]).

The function $\mathbf{b}(z)$ is defined by (cf. [7, p. 20])

$$\mathbf{b}(z):=\frac{1}{2}\left\{\psi\left(\frac{z+1}{2}\right)-\psi\left(\frac{z}{2}\right)\right\}. \tag{10}$$

Like the psi function $\psi(z)$, the function $\mathbf{b}(z)$ has a number of useful properties.

The Dirichlet beta function $\beta(z)$ is defined by

$$\beta(z):=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^z}\quad (\Re(z)>0). \tag{11}$$

Among various properties and formulas for $\beta(z)$, we recall the followings:

$$\begin{aligned}\beta(z) &= 4^{-z}\left\{\zeta\left(z,\tfrac{1}{4}\right)-\zeta\left(z,\tfrac{3}{4}\right)\right\}\\ &= \frac{1}{(-2)^{2z}\,\Gamma(z)}\left\{\psi^{(z-1)}\left(\tfrac{1}{4}\right)-\psi^{(z-1)}\left(\tfrac{3}{4}\right)\right\}\\ &= \frac{i}{2}\left\{\mathrm{Li}_z(-i)-\mathrm{Li}_z(i)\right\},\end{aligned} \tag{12}$$

where the polylogarithm function $\mathrm{Li}_z(u)$ of order z can be given as follows (see, e.g., [23, p. 198]):

$$\mathrm{Li}_z(u) := \sum_{m=1}^{\infty}\frac{u^m}{m^z} \tag{13}$$

$$(z\in\mathbb{C} \text{ and } |u|<1;\ \Re(z)>1 \text{ and } |u|=1).$$

The functional equation for the Dirichlet beta function $\beta(z)$ is expressed as follows:

$$\beta(1-z) = \left(\frac{2}{\pi}\right)^{z}\sin\left(\frac{\pi z}{2}\right)\Gamma(z)\,\beta(z). \tag{14}$$

This equation can be analytically extended to the left-half plane $\Re(z)\leqslant 0$ (refer to [25]). Euler first proposed this conjecture in 1749, and it was later proven by Malmsten in 1842 (see [4]). It is intriguing to juxtapose the functional equation of the Dirichlet beta function $\beta(z)$ in (14) with the renowned functional equation of the Riemann zeta function $\zeta(z)$ (refer to [23, p. 166]).

The Catalan constant G is given as

$$G = \beta(2) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\approx 0.91597. \tag{15}$$

There are also numerous identities for G. For example,

$$G = -\int_0^1 \frac{\ln x}{1+x^2}\,\mathrm{d}x = \int_0^1 \frac{\arctan x}{x}\,\mathrm{d}x = \Im\left(\mathrm{Li}_2(i)\right). \tag{16}$$

$H_\alpha^{(m)}$ are extended harmonic numbers of order $m\in\mathbb{N}$ with index $\alpha\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant -1}$ defined by (see [21])

$$H_\alpha^{(m)} := \begin{cases}\gamma+\psi(\alpha+1) & (m=1),\\ \zeta(m)+\frac{(-1)^{m-1}}{(m-1)!}\,\psi^{(m-1)}(\alpha+1) & (m\in\mathbb{Z}_{\geqslant 2}).\end{cases} \tag{17}$$

The case $m=1$ in (17) is given in (2). Employing (17) in (6) gives

$$H_\alpha^{(m)} = H_{\alpha-1}^{(m)}+\frac{1}{\alpha^m}\quad (m\in\mathbb{N},\ \alpha\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}). \tag{18}$$

Applying (17) to the multiplication formula for polygamma functions (see, e.g., [11, p. 14]):

$$\psi^{(n)}(mz) = \delta_{n,0}\log m + \frac{1}{m^{n+1}}\sum_{j=1}^{m}\psi^{(n)}\left(z+\tfrac{j-1}{m}\right) \quad (m\in\mathbb{N},\, n\in\mathbb{Z}_{\geqslant 0}), \tag{19}$$

$\delta_{n,j}$ being the Kronecker delta, can provide the following multiplication formula for the extended harmonic numbers:

$$\begin{gathered} H_{m\alpha}^{(p)} = \frac{1}{m^p}\sum_{j=1}^{m} H_{\alpha+\frac{j}{m}-1}^{(p)} + \left(1-m^{1-p}\right)\zeta(p) \\ \left(m\in\mathbb{N},\, p\in\mathbb{Z}_{\geqslant 2};\, m\alpha+1,\, \alpha+\frac{j}{m}\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}\right). \end{gathered} \tag{20}$$

The Eulerian numbers $A_{n,k}\ (n,\,k\in\mathbb{Z}_{\geqslant 0})$ are defined by (see, e.g., [1], [2], [15])

$$A_{n,k} := \sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k-j)^n. \tag{21}$$

These numbers gratify the following conditions

$$A_{0,0}=1, \quad A_{n,0}=0\ (n\in\mathbb{N}), \quad A_{n,k}=0\ (k>n), \tag{22}$$

and the recurrence relations

$$\begin{cases} A_{n,k} = A_{n,n-k+1}, \\ A_{n+1,k} = kA_{n,k} + (n-k+2)A_{n,k-1}. \end{cases} \tag{23}$$

These numbers are included in the expansion

$$x^m = \sum_{k=0}^{m} A_{m,k}\binom{x+m-k}{m} \quad (m\in\mathbb{Z}_{\geqslant 0}), \tag{24}$$

which is known as the Worpitzky identity (refer to [1, p. 4]).

