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ON THE LACUNARY–TYPE UNIVARIATE COMPLEX POLYNOMIALS

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Abstract. In this paper, we study the zeros of lacunary-type polynomials with complex coefficients. Here we present some results to locate the zeros of lacunary-type polynomials and discuss their importance with respect to existing results comparatively.

1. Introduction

The following result due to Cauchy [3] is classical in the theory of distribution of zeros of a polynomial

THEOREM A. *All the zeros of a polynomial*

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n, \quad a_n \neq 0$$

lie in

$$|z| \leqslant 1 + M,$$

where $M = \max\limits_{1 \leqslant j \leqslant n-1} \left| \frac{a_j}{a_n} \right|$.

Look at Theorem A, only leading coefficient a_n is restricted and rest are arbitrary from $\mathbb{C}$. This means that Theorem A guarantees us that whenever $a_n \neq 0$ and $a_k \in \mathbb{C}$, $1 \leqslant k \leqslant n-1$ are chosen arbitrary, all the zeros of $P(z)$ lie in $|z| \leqslant 1 + M$. As a result, in this theorem the underlying polynomial is liberated with respect to its coefficients except leading coefficient.

The following result which improves upon Theorem A and provide an annulus containing all the zeros of a polynomial by using special type of numbers and binomial coefficients is due to Diaz-Barrero [5].

THEOREM B. *Let* $P(z) = \sum\limits_{t=0}^{n} a_t z^t$ $(a_t \neq 0,\ 0 \leqslant t \leqslant n)$ *be a non-constant complex polynomial. Then all its zeros lie in the annulus* $\mathscr{C} = \{z : r_1 \leqslant |z| \leqslant r_2\}$, *where*

$$r_1 = \frac{3}{2} \min_{1 \leqslant t \leqslant n} \left\{ \frac{2^n F_t C(n,\, t)}{F_{4n}} \left| \frac{a_0}{a_t} \right| \right\}^{\frac{1}{t}} \tag{1}$$

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and

$$r_2 = \frac{2}{3} \max_{1 \leqslant t \leqslant n} \left\{ \frac{F_{4n}}{2^n F_t C(n,\ t)} \left| \frac{a_{n-t}}{a_n} \right| \right\}^{\frac{1}{t}}. \tag{2}$$

Here F_t is the t^{th} Fibonacci number, defined by, $F_0 = 0$, $F_1 = 1$ and for $t \geqslant 2$, $F_t = F_{t-1} + F_{t-2}$. Furthermore, $C(n,t) = \frac{n!}{t!(n-t)!}$ are the binomial coefficients. Another result in this connection providing annulus containing all the zeros of a polynomial $P(z)$ is the following, and is ascribed to Kim [10].

THEOREM C. *Let $P(z) = \sum_{t=0}^{n} a_t z^t$ $(a_t \neq 0,\ 0 \leqslant t \leqslant n)$ be a non-constant polynomial with complex coefficients. Then all its zeros lie in the annulus $A = \{z : r_1 \leqslant |z| \leqslant r_2\}$, where*

$$r_1 = \min_{1 \leqslant t \leqslant n} \left\{ \frac{C(n,t)}{2^n - 1} \left| \frac{a_0}{a_t} \right| \right\}^{\frac{1}{t}} \tag{3}$$

and

$$r_2 = \max_{1 \leqslant t \leqslant n} \left\{ \frac{2^n - 1}{C(n,t)} \left| \frac{a_{n-t}}{a_n} \right| \right\}^{\frac{1}{t}}. \tag{4}$$

Here $C(n,t)$ is the binomial coefficient.

We have following two more results due to Diaz-Barrero and Egozcue [7] regarding the zeros of $P(z)$.

