Journal of Classical Analysis
Volume 23, Number 1 (2024), 51–62
doi:10.7153/jca-2024-23-05

STATISTICAL CONVERGENCE AND CESÀRO SUMMABILITY OF DIFFERENCE SEQUENCES RELATIVE TO MODULUS FUNCTION

NAVEEN SHARMA AND SANDEEP KUMAR*

Abstract. In the present paper, we introduce and study the strong Cesàro summability of difference sequence spaces through fusion of modulus function. On the newly established sequence space, linear structure is imposed and a paranorm is established. Apart from various inclusion relations, a new variant of statistical convergence is investigated.

1. Introduction

Kizmaz [23] in 1981, initiated the theory of difference sequence spaces $E(\Delta)$ as follows

$$E(\Delta)=\{(\xi_k)\in s:(\Delta\xi_k)=(\xi_k-\xi_{k+1})\in E\},\ E\in\{\ell_\infty,c,c_0\}$$

where s, c_0, c and ℓ_∞ denotes the spaces of all, null, convergent and bounded scalar sequences.

Since 1981 to till date, a huge amount of research work has been performed by many more mathematicians with reference to various extensions/generalizations of difference sequence spaces.

Bhardwaj and Gupta [5] investigated a new difference sequence space with C_1 as a underlying space in the following ways:

$$\begin{aligned} C_1(\Delta)&=\{(\xi_k)\in s:(\Delta\xi_k)=(\xi_k-\xi_{k+1})\in C_1\}\\ &=\left\{(\xi_k)\in s:\left\langle\frac{1}{k}\sum_{i=1}^{k}\Delta\xi_i\right\rangle\in c\right\}\end{aligned}$$

where C_1 is a space of Cesàro summable sequence of order 1, i.e.,

$$C_1=\left\{(\xi_k)\in s:\lim_{k\to\infty}\frac{\xi_1+\xi_2+\ldots+\xi_k}{k}\ \text{ exists}\right\}.$$

Mathematics subject classification (2020): 46A45, 40A05, 40A35.
Keywords and phrases: Difference sequence space, modulus function, natural density, statistical convergence.

* Corresponding author.

Pictorial inclusions among various well known sequence spaces $\ell_\infty, c, c_0, C_1, \ell_\infty(\Delta)$, $c(\Delta), c_0(\Delta)$ and $C_1(\Delta)$ is shown as:

$$\begin{array}{ccccc} & & C_1 & \subset & C_1(\Delta) \\ & & \cup & & \cup \\ c_0 & \subset & c & \subset & \ell_\infty \\ \cap & & \cap & & \cap \\ c_0(\Delta) & \subset & c(\Delta) & \subset & \ell_\infty(\Delta) \\ & & \cap & & \\ & & C_1(\Delta) & & \end{array}$$

Figure 1.

Thus the space of Cesàro summable difference sequences, i.e., $C_1(\Delta)$ space turned out be much wider space than these spaces.

In order to generalize the notion of usual convergence, the concept of statistical convergence come into existence, the credit of which goes to Fast [15]. The notion of statistical convergence relies upon the natural density $\delta(M)$, $(M \subseteq \mathbb{N})$ and defined as (see, [30])

$$\delta(M) = \lim_{n\to\infty} \frac{1}{n} \text{card}(\{k \in M : k \leqslant n\})$$

provided the limit exists, where card$(\cdot)$ means numbers of elements in the enclosed set. It is observed that

(i) $\delta(M) = 0$ for $M \subseteq \mathbb{N}$ finite.

(ii) $\delta(M) + \delta(\mathbb{N} - M) = 1$ for all $M \subseteq \mathbb{N}$.

A sequence $\langle \xi_k \rangle$ is statistically convergent to ℓ if for every $\varepsilon \geqslant 0$, $\delta(\{k \leqslant n : |\xi_k - \ell| \geqslant \varepsilon\}) = 0$, "i.e., $|\xi_k - \ell| \leqslant \varepsilon$ a.a. k. i.e., $\lim_{n\to\infty} \frac{1}{n} \text{card}(\{k \leqslant n : |\xi_k - \ell| > \varepsilon\}) = 0$." And ℓ is referred as statistical limit of $\langle \xi_k \rangle$. By S we will refer the collection of all statistically convergent sequences.

For more insight into sequence spaces/difference sequence spaces and statistical convergence one may peep into [3–12, 14, 16, 17, 20, 22, 24, 27, 28, 31–33, 35–41].

The study of sequence spaces is considered to be incomplete without computation of duals. The stepping or introduction of dual spaces is due to Köthe and Toeplitz [25] and for a sequence space E, the following

$$E^\alpha = \left\{ \langle a_k \rangle \in s : \sum_{k=1}^{\infty} |a_k \xi_k| < \infty \ \forall \ \xi = \langle \xi_k \rangle \in E \right\}$$

and

$$E^\beta = \left\{ \langle a_k \rangle \in s : \sum_{k=1}^{\infty} a_k \xi_k < \infty \ \forall \ \xi = \langle \xi_k \rangle \in E \right\}$$

are called α- and β-duals spaces of E respectively. Also for $E \subseteq F$, we have $F^\Theta \subseteq E^\Theta$ for $\Theta \in \{\alpha, \beta\}$.