The Pochhammer symbol $(\lambda)_n$ is defined (for $\lambda\in\mathbb{C}$) by (see, e.g., [23, p. 2 and pp. 4–6])

$$\begin{aligned} (\lambda)_n &= \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N}) \end{cases} \\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}). \end{aligned}$$

Also $\{z\}_n$ (for $z\in\mathbb{C}$) is the falling factorial defined by

$$\{z\}_n = \begin{cases} z(z-1)\cdots(z-n+1) & (n\in\mathbb{N}), \\ 1 & (n=0). \end{cases} \tag{25}$$

During his contact with Goldbach, which began in 1742, Euler initiated a series of investigations for the linear harmonic sums (26), and he was the first to investigate the sums that followed (see, e.g., [6, 9])

$$\mathtt{S}_{p,q} := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}. \tag{26}$$

Euler, whose research was finished by Nielsen in 1906 (see [13]), revealed that the linear harmonic sums in (26) can be evaluated in the subsequent instances: $p=1$; $p=q$; $p+q$ odd; $p+q$ even, but with only the pair (p,q) being the set $\{(2,4),\,(4.2)\}$. Of these specific cases, in the ones with $p \neq q$, if $\mathtt{S}_{p,q}$ is assessed, then $\mathtt{S}_{q,p}$ can be decided by use of the shuffle relation

$$\mathtt{S}_{p,q} + \mathtt{S}_{q,p} = \zeta(p)\,\zeta(q) + \zeta(p+q) \tag{27}$$

and vice versa. After the investigation of Euler's linear harmonic sums, many researchers have focused their attention on this topic. They have delved into numerous Euler-type sums using a range of techniques. For instance, they have explored parametric linear Euler sums and their extensions, as evidenced in sources like [3], [14], [20], and the references mentioned therein. Additionally, they have ventured into nonlinear Euler harmonic sums, as documented in sources such as [6], [9], [24], and their respective references. Furthermore, other variant Euler harmonic sums have been examined, as can be seen in sources like [12], [17], [19], [21], and the accompanying references.

In this study, our objectives are fourfold. Firstly, we present integral formulae associated with the function $\mathbf{b}(z)$ in equation (10). Secondly, we delve into a series involving polygamma functions, represented as a finite series of double integrals, by incorporating Eulerian numbers from equation (21). Building on these second findings, we proceed to investigate various Euler harmonic sums. Lastly, we provide closed-form evaluations for several specific instances of these variant Euler harmonic sums.

2. Integrals involving the function $\mathbf{b}(z)$

Like the psi function $\psi(z)$, a number of integrals associated with the function $\mathbf{b}(z)$ have been presented (see, e.g., [7, p. 20], [8]). Take some examples.

$$\mathbf{b}(z) = \int_0^1 \frac{t^{z-1}}{1+t}\,\mathrm{d}t \quad (\Re(z) > 0). \tag{28}$$

$$\mathbf{b}\left(\frac{z}{p}\right) = p\int_0^1 \frac{t^{z-1}}{1+t^p}\,\mathrm{d}t \quad \left(\Re(z) > 0,\, p \in \mathbb{R}_{>0}\right). \tag{29}$$

$$\int_0^1 \frac{t^z}{(1+t)^2}\,\mathrm{d}t = z\mathbf{b}(z) - \frac{1}{2} \quad \left(\Re(z) > -1\right). \tag{30}$$

$$\int_0^1 \frac{t^z}{(1+t^p)^2}\,\mathrm{d}t = \frac{z+1-p}{p^2}\mathbf{b}\left(\frac{z+1-p}{p}\right) - \frac{1}{2p} \tag{31}$$

$$\left(\Re(z) > -1,\, p \in \mathbb{R}_{>0}\right).$$

$$\int_0^1 \frac{t^z}{(1+t)^3}\,\mathrm{d}t = \frac{z(z-1)}{2}\mathbf{b}(z-1) - \frac{z}{4} - \frac{1}{8} \quad (\Re(z) > -1)\,. \tag{32}$$

$$\int_0^1 \frac{t^z}{(1+t^p)^3}\,\mathrm{d}t = \frac{(z-p+1)(z-2p+1)}{2p^3}\mathbf{b}\left(\frac{z-2p+1}{p}\right) - \frac{z+1}{4p} + \frac{1}{8} \tag{33}$$

$$(\Re(z) > -1,\ p \in \mathbb{R}_{>0})\,.$$

$$\int_0^{\frac{\pi}{4}} \tan^z t\,\mathrm{d}t = \frac{1}{2}\mathbf{b}\left(\frac{z+1}{2}\right) \quad (\Re(z) > -1)\,. \tag{34}$$

$$\int_0^{\frac{\pi}{4}} \tan^z t\,\cos^2 t\,\mathrm{d}t = \frac{z-1}{4}\mathbf{b}\left(\frac{z-1}{2}\right) - \frac{1}{4} \quad (\Re(z) > -1)\,. \tag{35}$$

$$\int_0^{\frac{\pi}{4}} \tan^z t\,\cos^4 t\,\mathrm{d}t = \frac{(z-1)(z-3)}{16}\mathbf{b}\left(\frac{z-3}{2}\right) - \frac{z}{8} \quad (\Re(z) > -1)\,. \tag{36}$$

An observation of the integrals (28), (30), and (32) enables us to derive generic integral formulae as described in the following theorem.

THEOREM 1. *Let $n \in \mathbb{N}$ and $\{z\}_n$ be the falling factorial in (25). Then the following integral formulas hold true:*

$$\int_0^1 \frac{t^z}{(1+t)^n}\,\mathrm{d}t = \frac{\{z\}_{n-1}}{(n-1)!}\mathbf{b}(z-n+2) + Q_n(z) \quad (\Re(z) > -1)\,, \tag{37}$$

where $Q_n(z)$ is a polynomial in z of degree $n-2$ which is given by the following recursive relation:

$$Q_1(z) = 0, \quad \textit{and} \quad Q_{n+1}(z) = \frac{z}{n}Q_n(z-1) - \frac{1}{n\,2^n}. \tag{38}$$

$$\int_0^\infty \frac{e^{-zt}}{(1+e^{-t})^n}\,\mathrm{d}t = \frac{\{z-1\}_{n-1}}{(n-1)!}\mathbf{b}(z-n+1) + P_n(z) \quad (\Re(z) > 0)\,, \tag{39}$$

where $P_n(z)$ is a polynomial in z of degree $n-2$ which is given by the following recursive relation:

$$P_1(z) = 0, \quad \textit{and} \quad P_{n+1}(z) = \frac{z-1}{n}P_n(z-1) - \frac{1}{n\,2^n}. \tag{40}$$