THEOREM D. *Let $P(z) = \sum_{t=0}^{n} a_t z^t$ $(a_t \neq 0)$ be a non-constant complex polynomial. Then for $j \geqslant 2$ all its zeros lie in the annulus $C = \{z : r_1 \leqslant |z| \leqslant r_2\}$, where*

$$r_1 = \min_{1 \leqslant t \leqslant n} \left\{ \frac{C(n,t) A_t B_j^t (bB_{j-1})^{n-t}}{A_{jn}} \left| \frac{a_0}{a_t} \right| \right\}^{1/t} \tag{5}$$

and

$$r_2 = \max_{1 \leqslant t \leqslant n} \left\{ \frac{A_{jn}}{C(n,t) A_t B_j^t (bB_{j-1})^{n-t}} \left| \frac{a_{n-t}}{a_n} \right| \right\}^{1/t}. \tag{6}$$

Here $B_n = \sum_{t=0}^{n-1} r^t s^{n-1-k}$ and $A_n = cr^n + ds^n$, where c,d are real constants and r,s are the roots of the equation $x^2 - ax - b = 0$ in which a,b are strictly positive real numbers. For $j \geqslant 2$, $\sum_{t=0}^{n} C(n,t)(bB_{j-1})^{n-t} B_j^t A_t = A_{jn}$. Furthermore, $C(n,t)$ is the binomial coefficient.

THEOREM E. *Let* $P(z)=\sum_{t=0}^{n} a_t z^t$ $(a_t \neq 0)$ *be a non-constant polynomial with complex coefficients. Then all its zeros lie in the ring shaped region* $C=\{z: r_1 \leqslant |z| \leqslant r_2\}$*, where*

$$r_1=\min_{1\leqslant t\leqslant n}\left\{\frac{2^t P_t C(n,\, t)}{P_{2n}}\left|\frac{a_0}{a_t}\right|\right\}^{\frac{1}{t}} \tag{7}$$

and

$$r_2=\max_{1\leqslant t\leqslant n}\left\{\frac{P_{2n}}{2^t P_t C(n,\, t)}\left|\frac{a_{n-t}}{a_n}\right|\right\}^{\frac{1}{t}}. \tag{8}$$

Here P_t *is the* t^{th} *Pell number, defined by,* $P_0=0,\ P_1=1$ *and for* $t\geqslant 2,\ P_t=2P_{t-1}+P_{t-2}$.

Again we state the following result which is due to Diaz-Barrero [6] providing regions containing all the zeros of a polynomial $P(z)$.

THEOREM F. *Let* $P(z)=\sum_{t=0}^{n} a_t z^t$ *be a complex monic polynomial. Then all its zeros lie in the disks* $C_1=\{z: |z|\leqslant r_1\}$ *or* $C_2=\{z: |z|\leqslant r_2\}$*, where*

$$r_1=\max_{1\leqslant t\leqslant n}\left\{\frac{2^{n-1}C(n+1,\, 2)}{t^2 C(n,\, t)}|a_{n-t}|\right\}^{1/t} \tag{9}$$

and

$$r_2=\max_{1\leqslant t\leqslant n}\left\{\frac{F_{3n}}{C(n,\, t)2^t F_t}|a_{n-t}|\right\}^{1/t}. \tag{10}$$

Here $C(n,\, t)$ *is the binomial coefficient.*

Next, we state the following unified result due to Dalal and Govil [4] (see also [1]), which includes all the above results, Theorems B-F as special cases.

THEOREM G. *Let* $A_t>0$ *for* $1\leqslant t\leqslant n$*, and be such that* $\sum_{t=1}^{n} A_t=1$*. If* $P(z)=\sum_{t=0}^{n} a_t z^t$ $(a_t\neq 0,\ 0\leqslant t\leqslant n)$ *is a non-constant polynomial with complex coefficients, then all the zeros of* $P(z)$ *lie in the annulus* $C=\{z: r_1\leqslant |z|\leqslant r_2\}$*, where*

$$r_1=\min_{1\leqslant t\leqslant n}\left\{A_t\left|\frac{a_0}{a_t}\right|\right\}^{\frac{1}{t}} \tag{11}$$

and

$$r_2=\max_{1\leqslant t\leqslant n}\left\{\frac{1}{A_t}\left|\frac{a_{n-t}}{a_n}\right|\right\}^{\frac{1}{t}}. \tag{12}$$

As an application of Theorem G, Govil and Kumar [9] proved the following two results that gives annuli in terms of Narayana numbers [11] and Motzkin numbers [8].