We recall [13, 21, 26], a sequence space E is

(i) perfect if $E^{\alpha\alpha} = E$.

(ii) Solid (normal) if $\langle \eta_k \rangle \in E$ whenever $|\eta_k| \leqslant |\xi_k|$, $k \geqslant 1$, for $\langle \xi_k \rangle \in E$.

(iii) Monotone if it contains the canonical pre-image of all its stepspaces.

(iv) Convergence free if $\langle \xi_k \rangle \in E$ and $\eta_k = 0$ whenever $\xi_k = 0$ implies $\langle \eta_k \rangle \in E$.

Motivating from the definition of absolute value function, i.e., $|a|$; $a \in \mathbb{R}$

$$|a| = \begin{cases} a, & \text{if} \quad a \geqslant 0, \\ -a, & \text{if} \quad a < 0. \end{cases}$$

Nakano [29] in 1953, structured the image of modulus function. By Ruckle [34] and Maddox [26], a modulus function is a map $\phi : [0,\infty) \to [0,\infty)$ such that the following holds

(M_1) $\phi(\xi) = 0$ iff $\xi = 0$.

(M_2) $\phi(\xi + \eta) \leqslant \phi(\xi) + \phi(\eta)$ for all $\xi \geqslant 0,\ \eta \geqslant 0$.

(M_3) ϕ is monotonically increasing.

(M_4) $\lim\limits_{\xi \to 0^+} \phi(\xi) = \phi(0)$.

As an example, $\phi_1(\xi) = \dfrac{\xi}{1+\xi}$ and $\phi_2(\xi) = \xi^p$, $(0 < p \leqslant 1)$ are modulus functions where ϕ_1 is bounded and ϕ_2 is unbounded. It is observed that sum of two modulus functions is again a modulus function. Moreover, composition of a modulus function over itself is also a modulus function.

Aizpuru et al. [1], Altin [2], Connor [12], Ghosh and Srivastva [18], Gupta and Bhardwaj [19], Şengül and Et [36], Verma and Singh [40] and some others have used the idea of modulus function to enrich the theory of statistical convergence and structured some significant sequence spaces.

We here in this paper appeal the approach of statistical convergence to the newly introduced space $C_1(\Delta)$ and have the concept of Cesàro summabilty of difference sequences with the aid of modulus function.

Throughout the paper, let $\lambda = \langle \lambda_k \rangle$ is a bounded sequence of positive real numbers with $\tau = \inf_{k \geqslant 1} \lambda_k$, $\Omega = \sup_{k \geqslant 1} \lambda_k$ and $C = \max\{1, 2^{\Omega - 1}\}$. Also for $a_k, b_k \in \mathbb{C}$, we have $|a_k + b_k|^{\lambda_k} \leqslant C[|a_k|^{\lambda_k} + |b_k|^{\lambda_k}]\ \forall\, k \in \mathbb{N}$, and for any $\mu \in \mathbb{C}$, $|\mu|^{\lambda_k} \leqslant \max\{1, |\mu|^{\Omega}\}$ (see, for instance, Maddox [26]).

2. Statistical convergence of Cesàro means of difference sequences

We begin this section by extending the notion of statistical convergence for Cesàro means of difference sequences of scalars and hence having the concept of $C_1(\Delta)$-statistical convergence. Apart from this, the dual spaces of new originated sequence spaces are computed.

DEFINITION 1. A sequence $\xi = \langle \xi_k \rangle$ is said to be $C_1(\Delta)$-statistically convergent to ℓ if for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \text{card}(\{1 \leqslant k \leqslant n : |\mu_k - \ell| \geqslant \varepsilon\}) = 0$$

where $\mu_k = \frac{1}{k}\sum_{i=1}^{k} \Delta\xi_i$ is a sequence of Cesàro means for difference sequence of $\langle \xi_k \rangle$. In this case we write $\xi_k \xrightarrow{SC_1(\Delta)} \ell$. By $SC_1(\Delta)$ we will notate the class of all $C_1(\Delta)$-statistically convergent sequences.

THEOREM 1. $C_1(\Delta) \subset SC_1(\Delta)$, *inclusion is proper.*

Proof. Let $\langle \xi_k \rangle \in C_1(\Delta)$ with $\mu_k = \frac{1}{k}\sum_{i=1}^{k} \Delta\xi_i \to \ell$, for some $\ell \in \mathbb{C}$. Then for $\varepsilon > 0$, $\{k \in \mathbb{N} : |\mu_k - \ell| \geqslant \varepsilon\}$ is a finite set. As every finite set is of zero natural density, so $\langle \xi_k \rangle \in SC_1(\Delta)$.

For proper inclusion, consider the following example:

Let $\langle \xi_k \rangle = \{0, -1, -2, -3, -16, -5, -6, -7, -8, -81, -10, -11, \ldots\}$, i.e.,

$$\langle \xi_k \rangle = \begin{cases} 0, & \text{if } k = 1 \\ -(k-1)^2, & \text{if } k = n^2 + 1,\ n \geqslant 1 \\ -(k-1), & \text{if } k \neq n^2 + 1,\ n \geqslant 1. \end{cases}$$

Then $\langle \mu_k \rangle = \langle \frac{1}{k}\sum_{i=1}^{k} \Delta\xi_i \rangle = \{1, 1, 1, 4, 1, 1, 1, 1, 9, \ldots\} \notin c$ but $\langle \mu_k \rangle \in S$.