Proof. We prove (37) by induction on n. The case $n=1$ of (37) gives

$$\int_0^1 \frac{t^z}{1+t}\,\mathrm{d}t = \mathbf{b}(z+1),$$

which is equivalent to (28). Assume that (37) is true for some $n \in \mathbb{N}$. Integrating the left-sided integral in (37) by parts offers

$$\int_0^1 \frac{t^{z+1}}{(1+t)^{n+1}}\,\mathrm{d}t = \frac{(z+1)\cdot\{z\}_{n-1}}{n!}\,\mathbf{b}(z-n+2) + \frac{z+1}{n}Q_n(z) - \frac{1}{n\,2^n},$$

which, upon replacing z by $z-1$, yields

$$\int_0^1 \frac{t^z}{(1+t)^{n+1}}\,\mathrm{d}t = \frac{\{z\}_n}{n!}\,\mathbf{b}(z-(n+1)+2)+\frac{z}{n}Q_n(z-1)-\frac{1}{n\,2^n},$$

whose right member with the aid of (38) is equal to the same expression of the right member of (37) when n is replaced by $n+1$. This completes the proof of (37).

Substituting e^{-t} for t in (28) gives

$$\mathbf{b}(z)=\int_0^\infty \frac{e^{-zt}}{1+e^{-t}}\,\mathrm{d}t \quad (\Re(z)>0). \tag{41}$$

Applying the same method as the proof of (37) to (41) may prove (39). □

REMARK 1. The integral (41) can be equivalently expressed as follows:

$$\mathbf{b}\left(\frac{z+1}{2}\right)=\int_0^\infty \frac{e^{-zt}}{\cosh t}\,\mathrm{d}t \quad (\Re(z)>-1). \tag{42}$$

The first few of $Q_n(z)$ and $P_n(z)$ are

$$Q_2(z)=-\frac{1}{2},\quad Q_3(z)=-\frac{z}{4}-\frac{1}{8},\quad Q_4(z)=-\frac{z^2}{12}+\frac{z}{24}-\frac{1}{24};$$

$$P_2(z)=-\frac{1}{2},\quad P_3(z)=-\frac{z}{4}+\frac{1}{8},\quad P_4(z)=-\frac{z^2}{12}+\frac{5}{24}z-\frac{1}{6}.$$

Replacing e^{-t} by t in (39) gives

$$\int_0^1 \frac{t^z}{(1+t)^n}\,\mathrm{d}t=\frac{\{z\}_{n-1}}{(n-1)!}\,\mathbf{b}(z-n+2)+P_n(z+1) \quad (n\in\mathbb{N},\ \Re(z)>-1),$$

which, in view of (37), yields

$$Q_n(z)=P_n(z+1) \quad \Longleftrightarrow \quad P_n(z)=Q_n(z-1) \quad (n\in\mathbb{N}).$$

In this respect, integral formulas (37) and (39) might be considered equivalent.

COROLLARY 1. *Let $n\in\mathbb{N}$, $\Re(z)>-1$, and $p\in\mathbb{R}_{>0}$. Then the following formulas hold true:*

$$\int_0^1 \frac{t^z}{(1+t^p)^n}\,\mathrm{d}t=\frac{\left\{\frac{z-p+1}{p}\right\}_{n-1}}{p(n-1)!}\,\mathbf{b}\left(\frac{z+1}{p}-n+1\right)+\frac{1}{p}\,Q_n\left(\frac{z-p+1}{p}\right). \tag{43}$$

$$\int_0^1 \frac{t^z}{(1+t^2)^n}\,\mathrm{d}t=\frac{\left\{\frac{z-1}{2}\right\}_{n-1}}{2\,(n-1)!}\,\mathbf{b}\left(\frac{z+1}{2}-n+1\right)+\frac{1}{2}\,Q_n\left(\frac{z-1}{2}\right). \tag{44}$$

$$\int_0^{\frac{\pi}{4}} \tan^z t\,\cos^{2(n-1)} t\,\mathrm{d}t=\frac{\left\{\frac{z-1}{2}\right\}_{n-1}}{2\,(n-1)!}\,\mathbf{b}\left(\frac{z+1}{2}-n+1\right)+\frac{1}{2}\,Q_n\left(\frac{z-1}{2}\right). \tag{45}$$

Proof. Replacing t by t^p in (37), multiplying both sides of the resulting identity by $\frac{1}{p}$, and setting $pz+p-1=z'$ in the last resulting identity and, then, dropping the prime on z, we get (43). The particular case $p=2$ of (43) gives (44).

Substituting $\tan t$ for t in the left-sided integral in (44), we derive (45). □

3. Double integral for a series involving polygamma functions

This section establishes a double integral formula for a series associated with polygamma functions.

THEOREM 2. *Let $a \in \mathbb{R}_{>0}$, $(p,t) \in \mathbb{N}^2$ and $m \in \mathbb{Z}_{\geqslant 0}$ with $t > m-1$. Then*

$$\begin{aligned}\chi(a,m,p,t) :=& \sum_{n\geqslant 0} \frac{(-1)^n n^m \left\{\psi^{(p)}\left(\frac{an}{2}+1\right) - \psi^{(p)}\left(\frac{an}{2}+\frac{1}{2}\right)\right\}}{(2n+1)^{t+1}} \\ =& \frac{(-1)^t 2^{p+1}}{t!} \sum_{k=0}^{m} (-1)^k A_{m,k} \int_0^1\int_0^1 \frac{\ln^p(x)\ln^t(y)\left(x^{\frac{a}{2}}y\right)^{2k}}{(1+x)(1+x^a y^2)^{m+1}}\,\mathrm{d}x\mathrm{d}y \\ &+ (-1)^{p+1} p! \sum_{n=0}^{m-1} \frac{(-1)^n}{(2n+1)^{t+1}} \sum_{k=n+1}^{m} A_{m,k}\binom{n+m-k}{m} \\ &\times \left\{\zeta\left(p+1,\frac{an}{2}+1\right) - \zeta\left(p+1,\frac{an+1}{2}\right)\right\}.\end{aligned} \tag{46}$$