THEOREM H. *Let* $P(z) = \sum_{t=0}^{n} a_t z^t$ *be a non-constant polynomial with complex coefficients, with* $a_t \neq 0,\ 0 \leqslant t \leqslant n$. *Then all the zeros of* $P(z)$ *lie in the annulus* $C = \{z : r_1 \leqslant |z| \leqslant r_2\}$, *where*

$$r_1 = \min_{1 \leqslant t \leqslant n} \left\{ \frac{N(n,\ t)}{C_n} \left| \frac{a_0}{a_t} \right| \right\}^{\frac{1}{t}} \tag{13}$$

and

$$r_2 = \max_{1 \leqslant t \leqslant n} \left\{ \frac{C_n}{N(n,\ t)} \left| \frac{a_{n-t}}{a_n} \right| \right\}^{\frac{1}{t}}, \tag{14}$$

where $C_n = \frac{C(2n,\ n)}{n+1}$ *is the* n^{th} *Catalan number,* $N(n,\ t),\ (1 \leqslant t \leqslant n)$ *are Narayana numbers defined for any natural number* n *by* $N(n,\ t) = \frac{1}{n} C(n,\ t)\ C(n,\ t-1)$, *and* $C(n,\ t)$ *is the binomial coefficient.*

THEOREM I. *Let* $P(z) = \sum_{t=0}^{n} a_t z^t$ *be a non-constant polynomial with complex coefficients, with* $a_t \neq 0,\ 0 \leqslant t \leqslant n$. *Then all the zeros of* $P(z)$ *lie in the annulus* $C = \{z : r_1 \leqslant |z| \leqslant r_2\}$, *where*

$$r_1 = \min_{1 \leqslant t \leqslant n} \left\{ \frac{M_{t-1} M_{n-1-t}}{M_n} \left| \frac{a_0}{a_t} \right| \right\}^{\frac{1}{t}} \tag{15}$$

and

$$r_2 = \max_{1 \leqslant t \leqslant n} \left\{ \frac{M_n}{M_{t-1} M_{n-1-t}} \left| \frac{a_{n-t}}{a_n} \right| \right\}^{\frac{1}{t}}, \tag{16}$$

where M_n *is the* n^{th} *Motzkin number defined by* $M_0 = M_1 = M_{-1} = 1$, *and*

$$M_{n+1} = \frac{2n+3}{n+3} M_n + \frac{3n}{n+3} M_{n-1}, \quad n \geqslant 1.$$

Now, let us look at Theorem G, which is due to Dalal and Govil [4], includes all the Theorems B–F and many other results as special cases by choosing $A_t > 0$ appropriately with $\sum_{t=1}^{n} A_t = 1$. But in Theorem G, the polynomial is not liberated with respect to its coefficients, that is, if at least one $a_k = 0,\ 1 \leqslant k \leqslant n-1$, Theorem G does not hold good. In view of that, we consider the class of lacunary type polynomials

$$\mathbb{P}_{n,\ \mu} = \left\{ P : P(z) = a_0 + \sum_{t=\mu}^{n} a_t z^t,\ (a_t \neq 0\ \forall\, t),\ 1 \leqslant \mu \leqslant n \right\}$$

and make an endeavor to resolve this case while proving several results which provide annuli containing all the zeros of the polynomial $P \in \mathbb{P}_{n,\ \mu}$. Note that for $\mu = 1$, the lacunary polynomial reduces to a simple polynomial

$$P(z) = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} + a_n z^n.$$

2. Main results

We first prove the following result which provide an annulus to locate the zeros of a polynomial $P \in \mathbb{P}_{n,\,\mu}$.

THEOREM 1. *Let $A_t > 0$ be such that $\sum_{t=1}^{n} A_t = 1$, and let $P \in \mathbb{P}_{n,\,\mu}$. Then all the zeros of P lie in the annulus $\mathscr{K} = \{z : R_1 \leqslant |z| \leqslant R_2\}$, where*

$$R_1 = \min_{\mu \leqslant t \leqslant n} \left\{ A_t \left| \frac{a_0}{a_t} \right| \right\}^{\frac{1}{t}} \tag{17}$$

and

$$R_2 = \max_{\mu \leqslant t \leqslant n} \left\{ \frac{1}{A_t} \left| \frac{a_{n-t}}{a_n} \right| \right\}^{\frac{1}{t}}. \tag{18}$$

Since Theorem G does not hold if at least one $a_k = 0,\ 1 \leqslant k \leqslant n-1$, we make use of Theorem 1 by adapting the parameter μ. Have a look at the following.