Hence $\langle \xi_k \rangle \notin C_1(\Delta)$ although it is a member of $SC_1(\Delta)$. $\square$

REMARK 1. It is to be noted that not all the sequences are $C_1(\Delta)$-statistically convergent, i.e., $SC_1(\Delta) \subsetneqq s$.

Proof. For this, let $\langle \xi_k \rangle = \langle k^2 \rangle = \{1^2, 2^2, 3^2, \ldots\}$.

Then $\langle \mu_k \rangle = \langle \frac{1}{k}\sum_{i=1}^{k} \Delta\xi_i \rangle = \langle -k-2 \rangle = \langle -3, -4, -5, -6, \ldots \rangle \notin S$, implies that $\langle \xi_k \rangle \notin SC_1(\Delta)$. $\square$

DEFINITION 2. A sequence $\xi = \langle \xi_k \rangle$ is said to be $C_1(\Delta)$-statistically bounded if there exists $M > 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \text{card}(\{1 \leqslant k \leqslant n : |\mu_k| \geqslant M\}) = 0,$$

and by $SC_1(\Delta, b)$, we have the space of all $C_1(\Delta)$-statistically bounded sequences.

THEOREM 2. *Every $C_1(\Delta)$-statistically convergent sequence is $C_1(\Delta)$-statistically bounded, but not conversely, i.e., $SC_1(\Delta) \subsetneqq SC_1(\Delta, b)$.*

Proof. Let $\langle\xi_k\rangle \in SC_1(\Delta)$ with $\xi_k \xrightarrow{SC_1(\Delta)} \ell$. Then for $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n}\text{card}(\{k \leqslant n : |\mu_k - \ell| \geqslant \varepsilon\}) = 0, \text{ where } \mu_k = \frac{1}{k}\sum_{i=1}^{k} \Delta\xi_i.$$

Using the inclusion $\{k : |\mu_k| \geqslant \varepsilon + |\ell|\} \subset \{k : |\mu_k - \ell| \geqslant \varepsilon\}$, the result follows.

For proper inclusion, consider the sequence $\langle\xi_k\rangle = \langle 0, -1, 2, -3, 4, -5, 6, -7, 8, -9, \ldots\rangle$. Then $\langle\mu_k\rangle = \langle \frac{1}{k}\sum_{i=1}^{k}\Delta\xi_i\rangle = \langle(-1)^{k-1}\rangle \notin c$ but $\langle\mu_k\rangle \in \ell_\infty$. Thus $\langle\xi_k\rangle \notin SC_1(\Delta)$ but $\langle\xi_k\rangle \in SC_1(\Delta, b)$. $\square$

In view of Theorem 1 and Theorem 2, the pictorial representation (shown in Figure 1) takes the form as

$$\begin{array}{ccccccc}
 & & C_1 & \subset & C_1(\Delta) & \subset & SC_1(\Delta) \subset SC_1(\Delta,b) \\
 & & \cup & & \cup & & \\
c_0 & \subset & c & \subset & \ell_\infty & & \\
\cap & & \cap & & \cap & & \\
c_0(\Delta) & \subset & c(\Delta) & \subset & \ell_\infty(\Delta) & & \\
 & & \cap & & & & \\
 & & C_1(\Delta) & \subset & SC_1(\Delta) & \subset & SC_1(\Delta,b)
\end{array}$$

Figure 2.

Here from Figure 2, there is no doubt in declaring that $SC_1(\Delta, b)$ are much wider spaces than most of the already existing spaces.

THEOREM 3. $[SC_1(\Delta)]^\beta = [SC_1(\Delta)]^\alpha = \Gamma$, *the space of finitely non-zero scalar sequences.*

Proof. Obviously $\Gamma \subset [SC_1(\Delta)]^\beta$. Let, if possible $[SC_1(\Delta)]^\beta \not\subseteq \Gamma$. Then there exist some $\langle a_k\rangle \in [SC_1(\Delta)]^\beta$ such that $\langle a_k\rangle \notin \Gamma$, i.e., $\langle a_k\rangle$ has infinitely many non-zero terms. Then it is easy to construct an increasing sequence $\langle k_i\rangle$ of natural numbers such that $k_i > i^2$, $i \in \mathbb{N}$, such that $a_{k_i} \neq 0$. Consider a sequence $\langle\xi_k\rangle$ as follows

$$\xi_k = \begin{cases} \dfrac{1}{a_{k_i}}, & \text{if } k = k_i \\ 0, & \text{if } k \neq k_i \end{cases} \qquad k \in \mathbb{N},\ i \in \mathbb{N}.$$