In particular,

$$\begin{aligned}\chi(a,0,p,t) =& \sum_{n\geqslant 0} \frac{(-1)^n \left\{\psi^{(p)}\left(\frac{an}{2}+1\right) - \psi^{(p)}\left(\frac{an}{2}+\frac{1}{2}\right)\right\}}{(2n+1)^{t+1}} \\ =& \frac{(-1)^t 2^{p+1}}{t!} \int_0^1\int_0^1 \frac{\ln^p(x)\ln^t(y)}{(1+x)(1+x^a y^2)}\,\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{47}$$

Proof. From (10) and (28), one gets

$$\int_0^1 \frac{x^{an}}{1+x}\,\mathrm{d}x = \frac{1}{2}\left\{\psi\left(\frac{an}{2}+1\right) - \psi\left(\frac{an}{2}+\frac{1}{2}\right)\right\},$$

both sides of which, upon differentiating with respect to a, p times, gives

$$\begin{aligned}\int_0^1 \frac{x^{an}\ln^p(x)}{1+x}\,\mathrm{d}x =& \frac{1}{2^{p+1}}\left\{\psi^{(p)}\left(\frac{an}{2}+1\right) - \psi^{(p)}\left(\frac{an}{2}+\frac{1}{2}\right)\right\} \\ =& \frac{(-1)^{p+1}p!}{2^{p+1}}\left\{\zeta\left(p+1,\frac{an}{2}+1\right) - \zeta\left(p+1,\frac{an+1}{2}\right)\right\}.\end{aligned} \tag{48}$$

Recall the following integral formula (see, e.g., [16, p. 110, Entry 18.90])

$$\int_0^1 y^{2n}\ln^t(y)\,\mathrm{d}y = \frac{(-1)^t t!}{(2n+1)^{t+1}} \quad (2n \in \mathbb{Z}_{\geqslant 0},\ t \in \mathbb{Z}_{\geqslant 0}). \tag{49}$$

Setting (48) and (49) in the right member of the first equality in (46) offers

$$\begin{aligned}\chi(a,m,p,t) &= \frac{(-1)^t 2^{p+1}}{t!}\sum_{n\geqslant 0}(-1)^n n^m \int_0^1\int_0^1 \frac{\left(x^{\frac{a}{2}}y\right)^{2n}\ln^p(x)\ln^t(y)}{1+x}\,\mathrm{d}x\mathrm{d}y \\ &= \frac{(-1)^t 2^{p+1}}{t!}\int_0^1\int_0^1 \frac{\ln^p(x)\ln^t(y)}{1+x}\sum_{n\geqslant 0}(-1)^n n^m\left(x^{\frac{a}{2}}y\right)^{2n}\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{50}$$

Consider

$$E(m;a,x,y) := \sum_{k=0}^{m}(-1)^k A_{m,k}\frac{\left(x^{\frac{a}{2}}y\right)^{2k}}{(1+x^a y^2)^{m+1}}.$$

Using $A_{m,0}=0\ \ (m\in\mathbb{N})$, $A_{m,k}=0\ \ (k>m)$, and

$$\begin{aligned}\left(1+x^a y^2\right)^{-m-1} &= \sum_{n\geqslant 0}(-1)^n\frac{(m+1)_n}{n!}\left(x^{\frac{a}{2}}y\right)^{2n} \\ &= \sum_{n\geqslant 0}(-1)^n\binom{m+n}{m}\left(x^{\frac{a}{2}}y\right)^{2n},\end{aligned}$$

we have

$$\begin{aligned}E(m;a,x,y) &= \sum_{n\geqslant 0}\sum_{k=0}^{m}(-1)^{n+k}A_{m,k}\binom{m+n}{m}\left(x^{\frac{a}{2}}y\right)^{2(n+k)} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+k}A_{m,k}\binom{m+n}{m}\left(x^{\frac{a}{2}}y\right)^{2(n+k)}.\end{aligned}$$

Employing a double series manipulation, we get

$$\begin{aligned}E(m;a,x,y) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+k}A_{m,k}\binom{m+n}{m}\left(x^{\frac{a}{2}}y\right)^{2(n+k)} \\ &\overset{n\to n-k}{=} \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{n}A_{m,k}\binom{n+m-k}{m}\left(x^{\frac{a}{2}}y\right)^{2n}.\end{aligned}$$

Separate the first sum into two parts:

$$\begin{aligned}E(m;a,x,y) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+k}A_{m,k}\binom{m+n}{m}\left(x^{\frac{a}{2}}y\right)^{2(n+k)} \\ &= \left(\sum_{n=m}^{\infty}+\sum_{n=0}^{m-1}\right)\sum_{k=0}^{n}(-1)^{n}A_{m,k}\binom{n+m-k}{m}\left(x^{\frac{a}{2}}y\right)^{2n}.\end{aligned}$$

Since $A_{m,k}=0\ \ (k>m)$, we obtain

$$\begin{aligned}E(m;a,x,y) &= \sum_{n=m}^{\infty}\sum_{k=0}^{m}A_{m,k}\binom{n+m-k}{m}(-1)^n\left(x^{\frac{a}{2}}y\right)^{2n} \\ &\quad+\sum_{n=0}^{m-1}\sum_{k=0}^{n}A_{m,k}\binom{n+m-k}{m}(-1)^n\left(x^{\frac{a}{2}}y\right)^{2n}.\end{aligned}$$

We therefore have

$$\begin{aligned}E(m;a,x,y) &= \sum_{n=0}^{\infty}\sum_{k=0}^{m} A_{m,k}\binom{n+m-k}{m}(-1)^n\left(x^{\frac{a}{2}}y\right)^{2n}\\ &\quad -\sum_{n=0}^{m-1}\sum_{k=0}^{m} A_{m,k}\binom{n+m-k}{m}(-1)^n\left(x^{\frac{a}{2}}y\right)^{2n}\\ &\quad +\sum_{n=0}^{m-1}\sum_{k=0}^{n} A_{m,k}\binom{n+m-k}{m}(-1)^n\left(x^{\frac{a}{2}}y\right)^{2n}.\end{aligned}$$