REMARK 1. If $P(z) = a_0 + a_2 z^2 + a_3 z^3 \ldots + a_n z^n$, $(a_k \neq 0,\ 2 \leqslant k \leqslant n-1)$, then Theorem G does not give any information about the location of its zeros. In this case, take $\mu = 2$ in Theorem 1, we get all the zeros of $P(z)$ lie in $\mathscr{K} = \{z : R_1 \leqslant |z| \leqslant R_2\}$. Again if $P(z) = a_0 + a_3 z^3 \ldots + a_n z^n$, $(a_k \neq 0,\ 3 \leqslant k \leqslant n-1)$, then Theorem G does not hold and in this case we take $\mu = 3$ in Theorem 1 and so on similarly, we get finally all the zeros of polynomial $a_0 + a_n z^n$ lie in $\mathscr{K} = \{z : R_1 \leqslant |z| \leqslant R_2\}$, where

$$R_1 = \left\{ A_n \left| \frac{a_0}{a_n} \right| \right\}^{\frac{1}{n}}$$

and

$$R_2 = \left\{ \frac{1}{A_n} \left| \frac{a_0}{a_n} \right| \right\}^{\frac{1}{n}}.$$

REMARK 2. Theorem 1 is also true if $A_1, A_2, \ldots, A_n$ are any real or complex numbers such that $\sum_{t=1}^{n} |A_t| \leqslant 1$. If we take $\mu = 1$, in Theorem 1, we obtain Theorem G as a special case.

REMARK 3. Note that in Theorem 1 the selection of coefficients and μ is like that: when a_1 is absent, we take $\mu = 2$, when $a_1,\ a_2$ are absent, we take $\mu = 3$ and so on.

REMARK 4. In case, a_{n-1} is absent, then $a_{n-1},\ a_{n-2}$ are absent and so on, the lacunary polynomial takes the form

$$P(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}.$$

It is well established that for every choice of A_t in Table 1, A_t satisfy the two needed conditions $A_t > 0$ for $1 \leqslant t \leqslant n$, and $\sum_{t=1}^{n} A_t = 1$.

By making the right choice of A_t and μ, such that $A_t > 0$ and $\sum_{t=1}^{n} A_t = 1$, Theorem 1 include all the results listed in Table 1 as special cases and resolves them for $\mu \geqslant 2$.

REMARK 5. It is easy to verify that Theorem 1 is also an extension of Theorem F to the lacunary-type of polynomials, i.e., if $P \in \mathbb{P}_{n,\,\mu}$ be a complex monic polynomial of degree n and we take $A_t = \frac{t^2 C(n,\,t)}{2^{n-1} C(n+1,\,2)}$ and $\mu = 1$ in the bound (18) of Theorem 1 and note that $A_t > 0$, for all values of t and $\sum_{t=1}^{n} \frac{t^2 C(n,\,t)}{C(n+1,\,2)} = 2^{n-1}$, we obtain the bound (9) of Theorem F. Similarly, if we take $A_t = \frac{C(n,\,t) 2^t F_t}{F_{3n}}$ and $\mu = 1$ in the bound (18) of Theorem 1, and note the identity $\sum_{t=1}^{n} C(n,\,t) 2^t F_t = F_{3n}$, then we will obtain the bound (10) of Theorem F.

COROLLARY 1. *If $P \in \mathbb{P}_{n,\,\mu}$, then all the zeros of P lie in annulus $r_1 \leqslant |z| \leqslant r_2$, where*

$$r_1 = \min_{\mu \leqslant t \leqslant n} \left\{ \frac{L_t}{L_{n+2} - 3} \left| \frac{a_0}{a_t} \right| \right\}^{\frac{1}{t}} \tag{19}$$

and

$$r_2 = \max_{\mu \leqslant t \leqslant n} \left\{ \frac{L_{n+2} - 3}{L_t} \left| \frac{a_{n-t}}{a_n} \right| \right\}^{\frac{1}{t}}. \tag{20}$$

Here L_t is the t^{th} Lucas number defined by $L_0 = 2,\ L_1 = 1$ and for $t \geqslant 0,\ L_{t+2} = L_t + L_{t+1}$.