Then

$$\mu_k = \frac{1}{k}\sum_{i=1}^{k}\Delta\xi_i = \begin{cases} \dfrac{-1}{k}\dfrac{1}{a_{k_i}}, & \text{if } k = k_i \\ 0, & \text{if } k \neq k_i \end{cases}$$

and $\lim_{n\to\infty}\frac{1}{n}\text{card}(\{1\leqslant k\leqslant n : |\mu_k - 0| > M\}) \leqslant \frac{\sqrt{n}}{n} = 0$ so $\langle\xi_k\rangle \in SC_1(\Delta)$. But $\sum_i a_{k_i}\xi_{k_i} = \sum_i 1 = \infty$, a contradiction to the fact that $\langle a_k\rangle \in [SC_1(\Delta)]^\beta$. Thus $[SC_1(\Delta)]^\beta = \Gamma$. As $\Gamma \subseteq [SC_1(\Delta)]^\alpha$ and $[SC_1(\Delta)]^\alpha \subset [SC_1(\Delta)]^\beta$, hence $[SC_1(\Delta)]^\alpha = \Gamma$. $\square$

In view of the fact $SC_1(\Delta) \subset SC_1(\Delta, b)$ and $Y^\Theta \subset X^\Theta$ for $(\Theta = \alpha, \beta)$ for $X \subset Y$ we have the following

COROLLARY 1. $[SC_1(\Delta, b)]^\alpha = [SC_1(\Delta, b)]^\beta = \Gamma$.

COROLLARY 2. *$SC_1(\Delta)$ and $SC_1(\Delta, b)$ are not perfect spaces.*

Proof. As $(\Gamma)^\alpha = s$ so we have $[SC_1(\Delta)]^{\alpha\alpha} = (\Gamma)^\alpha = s \neq SC_1(\Delta)$.
Similarly we have $[SC_1(\Delta, b)]^{\alpha\alpha} \neq SC_1(\Delta, b)$. $\square$

THEOREM 4. *$SC_1(\Delta)$ is not normal (solid) space.*

Proof. Let $\langle\xi_k\rangle = \langle k\rangle = \langle 1, 2, 3, \ldots\rangle$ then $\mu_k = \frac{1}{k}\sum_{i=1}^k \Delta\xi_i = -1$ and so $\langle\mu_k\rangle \in S$ and this implies $\langle\xi_k\rangle \in SC_1(\Delta)$. Now if we take $\langle\xi_k^{'}\rangle = \langle(-1)^{k-1}k\rangle = \langle 1, -2, 3, -4, \ldots\rangle$. Then

$$\mu_k^{'} = \frac{1}{k}\sum_{i=1}^k \Delta\xi_i^{'} = \begin{cases} \dfrac{k+2}{k}, & \text{if } k \text{ is odd} \\ -1, & \text{if } k \text{ is even} \end{cases}$$

and so $\langle\mu_k^{'}\rangle \notin S$, i.e., $\langle\xi_k^{'}\rangle \notin SC_1(\Delta)$ although $|\xi_k^{'}| \leqslant |\xi_k|\ \forall\, k$. $\square$

COROLLARY 3. *$SC_1(\Delta)$ is not convergence free space.*

Proof. The result follows from the fact that every convergence free space is normal. $\square$

THEOREM 5. *$SC_1(\Delta)$ is not monotone space.*

Proof. Let $\langle\xi_k\rangle = \langle k\rangle$. Then as in Theorem 4, $\langle\xi_k\rangle \in SC_1(\Delta)$. Now take $\langle\xi_k^{'}\rangle = \langle 1, 0, 3, 0, 5, \ldots\rangle$. Then

$$\mu_k^{'} = \frac{1}{k}\sum_{i=1}^k \Delta\xi_i^{'} = \begin{cases} \frac{1}{k}, & \text{if } k \text{ is odd} \\ -1, & \text{if } k \text{ is even.} \end{cases}$$

As $\langle\mu_k^{'}\rangle \notin S$ so $\langle\xi_k^{'}\rangle \notin SC_1(\Delta)$. $\square$

3. Cesàro summability of difference sequences via modulus function

It is well known that for a bounded scalar sequences, both the concepts, i.e., statistical convergence and strongly Cesàro summability coincide. In the present section, we introduce and study the concept of (ϕ,λ)-Cesàro summability of difference sequences where ϕ is a modulus function and have the following sequence space

$$C_1(\Delta,\phi,\lambda)=\left\{(\xi_k)\in s:\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[\phi(|\mu_k-\ell|)]^{\lambda_k}=0\right.$$
$$\left.\text{for some } \ell, \text{ where } \mu_k=\frac{1}{k}\sum_{i=1}^{k}\Delta\xi_i\right\}.$$

Here we investigate that for bounded modulus function ϕ, again both the concepts, i.e., $C_1(\Delta)$-statistical convergence (introduced in section-2) and (ϕ,λ)-Cesàro summability coincide.

THEOREM 6. *$C_1(\Delta,\phi,\lambda)$ has linear structure when equipped with operation of coordinate wise addition and scalar multiplication over complex field.*

Proof. For $\langle\xi_k\rangle$, $\langle\xi_k'\rangle\in C_1(\Delta,\phi,\lambda)$, there exists ℓ, $\ell'\in\mathbb{C}$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[\phi(|\mu_k-\ell|)]^{\lambda_k}=0=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[\phi(|\mu_k'-\ell'|)]^{\lambda_k} \tag{1}$$

where $\mu_k=\frac{1}{k}\sum_{i=1}^{k}\Delta\xi_i$ and $\mu_k'=\frac{1}{k}\sum_{i=1}^{k}\Delta\xi_i'$.