Using (24) gives

$$\begin{aligned}\sum_{n=0}^{\infty} n^m(-1)^n\left(x^{\frac{a}{2}}y\right)^{2n} &= E(m;a,x,y)\\ &\quad +\sum_{n=0}^{m-1}\sum_{k=n+1}^{m} A_{m,k}\binom{n+m-k}{m}(-1)^n\left(x^{\frac{a}{2}}y\right)^{2n}.\end{aligned} \tag{51}$$

Substituting (51) into the summation in the second equality in (50), with the aid of the integral formulas (48) and (49), we can readily obtain the desired identity (46). □

4. Certain variant Euler harmonic sums

This section delves into specific variants of Euler harmonic sums, examining their relevance to the content discussed in Section 3.

THEOREM 3. *Let $a\in\mathbb{R}_{>0}$, $(p,t)\in\mathbb{N}^2$ and $m\in\mathbb{Z}_{\geqslant 0}$ with $t>m-1$. Then*

$$\begin{aligned}&\sum_{n\geqslant 0}\frac{(-1)^n n^m\left(H^{(p+1)}_{\frac{an}{2}}-H^{(p+1)}_{\frac{an}{2}-\frac{1}{2}}\right)}{(2n+1)^{t+1}}\\ &= \frac{(-1)^{t+p}2^{p+1}}{t!p!}\sum_{k=0}^{m}(-1)^k A_{m,k}\int_0^1\!\!\int_0^1 \frac{\ln^p(x)\ln^t(y)\left(x^{\frac{a}{2}}y\right)^{2k}}{(1+x)(1+x^a y^2)^{m+1}}\,\mathrm{d}x\mathrm{d}y\\ &\quad -\sum_{n=0}^{m-1}\frac{(-1)^n}{(2n+1)^{t+1}}\sum_{k=n+1}^{m} A_{m,k}\binom{n+m-k}{m}\\ &\qquad \times\left\{\zeta\left(p+1,\tfrac{an}{2}+1\right)-\zeta\left(p+1,\tfrac{an+1}{2}\right)\right\}.\end{aligned} \tag{52}$$

Particularly,

$$\begin{aligned}&\sum_{n\geqslant 0}\frac{(-1)^n\left(H^{(p+1)}_{\frac{an}{2}}-H^{(p+1)}_{\frac{an}{2}-\frac{1}{2}}\right)}{(2n+1)^{t+1}}\\ &= \frac{(-1)^{t+p}2^{p+1}}{t!p!}\int_0^1\!\!\int_0^1\frac{\ln^p(x)\ln^t(y)}{(1+x)(1+x^a y^2)}\,\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{53}$$

Proof. The identity (17) provides

$$\psi^{(p)}\left(\tfrac{an}{2}+1\right)-\psi^{(p)}\left(\tfrac{an}{2}+\tfrac{1}{2}\right)=(-1)^p\,p!\left(H^{(p+1)}_{\frac{an}{2}}-H^{(p+1)}_{\frac{an}{2}-\frac{1}{2}}\right),$$

which is utilized in Theorem 2 to yield the results here. □

COROLLARY 2. *Let $a\in\mathbb{R}_{>0}$, $(p,t)\in\mathbb{N}^2$ and $m\in\mathbb{Z}_{\geqslant 0}$ with $t>m-1$. Then*

$$\begin{aligned}
&\sum_{n\geqslant 0}\frac{(-1)^n n^m\left(2H^{(p+1)}_{\frac{an}{2}}-2^{p+1}H^{(p+1)}_{an}\right)}{(2n+1)^{t+1}}\\
&\quad=\frac{(-1)^{t+p}2^{p+1}}{t!p!}\sum_{k=0}^{m}(-1)^k A_{m,k}\int_0^1\int_0^1\frac{\ln^p(x)\ln^t(y)\left(x^{\frac{a}{2}}y\right)^{2k}}{(1+x)(1+x^a y^2)^{m+1}}\,\mathrm{d}x\mathrm{d}y\\
&\qquad-\sum_{n=0}^{m-1}\frac{(-1)^n}{(2n+1)^{t+1}}\sum_{k=n+1}^{m}A_{m,k}\binom{n+m-k}{m}\\
&\qquad\quad\times\left\{\zeta\left(p+1,\tfrac{an}{2}+1\right)-\zeta\left(p+1,\tfrac{an+1}{2}\right)\right\}\\
&\qquad-2^{p+1-m}\eta(p+1)\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\beta(t+1-j),
\end{aligned}\tag{54}$$

where $\eta(s)$ is Dirichlet eta function defined by

$$\eta(s):=\sum_{n\geqslant 1}\frac{(-1)^{n-1}}{n^s}=\left(1-2^{1-s}\right)\zeta(s)\quad(\Re(s)>0).\tag{55}$$

Particularly,

$$\begin{aligned}
&\sum_{n\geqslant 0}\frac{(-1)^n\left(2H^{(p+1)}_{\frac{an}{2}}-2^{p+1}H^{(p+1)}_{an}\right)}{(2n+1)^{t+1}}\\
&\quad=-2^{p+1}\eta(p+1)\beta(t+1)+\frac{(-1)^{p+t}2^{p+1}}{p!t!}\int_0^1\int_0^1\frac{\ln^p(x)\ln^t(y)}{(1+x)(1+x^a y^2)}\,\mathrm{d}x\mathrm{d}y.
\end{aligned}\tag{56}$$

Proof. Let $\mathscr{L}$ be the left member of (52). Using (20) gives

$$H^{(p+1)}_{\frac{an}{2}-\frac{1}{2}}=2^{p+1}H^{(p+1)}_{an}-H^{(p+1)}_{\frac{an}{2}}-2^{p+1}\eta(p+1),$$

which is employed to offer

$$\mathscr{L}=\sum_{n\geqslant 0}\frac{(-1)^n n^m\left(2H^{(p+1)}_{\frac{an}{2}}-2^{p+1}H^{(p+1)}_{an}\right)}{(2n+1)^{t+1}}+2^{p+1}\eta(p+1)\sum_{n\geqslant 0}\frac{(-1)^n n^m}{(2n+1)^{t+1}}.\tag{57}$$