REMARK 6. Corollary 1 can be obtained from Theorem 1 by simply taking $A_t = \frac{L_t}{L_{n+2}-3}$, and from the definition of Lucas numbers, we have

$$\sum_{t=1}^{n} L_t = \sum_{t=1}^{n} \{L_{t+2} - L_{t+1}\} = L_{n+2} - L_2 = L_{n+2} - 3,$$

since $L_2 = L_0 + L_1 = 3$.

If we take $\mu = 1$ in Corollary 1, it immediately gives us the result due to Dalal and Govil [4, Corollary 2.1].

For example, if we consider the polynomial $P(z) = z^4 + 0.01z^3 + 0.1z^2 + 0.2z + 0.4$, then by taking $A_t = \frac{L_t}{L_{n+2}-3}$ in Theorem G, we get all the zeros of polynomial $P(z)$ lie in the annulus $r_1 \leqslant |z| \leqslant r_2$, where $r_1 \approx 0.1333$ and $r_2 \approx 0.9621$, and area of annulus comes out to be 2.8512 approximately. Now, if we consider the polynomial

Table 1.

Value of μ	$\mathscr{X}_t$	**Theorem**
1	$\frac{2^{n-t}3^t F_t C(n,\,t)}{F_{4n}}$	B
1	$\frac{C(n,\,t)}{2^n-1}$	C
1	$\frac{C(n,t)A_t B_j^t (bB_{j-1})^{n-t}}{A_{jn}}$	D
1	$\frac{2^n P_t C(n,\,t)}{P_{3n}}$	E

$P(z) = z^4 + 0.01z^3 + 0.1z^2 + 0.4$, then Theorem G does not give any annulus to locate the zeros of the polynomial $P(z)$ because the coefficient a_1 is absent. In this case, take $\mu = 2$ in Corollary 1, we get all the zeros of the polynomial $P(z)$ lie in the annulus $r_1 \leqslant |z| \leqslant r_2$, where $r_1 \approx 0.6573$ and $r_2 \approx 0.9621$, and area of annulus comes out to be 1.5498 approximately, which is also a significant improvement over the area obtained by Theorem G.

Catalan numbers, which are defined as $C_k = \frac{C(2k,\,k)}{k+1}$, where $C(2k,\,k)$ being the binomial coefficients, are well known in the field of combinatorics. We state the following result in terms of Catalan numbers as a corollary of Theorem 1, which resolves the result of Dalal and Govil [4, Corollary 2.2].

COROLLARY 2. *If $P \in \mathbb{P}_{n,\,\mu}$, then all the zeros of P lie in the annulus $r_1 \leqslant |z| \leqslant r_2$, where*

$$r_1 = \min_{\mu \leqslant k \leqslant n} \left\{ \frac{C_{k-1}\,C_{n-k}}{C_n} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}} \tag{21}$$

and

$$r_2 = \max_{\mu \leqslant k \leqslant n} \left\{ \frac{C_n}{C_{k-1}\,C_{n-k}} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}. \tag{22}$$

Here, as defined above, C_k is the k^{th} Catalan number.

REMARK 7. Corollary 2 is also an immediate consequence of Theorem 1 by taking $A_k = \frac{C_{k-1}\,C_{n-k}}{C_n}$, for $k = 1, 2, \ldots, n$, and noting that $\frac{C_{k-1}\,C_{n-k}}{C_n} > 0$ and

$$\sum_{k=1}^{n} C_{k-1}\,C_{n-k} = C_n.$$

Next we present some of the applications of Theorem 1 and obtain annuli containing all the zeros of a polynomial $P \in \mathbb{P}_{n,\,\mu}$. The first result in this connection, stated below gives an annular region for the zeros of a polynomial $P \in \mathbb{P}_{n,\,\mu}$ in terms of Narayana numbers.