Now for $a,b\in\mathbb{C}$

$$\begin{aligned}\left[\phi\left(\left|\frac{1}{k}\sum_{i=1}^{k}\Delta(a\xi_i+b\xi_i')-(a.\ell+b.\ell')\right|\right)\right]^{\lambda_k}&=\left[\phi\left(\left|a(\mu_k-\ell)+b(\mu_k'-\ell')\right|\right)\right]^{\lambda_k}\\&\leqslant C[\phi(|a||\mu_k-\ell|)]^{\lambda_k}\\&\quad+C[\phi(|b||\mu_k'-\ell'|)]^{\lambda_k}\\&\leqslant([|a|]+1)^{\Omega}C(\phi(|\mu_k-\ell|))^{\lambda_k}\\&\quad+([|b|]+1)^{\Omega}C(\phi(|\mu_k'-\ell'|))^{\lambda_k}.\end{aligned}$$

The result follows in view of (1). $\square$

It is observed that $C_1(\Delta,\phi)$ has paranorm structure q, where

$$q(\xi)=q(\langle\xi_k\rangle)=\sup_n\left(\frac{1}{n}\sum_{i=1}^{n}[\phi(|\mu_k|)]^{\lambda_k}\right)^{\frac{1}{M}},\quad\text{where } M=\max\{1,\sup_k\lambda_k\}.$$

THEOREM 7. *For any modulus function ϕ, $C_1(\Delta,\phi,\lambda) \subset SC_1(\Delta)$, inclusion being proper for unbounded modulus function ϕ.*

Proof. Let $\langle \xi_k \rangle \in C_1(\Delta,\phi,\lambda)$ with $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}[\phi(|\mu_k-\ell|)]^{\lambda_k} = 0$ for some $\ell \in \mathbb{C}$. Then

$$\frac{1}{n}\sum_{k=1}^{n}[\phi(|\mu_k-\ell|)]^{\lambda_k} \geqslant \frac{1}{n}\sum_{\substack{k=1\\|\mu_k-\ell|\geqslant\varepsilon}}^{n}[\phi(|\mu_k-\ell|)]^{\lambda_k}$$
$$\geqslant \frac{1}{n}\min\{\phi(\varepsilon)^{\tau},\phi(\varepsilon)^{\Omega}\}.\mathrm{card}(\{1\leqslant k\leqslant n : |\mu_k-\ell|\geqslant\varepsilon\}).$$

This implies $\lim_{n\to\infty}\frac{1}{n}\mathrm{card}(\{1\leqslant k\leqslant n : |\mu_k-\ell|\geqslant\varepsilon\}) = 0$ and hence $\langle \xi_k \rangle \in SC_1(\Delta)$.

For proper inclusion, let ϕ be an unbounded modulus function and $\lambda_k = 1$ for all $k \in \mathbb{N}$. Then $\exists$ a positive integral sequence $\{t_1 < t_2 < t_3 < \ldots\}$ such that $\phi(t_n) = n^3$, $n = 1,2,3,\ldots$.

Consider a sequence $\langle \xi_k \rangle$ for which

$$\mu_k = \frac{1}{k}\sum_{i=1}^{k}\Delta\xi_i = \begin{cases} t_n, & \text{if } k = n^3 \\ 0, & \text{if } k \neq n^3. \end{cases}$$

Here $\mu_k = \{t_1,0,0,0,0,0,0,t_8,0,0,\ldots\}$.

Then $\frac{1}{n}\mathrm{card}(\{1\leqslant k\leqslant n : \phi(|\mu_k-0|)\geqslant\varepsilon\}) \leqslant \frac{n^{\frac{1}{3}}}{n} \to 0$ as $n\to\infty$ and hence $\xi_k \xrightarrow{SC_1(\Delta)} 0$.

Now

$$\frac{1}{n^3}\sum_{k=1}^{n^3}[\phi(|\mu_k-0|)] = \frac{\phi(\mu_1)+\phi(\mu_2)+\ldots+\phi(\mu_{n^3})}{n^3}$$
$$= \frac{\phi(t_1)+\phi(t_2)+\ldots+\phi(t_n)}{n^3}$$
$$= \frac{1^3+2^3+\ldots+n^3}{n^3}$$
$$= \frac{1}{n^3}\frac{n^2(n+1)^2}{4} \longrightarrow \infty \text{ as } n\to\infty.$$

This implies $\langle \frac{1}{n^3}\sum_{k=1}^{n^3}[\phi(|\mu_k-0|)]\rangle$ is not convergent and so $\langle \xi_k \rangle \notin C_1(\Delta,\phi,\lambda)$. □

THEOREM 8. *For a bounded modulus function* ϕ, $SC_1(\Delta) \subset C_1(\Delta, \phi, \lambda)$.