Also

$$\sum_{n\geqslant 0}\frac{(-1)^n n^m}{(2n+1)^{t+1}}=\frac{1}{2^m}\sum_{n\geqslant 0}\frac{(-1)^n (2n)^m}{(2n+1)^{t+1}}$$

and

$$(2n)^m=\{(2n+1)-1\}^m=\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(2n+1)^j.$$

We thus have

$$\sum_{n\geqslant 0}\frac{(-1)^n n^m}{(2n+1)^{t+1}}=\frac{1}{2^m}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\beta(t+1-j). \tag{58}$$

Substituting (58) into (57) affords

$$\begin{aligned}\mathscr{L}=&\sum_{n\geqslant 0}\frac{(-1)^n n^m\left(2H_{\frac{an}{2}}^{(p+1)}-2^{p+1}H_{an}^{(p+1)}\right)}{(2n+1)^{t+1}}\\&+2^{p+1-m}\eta(p+1)\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\beta(t+1-j).\end{aligned} \tag{59}$$

Finally, the new expression of $\mathscr{L}$ in (59) is incorporated in the results in Theorem 3 to provide the results here. $\square$

THEOREM 4. *Let $a\in\mathbb{R}_{>0}$, $(p,t)\in\mathbb{N}^2$ and $m\in\mathbb{Z}_{\geqslant 0}$ with $t>m-1$. Then*

$$\begin{aligned}\upsilon(a,m,p,t):=&\sum_{n\geqslant 0}\frac{(-1)^n\ n^m\ H_{an}^{(p+1)}}{(2n+1)^{t+1}}\\=&\frac{(-1)^{p+t+1}}{p!t!}\sum_{k=0}^{m}(-1)^k A_{m,k}\int_0^1\!\!\int_0^1\frac{\ln^p(x)\ln^t(y)\left(x^{\frac{a}{2}}y\right)^{2k}}{(1-x)(1+x^a y^2)^{m+1}}\,\mathrm{d}x\mathrm{d}y\\&+\frac{(-1)^t}{t!}\zeta(p+1)\sum_{k=0}^{m}(-1)^k A_{m,k}\int_0^1\frac{y^{2k}\ln^t(y)}{(1+y^2)^{m+1}}\,\mathrm{d}y\\&+\zeta(p+1)\sum_{n=0}^{m-1}\frac{(-1)^n}{(2n+1)^{t+1}}\sum_{k=n+1}^{m}A_{m,k}\binom{n+m-k}{m}\\&+\frac{(-1)^p}{p!}\sum_{n=0}^{m-1}\frac{(-1)^n\psi^{(p)}(an+1)}{(2n+1)^{t+1}}\sum_{k=n+1}^{m}A_{m,k}\binom{n+m-k}{m}.\end{aligned} \tag{60}$$

Particulary,

$$\begin{aligned}
\upsilon(a,0,p,t) &= \sum_{n\geqslant 0} \frac{(-1)^n\ H_{an}^{(p+1)}}{(2n+1)^{t+1}} \\
&= \frac{(-1)^{p+t+1}}{p!t!} \int_0^1\int_0^1 \frac{\ln^p(x)\ln^t(y)}{(1-x)(1+x^a y^2)}\,\mathrm{d}x\mathrm{d}y \\
&\quad + \frac{(-1)^t}{t!}\zeta(p+1)\int_0^1 \frac{\ln^t(y)}{1+y^2}\,\mathrm{d}y.
\end{aligned} \tag{61}$$

Proof. From (17), we get

$$(-1)^p\,p!\,H_{an}^{(p+1)} = \psi^{(p)}(an+1) + (-1)^p p!\,\zeta(p+1). \tag{62}$$

Recall the following integral formulas (cf. [8, p. 546, Entry 4.272-4])

$$\int_0^1 \frac{\ln^p(x)}{1-x}\,dx = (-1)^p\,p!\,\zeta(p+1) \quad (p\in\mathbb{N}) \tag{63}$$

and (see, e.g., [23, p. 25, Eq. (13)])

$$\psi(z) = -\gamma + \int_0^1 \frac{1-x^{z-1}}{1-x}\,dx \quad (\Re(z)>0). \tag{64}$$

Differentiating both sides of (64) with respect to z, p times, and putting $z=an+1$ in the resultant identity, we derive

$$\psi^{(p)}(an+1) = -\int_0^1 \frac{x^{an}\ln^p(x)}{1-x}\,dx \quad (p\in\mathbb{N}). \tag{65}$$

Using (63) and (65) in the right side of (62), we obtain

$$(-1)^p\,p!\,H_{an}^{(p+1)} = \int_0^1 \frac{(1-x^{an})\ln^p(x)}{1-x}\,dx. \tag{66}$$

Using (49) and (66), we find

$$\begin{aligned}
\upsilon(a,m,p,t) &:= \sum_{n\geqslant 0} \frac{(-1)^n\ n^m\ H_{an}^{(p+1)}}{(2n+1)^{t+1}} \\
&= \frac{(-1)^{p+t}}{p!t!}\int_0^1\int_0^1 \frac{\ln^p(x)\ln^t(y)}{1-x} \sum_{n\geqslant 0}(-1)^n\,n^m\left\{y^{2n} - \left(x^{\frac{a}{2}}y\right)^{2n}\right\} dxdy.
\end{aligned} \tag{67}$$