THEOREM 2. *All the zeros of the polynomial $P \in \mathbb{P}_{n,\,\mu}$ lie in $C = \{z : K_1 \leqslant |z| \leqslant K_2\}$, where*

$$K_1 = \min_{\mu \leqslant k \leqslant n} \left\{ \frac{N(n,\,k)}{C_n} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}} \tag{23}$$

and

$$K_2 = \max_{\mu \leqslant k \leqslant n} \left\{ \frac{C_n}{N(n,\,k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}. \tag{24}$$

Here $C_n = \frac{C(2n,\,n)}{n+1}$ is the n^{th} Catalan number, $N(n,\,k) = \frac{1}{n} C(n,\,k)\, C(n,\,k-1)$ are Narayana numbers for any natural number n and $C(n,\,k)$ is the binomial coefficient.

REMARK 8. For $\mu = 1$, the polynomial $P \in \mathbb{P}_{n,\,\mu}$ reduces to a simple polynomial of degree n. In this case, Theorem 2 reduces to Theorem H. For $\mu \geqslant 2$, it resolves Theorem H if at least one $a_k = 0,\ 1 \leqslant k \leqslant n-1$ sequentially.

The Motzkin numbers M_n are defined by $M_0 = M_1 = M_{-1} = 1$ and

$$M_{n+1} = \frac{2n+3}{n+3} M_n + \frac{3n}{n+3} M_{n-1}, \quad n \geqslant 1.$$

The next result is based on the application of Motzkin numbers to get an annular region containing all the zeros of a polynomial $P \in \mathbb{P}_{n,\,\mu}$.

THEOREM 3. *Let $P \in \mathbb{P}_{n,\,\mu}$ be a complex polynomial of degree n. Then all the zeros of P lie in the annulus $C = \{z : K_1 \leqslant |z| \leqslant K_2\}$, where*

$$K_1 = \min_{\mu \leqslant k \leqslant n} \left\{ \frac{M_{k-1} M_{n-1-k}}{M_n} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}} \tag{25}$$

and

$$K_2 = \max_{\mu \leqslant k \leqslant n} \left\{ \frac{M_n}{M_{k-1} M_{n-1-k}} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}. \tag{26}$$

REMARK 9. For $\mu = 1$, Theorem 3 reduces to Theorem I.

Now, we present the following result which is based on generalized Fibonacci numbers. More precisely we prove.

THEOREM 4. *If $P \in \mathbb{P}_{n,\,\mu}$, then for $j \geqslant 1$, all the zeros of P lie in the annulus $\mathscr{R} = \{z : R_1 \leqslant |z| \leqslant R_2\}$ with*

$$R_1 = \min_{\mu \leqslant k \leqslant n} \left\{ \frac{C(n,\,k) F_{p,s,k} (F_{p,s,2^j})^k\ (s F_{p,s,2^j-1})^{n-k}}{F_{p,s,2^j n}} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}} \tag{27}$$

and

$$R_2 = \max_{\mu \leqslant k \leqslant n} \left\{ \frac{F_{p,s,2^j n}}{C(n,\ k) F_{p,s,k} (F_{p,s,2^j})^k \ (sF_{p,s,2^j-1})^{n-k}} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}, \tag{28}$$

where (p,s)*-Fibonacci sequence* $\{F_{p,s,n}\}_{n\in\mathbb{N}}$*, for any positive real numbers* $p,\ s,$ *is defined by*

$$F_{p,s,n+1} = pF_{p,s,n} + sF_{p,s,n-1}, \ \ n \geqslant 1$$

with initial conditions

$$F_{p,s,0} = 0, \quad F_{p,s,1} = 1.$$

REMARK 10. Since for $\mu = 1$, the lacunary polynomial $P \in \mathbb{P}_{n,\ \mu}$ reduces to a simple polynomial of degree n, Theorem 4 reduces to a result due to Bidkham et al. [2, Theorem 1]. For $\mu = 1,\ p = s = 1$ and $j = 2$, Theorem 4 reduces to Theorem B. If we take $p = 2,\ s = 1$ in Theorem 4, we get the following more general version of Theorem E.