Proof. As ϕ is bounded so $\exists$ positive integer K such that $\phi(x) \leqslant K \ \forall\, x \in [0, \infty)$. Let $\langle \xi_k \rangle \in SC_1(\Delta)$. Now

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}[\phi(|\mu_k-\ell|)]^{\lambda_k} &= \frac{1}{n}\sum_{\substack{1\leqslant k\leqslant n\\ |\mu_k-\ell|\geqslant\varepsilon}}[\phi(|\mu_k-\ell|)]^{\lambda_k}+\frac{1}{n}\sum_{\substack{1\leqslant k\leqslant n\\ |\mu_k-\ell|<\varepsilon}}[\phi(|\mu_k-\ell|)]^{\lambda_k}\\
&\leqslant \frac{1}{n}\sum_{\substack{1\leqslant k\leqslant n\\ |\mu_k-\ell|\geqslant\varepsilon}}K^{\lambda_k}+\frac{1}{n}\max\{\phi(\varepsilon)^{\tau},\phi(\varepsilon)^{\Omega}\}.n\\
&\leqslant \frac{1}{n}K^{\Omega}\mathrm{card}(\{1\leqslant k\leqslant n:|\mu_k-\ell|\geqslant\varepsilon\})+\max\{\phi(\varepsilon)^{\tau},\phi(\varepsilon)^{\Omega}\}.
\end{aligned}$$

Using the facts $\lim\limits_{n\to\infty}\frac{1}{n}\mathrm{card}(\{1\leqslant k\leqslant n:|\mu_k-\ell|\geqslant\varepsilon\})=0$, $\phi(0)=0$ and continuity of ϕ at 0, we get $\lim\limits_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[\phi(|\mu_k-\ell|)]^{\lambda_k}=0$. This implies $\langle \xi_k \rangle \in C_1(\Delta,\phi,\lambda)$. □

In view of Theorem 7 and Theorem 8, we have

THEOREM 9. $SC_1(\Delta) = C_1(\Delta,\phi,\lambda)$ *iff* ϕ *is bounded modulus function.*

THEOREM 10. *Let* ϕ_1, ϕ_2 *are modulus functions. Then*

(i) $C_1(\Delta,\phi_1,\lambda) \subseteq C_1(\Delta,\phi_2\circ\phi_1,\lambda)$.

(ii) $C_1(\Delta,\phi_1,\lambda)\cap C_1(\Delta,\phi_2,\lambda) \subseteq C_1(\Delta,\phi_1+\phi_2,\lambda)$.

Proof.

(i) Let $\langle \xi_k \rangle \in C_1(\Delta,\phi_1,\lambda)$ with $\lim\limits_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[\phi_1(|\mu_k-\ell|)]^{\lambda_k}=0$, where $\mu_k=\frac{1}{k}\sum_{i=1}^{k}\Delta\xi_i$, for some $\ell\in\mathbb{C}$. As ϕ_2 is continuous at 0, so for given $\varepsilon>0$ we can choose $0<\delta<1$ such that $\phi_2(x)<\varepsilon$ for all $0\leqslant x\leqslant\delta$. Put $t_k=\phi_1(|\mu_k-\ell|)$. Now

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}\left[\phi_2\left(\phi_1\left|\frac{1}{k}\sum_{i=1}^{k}\Delta\xi_i-\ell\right|\right)\right]^{\lambda_k} &= \frac{1}{n}\sum_{1\leqslant k\leqslant n}[\phi_2(\phi_1|\mu_k-\ell|)]^{\lambda_k}\\
&= \frac{1}{n}\sum_{1\leqslant k\leqslant n}[\phi_2(t_k)]^{\lambda_k}\\
&= \frac{1}{n}\sum_{\substack{1\leqslant k\leqslant n\\ t_k\leqslant\delta}}[\phi_2(t_k)]^{\lambda_k}+\frac{1}{n}\sum_{\substack{1\leqslant k\leqslant n\\ t_k>\delta}}[\phi_2(t_k)]^{\lambda_k}\\
&= \frac{1}{n}\sum_{\substack{1\leqslant k\leqslant n\\ t_k\leqslant\delta}}\varepsilon^{\lambda_k}+\frac{1}{n}\sum_{\substack{1\leqslant k\leqslant n\\ t_k>\delta}}[\phi_2(t_k)]^{\lambda_k}\\
&< \max\{\varepsilon^{\tau},\varepsilon^{\Omega}\}+\frac{1}{n}\sum_{\substack{1\leqslant k\leqslant n\\ t_k>\delta}}[\phi_2(t_k)]^{\lambda_k}.
\end{aligned}$$

Now for $t_k > \delta$, we use the fact $t_k < \frac{t_k}{\delta} < 1 + \left[\frac{t_k}{\delta}\right]$. As ϕ_2 is increasing function so

$$\begin{aligned}
\phi_2(t_k) &\leqslant \phi_2\left(1 + \left[\frac{t_k}{\delta}\right]\right) \\
&\leqslant \left(1 + \left[\frac{t_k}{\delta}\right]\right)\phi_2(1) \\
&\leqslant 2\phi_2(1)\frac{t_k}{\delta}.
\end{aligned}$$