Now, as in the proof of Theorem 2, we can obtain the identity here. □

5. Particular cases

This section provides ceratin particular instances of the variant Euler harmonic sums in Section 4. To do this, we begin by recalling an intriguing and useful mathematical constant $\mathscr{G}$ defined by

$$\mathscr{G} := \Im\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right) \approx .570077, \tag{68}$$

which was confirmed among useful mathematical constants and investigated in [5] (see also [19]). This constant $\mathscr{G}$ is a natural companion to the Catalan's constant G in (15) in many ways and has appeared in various literature (for example, see the references in [5]). One can find from (68) that (see, e.g., [19]; see also [5])

$$\mathscr{G} = \sum_{n\geqslant 1} \frac{\sin\left(\frac{\pi n}{4}\right)}{2^{\frac{n}{2}} n^3} = \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{2^{2n}} \left(\frac{2}{(4n-3)^3} + \frac{2}{(4n-2)^3} + \frac{1}{(4n-1)^3}\right). \tag{69}$$

We also recall the real part expression of $\mathrm{Li}_3\left(\frac{1+i}{2}\right)$

$$\Re\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right) = \frac{35\zeta(3)}{64} - \frac{5\pi^2 \ln 2}{192} + \frac{\ln^3 2}{48}, \tag{70}$$

which can be derived by setting $\theta = \frac{\pi}{2}$ in the formula [10, Eq. (6.54)] and using [10, Eq. (6.6)] (or [10, p. 296, Entry A.2.6-(5)]). A simple and elegant integral expression for $\mathscr{G}$ is recalled (see, e.g., [5], [19])

$$\mathscr{G} = \frac{1}{2} \int_0^1 \frac{\ln^2(1-x)}{1+x^2}\, \mathrm{d}x. \tag{71}$$

The following lemma gives a new association of the mathematical constant $\mathscr{G}$ to the average value of two variant Euler sums, one containing the square of the harmonic numbers and the other containing harmonic numbers of order two.

LEMMA 1. *The following formulas hold true.*

$$\mathscr{G} = \frac{1}{2} \sum_{n\geqslant 0} \frac{(-1)^n \left(H_{2n+1}^2 + H_{2n+1}^{(2)}\right)}{2n+1}; \tag{72}$$

$$\sum_{r=0}^{n} \frac{(-1)^r \binom{n}{r}}{(r+1)^3} = \frac{H_{n+1}^2 + H_{n+1}^{(2)}}{2(n+1)} \quad (n \in \mathbb{N}). \tag{73}$$

Proof. Applying the Maclaurin-Taylor series expansion of $\frac{1}{1+x^2}$ in (71), we have

$$\begin{aligned}\mathscr{G} &= \frac{1}{2}\sum_{n\geqslant 0}(-1)^n\int_0^1 x^{2n}\ \ln^2(1-x)\,\mathrm{d}x \\ &= \frac{1}{2}\sum_{n\geqslant 0}(-1)^n\int_0^1 (1-x)^{2n}\ \ln^2(x)\,\mathrm{d}x.\end{aligned} \tag{74}$$

For the second integral in (74), setting $\mu = 1$ and $\nu = 2n+1$ in (9) and using (4), we can obtain (72). Also

$$\begin{aligned}\int_0^1 (1-x)^{2n}\ \ln^2(x)\,\mathrm{d}x &= \sum_{r=0}^{2n}(-1)^r\binom{2n}{r}\int_0^1 x^r\ \ln^2(x)\,\mathrm{d}x \\ &= \sum_{r=0}^{2n}(-1)^r\binom{2n}{r}\frac{2}{(r+1)^3},\end{aligned}$$

which, upon matching the corresponding part in (72), can provide (73). □

Setting $t=0$, $p=1$, and $a=2$ in (56), we obtain

$$\begin{aligned}\sum_{n\geqslant 0}\frac{(-1)^n\left(2H_n^{(2)}-4H_{2n}^{(2)}\right)}{2n+1} &= -4\eta(2)\beta(1) - 4\int_0^1\int_0^1\frac{\ln x}{(1+x)(1+x^2y^2)}\,\mathrm{d}x\mathrm{d}y \\ &= -4\eta(2)\beta(1) - 4\int_0^1\frac{(\ln x)\arctan x}{x(1+x)}\,\mathrm{d}x\mathrm{d}y,\end{aligned}$$

the last integral of which can be evaluated by the methods developed in [19] and one can get

$$\sum_{n\geqslant 0}\frac{(-1)^n\left(2H_n^{(2)}-4H_{2n}^{(2)}\right)}{2n+1} = 4\eta(2)\beta(1) + 2G\ln 2 + \frac{\pi^3}{16}. \tag{75}$$

Recall the following formula (see [18, Lemma 2]):

$$\sum_{n\geqslant 0}\frac{(-1)^n H_n^{(2)}}{2n+1} = 4\mathscr{G} + 2G\ln 2 - \frac{\pi}{8}\ln^2 2 - \frac{11\pi^3}{96}, \tag{76}$$

which is used in (75) to yield

$$\sum_{n\geqslant 0}\frac{(-1)^n H_{2n}^{(2)}}{2n+1} = 2\mathscr{G} + \frac{G\ln 2}{2} - \frac{\pi}{16}\ln^2 2 - \frac{3\pi^3}{32}. \tag{77}$$

Consider

$$H_{2n+1}^{(2)} = H_{2n}^{(2)} + \frac{1}{(2n+1)^2},$$

which is employed in (77) to provide

$$\sum_{n\geqslant 0}\frac{(-1)^n H_{2n+1}^{(2)}}{2n+1}=2\mathscr{G}+\frac{G\ln 2}{2}+\beta(3)-\frac{\pi}{16}\ln^2 2-\frac{3\pi^3}{32}. \tag{78}$$

Applying (78) to (72), one can obtain

$$\sum_{n\geqslant 0}\frac{(-1)^n H_{2n+1}^2}{2n+1}=\frac{3\pi^3}{32}+\frac{\pi}{16}\ln^2 2-\frac{G\ln 2}{2}-\beta(3). \tag{79}$$