COROLLARY 3. *If* $P \in \mathbb{P}_{n,\ \mu}$*, then for* $j \geqslant 1$*, all the zeros of* P *lie in the annulus* $\mathscr{R} = \{z : r_1 \leqslant |z| \leqslant r_2\}$ *with*

$$r_1 = \min_{\mu \leqslant k \leqslant n} \left\{ \frac{C(n,\ k) P_k (P_{2^j})^k (P_{2^j-1})^{n-k}}{P_{2^j n}} \left| \frac{a_0}{a_k} \right| \right\}^{\frac{1}{k}}$$

and

$$r_2 = \max_{\mu \leqslant k \leqslant n} \left\{ \frac{P_{2^j n}}{C(n,\ k) P_k (P_{2^j})^k (P_{2^j-1})^{n-k}} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{\frac{1}{k}}.$$

REMARK 11. If $\mu = 1$ and $j = 1$, then Corollary 3 reduces to Theorem E which is based on Pell numbers.

3. Lemmas

To prove Theorem 4, we need the following lemma.

LEMMA 1. *For* $j \geqslant 1$*,*

$$\sum_{k=1}^{n} C(n,\ k)(sF_{p,s,2^j-1})^{n-k} (F_{p,s,2^j})^k F_{p,s,k} = F_{p,s,2^j n}$$

holds. This Lemma is a special case of a result due to Diaz-Barrero and Egozcue [7, Theorem 1].

4. Proofs of the Theorems

Proof of Theorem 1. If $a_0 = 0$, then $R_1 = 0$ and $P(z)$ has a zero at origin. Following Cauchy's method, if we assume that $a_0 \neq 0$ and $|z| < R_1$. We shall prove (17) by principle of mathematical induction. Result is true for $\mu = 1$ by Theorem G. Now for $\mu = 2$. Let

$$P(z) = a_0 + \sum_{t=2}^{n} a_t\, z^t .$$

Now, by the application of triangle inequality, we have

$$\begin{aligned}|P(z)| &\geqslant |a_0| - \left|\sum_{t=2}^{n} a_t z^t\right| \\ &\geqslant |a_0| - \sum_{t=2}^{n} |a_t|\; |z|^t \\ &> |a_0| - \sum_{t=2}^{n} |a_t|\; R_1^t \\ &= |a_0| \left(1 - \sum_{t=2}^{n} \left|\frac{a_t}{a_0}\right| R_1^t\right),\end{aligned}$$

i.e.,

$$|P(z)| > |a_0| \left(1 - \sum_{t=2}^{n} \left|\frac{a_t}{a_0}\right| R_1^t\right). \tag{29}$$

Now, from equation (17), we have for $2 \leqslant t \leqslant n$

$$\left|\frac{a_t}{a_0}\right| R_1^t \leqslant A_t, \tag{30}$$

hence using (30) in (29), we get

$$\begin{aligned}|P(z)| &> |a_0| \left(1 - \sum_{t=2}^{n} \left|\frac{a_t}{a_0}\right| R_1^t\right) \\ &> |a_0| \left(1 - \sum_{t=2}^{n} A_t\right) \\ &= |a_0|\,(1 - 1 + A_1) > 0,\end{aligned}$$

as by hypothesis $\sum_{t=1}^{n} A_t = 1$. Thus $P(z)$ does not have any zero in $|z| < R_1$. Therefore, we conclude that all the zeros of $P(z)$ lie in $|z| \geqslant R_1$, and (17) is thus proved for $\mu = 2$. We assume that (17) is true for $\mu = s$, i.e., all the zeros of $P(z)$ lie in $|z| \geqslant R_1$, where

$$R_1 = \min_{s \leqslant t \leqslant n} \left\{ A_t \left|\frac{a_0}{a_t}\right| \right\}^{\frac{1}{t}}$$

and $\sum_{t=1}^{s} A_t = 1 - \sum_{t=s+1}^{n} A_t$.