This implies

$$\begin{aligned}
\frac{1}{n}\sum_{\substack{1\leqslant k\leqslant n\\ t_k>\delta}} [\phi_2(t_k)]^{\lambda_k} &\leqslant \frac{1}{n}\sum_{\substack{1\leqslant k\leqslant n\\ t_k>\delta}} \left(\frac{2\phi_2(1)}{\delta}\right)^{\lambda_k} {t_k}^{\lambda_k} \\
&\leqslant \max\left\{1, \left(\frac{2\phi_2(1)}{\delta}\right)^{\Omega}\right\}.\frac{1}{n}\sum_{\substack{1\leqslant k\leqslant n\\ t_k>\delta}} |t_k|^{\lambda_k} \\
&= \max\left\{1, \left(\frac{2\phi_2(1)}{\delta}\right)^{\Omega}\right\}.\frac{1}{n}\sum_{\substack{1\leqslant k\leqslant n\\ t_k>\delta}} [\phi_1(|\mu_k - \ell|)]^{\lambda_k} \\
&\leqslant \max\left\{1, \left(\frac{2\phi_2(1)}{\delta}\right)^{\Omega}\right\}.\frac{1}{n}\sum_{1\leqslant k\leqslant n} [\phi_1(|\mu_k - \ell|)]^{\lambda_k} \\
&\longrightarrow 0 \quad \text{as} \quad n \to \infty
\end{aligned}$$

and this proves the result.

(ii) Let $\langle \xi_k \rangle \in C_1(\Delta, \phi_1, \lambda) \cap C_1(\Delta, \phi_2, \lambda)$. So $\exists\ \ell \in \mathbb{C}$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[\phi_1(|\mu_k - \ell|)]^{\lambda_k} = 0 = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}[\phi_2(|\mu_k - \ell|)]^{\lambda_k}.$$

Now the result follows in view of the inequality

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}\left[(\phi_1+\phi_2)\left(\left|\frac{1}{k}\sum_{i=1}^{k}\Delta\xi_i - \ell\right|\right)\right]^{\lambda_k} &= \frac{1}{n}\sum_{k=1}^{n}[(\phi_1+\phi_2)|\mu_k - \ell|]^{\lambda_k} \\
&= \frac{1}{n}\sum_{k=1}^{n}[\phi_1|\mu_k - \ell| + \phi_2|\mu_k - \ell|]^{\lambda_k} \\
&\leqslant C.\frac{1}{n}\sum_{k=1}^{n}[\phi_1|\mu_k - \ell|]^{\lambda_k} \\
&\quad + C.\frac{1}{n}\sum_{k=1}^{n}[\phi_2|\mu_k - \ell|]^{\lambda_k} \\
&\longrightarrow 0 \quad \text{as} \quad n \to \infty
\end{aligned}$$

that is,

$$\frac{1}{n}\sum_{k=1}^{n}\left[(\phi_1+\phi_2)\left(\left|\frac{1}{k}\sum_{i=1}^{k}\Delta\xi_i-\ell\right|\right)\right]^{\lambda_k}=0.$$

This implies $\langle\xi_k\rangle\in C_1(\Delta,\phi_1+\phi_2,\lambda)$. □

Acknowledgement. The authors thank to the referee for valuable comments and fruitful suggestions which enhanced the readability of the paper.