Likewise one can obtain closed form expressions of numerous variant Euler harmonic sums. Here, the following examples are offered:

$$\sum_{n\geqslant 0}\frac{(-1)^n n\left(H_{\frac{n}{2}}-H_{\frac{n}{2}-\frac{1}{2}}\right)}{(2n+1)^2}=\frac{\pi}{2}G+\frac{\pi^2}{16}-\frac{7}{4}\zeta(3); \tag{80}$$

$$\sum_{n\geqslant 0}\frac{(-1)^n n\left(H_{n-\frac{1}{2}}-H_n\right)}{(2n+1)^2}=3\mathscr{G}-\frac{5\pi^3}{128}-G+\frac{\pi}{8}\ln 2+\frac{1}{2}G\ln 2-\frac{3\pi}{32}\ln^2 2; \tag{81}$$

$$\sum_{n\geqslant 0}\frac{(-1)^n n\left(H_{2n}-H_n\right)}{(2n+1)^2}=\frac{3}{2}\mathscr{G}-\frac{1}{4}G\ln 2-\frac{5\pi^3}{256}-\frac{1}{2}G+\frac{3\pi}{16}\ln 2-\frac{3\pi}{64}\ln^2 2; \tag{82}$$

$$\sum_{n\geqslant 0}\frac{(-1)^n n\left(H_n-H_{\frac{n}{2}}\right)}{(2n+1)^2}=\frac{\pi}{8}\ln 2-\frac{1}{2}G\ln 2+\frac{7}{8}\zeta(3)-\frac{\pi^2}{32}-\frac{\pi}{4}G; \tag{83}$$

$$\begin{aligned}\sum_{n\geqslant 0}\frac{(-1)^n n\left(H_{2n}-H_{\frac{n}{2}}\right)}{(2n+1)^2}=&\frac{3}{2}\mathscr{G}-\frac{5\pi^3}{256}-\frac{1}{2}G-\frac{3}{4}G\ln 2-\frac{3\pi}{64}\ln^2 2\\&+\frac{5\pi}{16}\ln 2+\frac{7}{8}\zeta(3)-\frac{\pi^2}{32}-\frac{\pi}{4}G;\end{aligned} \tag{84}$$

$$\sum_{n\geqslant 0}\frac{(-1)^n n\left(H_{n-\frac{1}{2}}+H_n\right)}{(2n+1)^2}=\mathscr{G}+\frac{3}{2}G\ln 2-\frac{3\pi^3}{128}-\frac{3\pi}{8}\ln 2-\frac{\pi}{32}\ln^2 2; \tag{85}$$

$$\sum_{n\geqslant 0}\frac{(-1)^n n\, H_n}{(2n+1)^2}=\frac{\pi^3}{128}+\frac{1}{2}G\ln 2+\frac{1}{2}G-\frac{\pi}{4}\ln 2+\frac{\pi}{32}\ln^2 2-\mathscr{G}; \tag{86}$$

$$\sum_{n\geqslant 0}\frac{(-1)^n n\, H_{n-\frac{1}{2}}}{(2n+1)^2}=2\mathscr{G}-\frac{1}{2}G+G\ln 2-\frac{\pi^3}{32}-\frac{\pi}{8}\ln 2-\frac{\pi}{16}\ln^2 2; \tag{87}$$

$$\begin{aligned}\sum_{n\geqslant 0}\frac{(-1)^n n\, H_{\frac{n}{2}}}{(2n+1)^2}=&\mathscr{G}-\frac{\pi^3}{128}-\frac{1}{2}G-G\ln 2-\frac{\pi}{32}\ln^2 2\\&+\frac{3\pi}{8}\ln 2+\frac{7}{8}\zeta(3)-\frac{\pi^2}{32}-\frac{\pi}{4}G;\end{aligned} \tag{88}$$

$$\sum_{n\geqslant 0} \frac{(-1)^n\, n\, H_{2n}}{(2n+1)^2} = \frac{1}{2}\mathscr{G} - \frac{3\pi^3}{256} + \frac{1}{4}G\ln 2 - \frac{\pi}{16}\ln 2 - \frac{\pi}{64}\ln^2 2; \tag{89}$$

$$\sum_{n\geqslant 0} \frac{(-1)^n\, n\, H_n^{(2)}}{(2n+1)^2} = 2\mathscr{G} + G\ln 2 - \frac{11\pi^3}{192} - \frac{\pi}{16}\ln^2 2 + 3\beta(4) - \frac{7\pi}{8}\zeta(3) + \frac{\pi^2 G}{24}. \tag{90}$$

Remarks

We offered integral formulae connected with the function $\mathbf{b}(z)$ in (10), which is also signified by $G(z) = 2\mathbf{b}(z)$ (cf. [7, p. 20]) and, mostly, by $\beta(z)$ (see, e.g., [8, Section 8.37]). Yet, in order to avoid notation-overlap of the Dirichlet beta function $\beta(z)$ in (11), we chose to use $\mathbf{b}(z)$. We investigated a series involving polygamma functions that is represented as double integrals, by incorporating the Eulerian numbers $A_{n,k}$ $(n, k \in \mathbb{Z}_{\geqslant 0})$ in (21). We employed the double integral formula of series involving polygamma functions to explore certain variant Euler harmonic sums. Finally, we provided closed-form evaluations for a number of specific examples of the variant Euler harmonic sums, by using mathematical constants, in particular, the constant $\mathscr{G}$ in (68). The interested researcher is encouraged to prove the identities (80)–(90). For example, using $n = \frac{1}{2}(2n+1) - \frac{1}{2}$, one gets

$$\sum_{n\geqslant 0} \frac{(-1)^n\, n\, H_n^{(2)}}{(2n+1)^2} = \frac{1}{2}\sum_{n\geqslant 0} \frac{(-1)^n\, H_n^{(2)}}{2n+1} - \frac{1}{2}\sum_{n\geqslant 0} \frac{(-1)^n\, H_n^{(2)}}{(2n+1)^2}$$
$$= \left(2\mathscr{G} + G\ln 2 - \frac{11\pi^3}{192} - \frac{\pi}{16}\ln^2 2\right) + \left(3\beta(4) - \frac{7\pi}{8}\zeta(3) + \frac{\pi^2 G}{24}\right),$$

which can be rearranged to yield the last identity (90).

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