We will prove that (17) is true for $\mu = s+1$. Let

$$P(z) = a_0 + \sum_{t=s+1}^{n} a_t\, z^t.$$

Then it is easy to verify that

$$|P(z)| \geqslant |a_0| - \left|\sum_{t=s+1}^{n} a_t z^t\right| > |a_0| - \sum_{t=s+1}^{n} |a_t|\, R_1^t = |a_0| \left(1 - \sum_{t=s+1}^{n} \left|\frac{a_t}{a_0}\right| R_1^t\right). \tag{31}$$

Since, by (17) we have for $s+1 \leqslant t \leqslant n$, the inequality

$$\left|\frac{a_t}{a_0}\right| R_1^t \leqslant A_t, \tag{32}$$

hence using (32) in (31), we get

$$\begin{aligned} |P(z)| &> |a_0| \left(1 - \sum_{t=s+1}^{n} A_t\right) \\ &= |a_0| \left(1 - 1 + \sum_{t=1}^{s} A_t\right) \\ &> 0 \end{aligned}$$

Thus $P(z)$ does not have any zero in $|z| < R_1$. Hence (17) is true for $\mu = s+1$.

To prove the bound (18), we consider the polynomial

$$S(z) = z^n P(1/z) = a_n + a_{n-\mu}\, z^{\mu} + a_{n-\mu-1}\, z^{\mu+1} + \ldots + a_{\mu} z^{n-\mu} + a_0 z^n.$$

By the first part of the theorem, all the zeros of the polynomial $S(z)$ lie in

$$\begin{aligned} |z| &\geqslant \min_{\mu \leqslant t \leqslant n} \left\{ A_t \left|\frac{a_n}{a_{n-t}}\right| \right\}^{\frac{1}{t}} \\ &= \min_{\mu \leqslant t \leqslant n} \left\{ \frac{1}{\frac{1}{A_t}\left|\frac{a_{n-t}}{a_n}\right|} \right\}^{\frac{1}{t}} \\ &= \frac{1}{\max_{\mu \leqslant t \leqslant n} \left\{ \frac{1}{A_t}\left|\frac{a_{n-t}}{a_n}\right| \right\}^{\frac{1}{t}}} \\ &= \frac{1}{R_2}. \end{aligned}$$

Replacing z by $\frac{1}{z}$ and noting that $P(z) = z^n S(1/z)$, we conclude that all the zeros of $P(z)$ lie in

$$|z| \leqslant R_2 = \max_{\mu \leqslant t \leqslant n} \left\{ \frac{1}{A_t} \left| \frac{a_{n-t}}{a_n} \right| \right\}^{\frac{1}{t}},$$

which is (18). This proves Theorem 1 completely. □

Proof of Theorem 2. Let $C(n, k)$ denote the binomial coefficients, then Narayana numbers are given by the identity

$$N(n, k) = \frac{1}{n} C(n, k)\, C(n, k-1),$$

and Catalan numbers by the identity $C_n = \frac{C(2n, n)}{n+1}$. Therefore,

$$\begin{aligned} \sum_{k=1}^{n} N(n, k) &= \frac{1}{n} \sum_{k=1}^{n} C(n, k)\, C(n, k-1) \\ &= \frac{1}{n} C(2n, n-1) \\ &= \frac{1}{n} \frac{(2n)!}{(n-1)!\,(2n-(n-1))!} \\ &= \frac{C(2n, n)}{n+1} = C_n. \end{aligned}$$

Thus, if we take $A_k = \frac{N(n, k)}{C_n}$, then $A_k > 0$ for each k and $\sum\limits_{k=1}^{n} A_k = 1$. Hence, applying Theorem 1 for this set of A_k, $(1 \leqslant k \leqslant n)$, we get the desired result. This completes the proof of Theorem 2. □

Proof of Theorem 3. Let M_n be the n^{th} Motzkin number, then we have

$$\sum_{k=1}^{n} M_{k-1}\, M_{n-1-k} = M_n,$$

with $M_0 = M_1 = M_{-1} = 1$. Now, if we take $A_k = \frac{M_{k-1}\, M_{n-1-k}}{M_n}$, then $A_k > 0$ for each k and $\sum\limits_{k=1}^{n} A_k = 1$, and hence applying Theorem 1 for this set of A_k, $(1 \leqslant k \leqslant n)$, we get the desired result and the proof of Theorem 3 is thus complete. □

Proof of Theorem 4. The proof of this theorem follows by applying Lemma 1 and then Theorem 1. □

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