REFERENCES

[1] A. AIZPURU, M. C. LISTAN-GARCIA AND F. RAMBLA-BARRENO, *Density by moduli and statistical convergence*, Quaest. Math., **37**, 4 (2014), 525–530.
[2] Y. ALTIN, *Properties of some sets of sequences defined by a modulus function*, Acta Math. Sci., **29**, 2 (2009), 427–434.
[3] M. ARSLAN AND E. DÜNDAR, *Rough statistical convergence in 2-normed spaces*, Honam Math. J., **43**, 3 (2021), 417–431.
[4] Ç. A. BEKTAŞ, M. ET AND R. ÇOLAK, *Generalized difference sequence spaces and their dual spaces*, J. Math. Anal. Appl., **292**, 2 (2004), 423–432.
[5] V. K. BHARDWAJ AND S. GUPTA, *Cesàro summable difference sequence space*, J. Inequal. Appl., 1 (2013), 1–9.
[6] V. K. BHARDWAJ AND I. BALA, *On lacunary generalized difference sequence spaces defined by orlicz functions in a seminormed space and Δ_q^m-lacunary statistical convergence*, Demonstr. Math., **41**, 2 (2008), 415–424.
[7] V. K. BHARDWAJ AND N. SINGH, *Some sequence spaces defined by Orlicz functions*, Demonstr. Math., **33**, 3 (2000), 571–582.
[8] R. C. BUCK, *Generalized asymptotic density*, Amer. J. Math., **75**, 2 (1953), 335–346.
[9] M. BURGIN AND O. DUMAN, *Statistical convergence and convergence in statistics*, arXiv preprint Math., (2006) /0612179.
[10] R. ÇOLAK, *On some generalized sequence spaces*, Commun. Fac. Sci. Uni. Ank. Ser. A1. Math. Stat., **38**, (1989), 35–46.
[11] J. CONNOR, *The statistical and strong p-Cesàro convergence of sequences*, Analysis, **8**, (1–2) (1988), 47–64.
[12] J. CONNOR, *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull., **32**, 2 (1989), 194–198.
[13] R. G. COOKE, *Infinite matrices and sequence spaces*, Macmillan, London, (1950).
[14] M. ET, V. K. BHARDWAJ AND S. GUPTA, *On deferred statistical boundedness of order α*, Comm. Statist. Theory Methods, **51**, 24 (2022), 8786–8798.
[15] H. FAST, *Sur la convergence statistique*, Colloq. Math., **2**, (3–4) (1951), 241–244.
[16] A. R. FREEDMAN, J. J. SEMBER AND M. RAPHAEL, *Some Cesàro-type summability spaces*, Proc. Lond. Math. Soc., **37**, 3 (1978), 508–520.
[17] J. A. FRIDY, *On statistical convergence*, Analysis, **5**, 4 (1985), 301–314.
[18] D. GHOSH AND P. D. SRIVASTAVA, *On some vector valued sequence spaces defined using a modulus function*, Indian J. Pure Appl. Math., **30**, 8 (1999), 819–826.
[19] S. GUPTA AND V. K. BHARDWAJ, *On deferred f-statistical convergence*, Kyungpook Math. J., **58**, (2018), 91–103.
[20] M. IŞIK, *On statistical convergence of generalized difference sequences*, Soochow J. Math., **30**, 2 (2004), 197–206.
[21] P. K. KAMTHAN AND M. GUPTA, *Sequence Spaces and Series*, Marcel Deker. Inc., New York and Basel (1981).
[22] G. KARABACAK AND A. OR, *Rough Statistical Convergence for Generalized Difference Sequences*, Electron. J. Math. Anal. Appl., **11**, 1 (2023), 222–230
[23] H. KIZMAZ, *On certain sequence spaces*, Canad. Math. Bull., **24**, 2 (1981), 169–176.

[24] E. KOLK, *The statistical convergence in Banach spaces*, Acta Comment. Univ. Tartu. Math., **928**, (1991), 41–52.
[25] G. KÖTHE AND O. TOEPLITZ, *Lineare Räumemitunendlichvielen Koordinaten und Ringeunendlicher Matrizen*, J. Reine Angew. Math., **171**, (1934), 193–226.
[26] I. J. MADDOX, *Elements of functional analysis*, Camb. Univ. Press, (1970).
[27] I. J. MADDOX, *Spaces of strongly summable sequences*, Q. J. Math., **18**, 1 (1967), 345–355.
[28] M. MURSALEEN, *λ-statistical convergence*, Math. Slovaca, **50**, 1 (2000), 111–115.
[29] H. NAKANO, *Concave modulars*, J. Math. Soc. Japan, **5**, 1 (1953), 29–49.
[30] I. NIVEN AND H. S. ZUCKERMAN, *An Introduction to Theory of Numbers*, Fourth Ed., New York John Willey and Sons, 1980.
[31] F. NURAY, E. DÜNDAR, AND U. ULUSU, *Deffered strongly Cesàro summable and statistically convergent functions*, Honam Math. J., **44**, 4 (2022), 560–571.
[32] E. ÖZTÜRK AND T. BILGIN, *Strongly summable sequence spaces defined by a modulus*, Indian J. Pure Appl. Math., **25**, 6 (1994), 621–625.
[33] D. RATH AND B. C. TRIPATHY, *On statistically convergent and statistically Cauchy sequences*, Indian J. Pure Appl. Math., **25**, (1994), 381–381.
[34] W. H. RUCKLE, *Sequence Spaces*, Pitman Advanced Publishing Program, (1981).
[35] T. ŠALÁT, *On statistically convergent sequences of real numbers*, Math. Slovaca, **30**, 2 (1980), 139–150.
[36] H. ŞENGÜL AND M. ET, *f-lacunary statistical convergence and strong f-lacunary summability of order α*, Filomat, **32**, 13 (2018), 4513–4521.
[37] H. STEINHAUS, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., **2**, 1 (1951), 73–74.
[38] B. C. TRIPATHY AND A. ESI, *On some new type of generalized difference Cesàro sequence spaces*, Soochow J. Math., **31**, 3 (2005), 333–341.
[39] B. C. TRIPATHY AND H. DUTTA, *On some lacunary difference sequence spaces defined by a sequence of Orlicz functions and q-lacunary Δ-statistical convergence*, An. ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat., **20**, 1 (2012), 417–430.
[40] A. K. VERMA AND L. K. SINGH, *(Δ_v^m, f)-lacunary statistical convergence of order α*, Proyecciones, **41**, 4 (2022), 791–804.
[41] A. ZYGMUND, *Trigonometric Series*, Cambridge Univ. Press, UK (1979).

(Received January 25, 2024)

Naveen Sharma
D.A.V. College Muzaffarnagar
Chaudhary Charan Singh University Meerut
India
e-mail: ns2000dav@gmail.com

Sandeep Kumar
D.A.V. College Muzaffarnagar
Chaudhary Charan Singh University Meerut
India
e-mail: sandeepkharb1989@gmail.com