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ON THE ZEROS OF QUATERNIONIC POLYNOMIAL WITH RESTRICTED COEFFICIENTS

BILAL DAR* AND ABDUL LIMAN

Abstract. Location of the zeros of regular polynomial of a quaternionic variable with quaternionic coefficients is addressed in this study. We derive new bounds for the zeros of such polynomials by virtue of the structure of the zero sets in the newly developed theory of polynomials of a quaternionic variable. We will generalize some recently proven results concerning the distribution of zeros of a quaternionic polynomial with restricted coefficients.

1. Introduction

In geometric function theory, a key focus is on finding the zeros of a polynomial in the plane, using various methods and techniques. This study has profoundly impacted the development of mathematics and its practical applications, including physical systems. It has also inspired a considerable amount of further research, both theoretically and practically.

The need to estimate the zeros of a polynomial arises frequently in various applications. However, deriving bounds on the norms of zeros for a general algebraic polynomial is quite complex. To obtain more precise estimates, it is helpful to impose restrictions on the coefficients of the polynomial.

One classical result in the distribution of zeros of complex polynomials is the Eneström–Kakeya theorem. This theorem plays a crucial role in geometric function theory and is particularly important in the study of numerical methods for solving differential equations.

THEOREM A. (Eneström–Kakeya) *If* $p(z)=\sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *such that* $0<a_0 \leqslant a_1 \leqslant \ldots \leqslant a_n$*, then all the zeros of* p *lie in* $|z| \leqslant 1$.

Several extensions of the Eneström-Kakeya theorem can be found in the literature (see references [10], [11]). A comprehensive survey of the Eneström-Kakeya theorem and its various generalizations is provided in the thorough books by Marden [13] and Milovanović et al. [19].

In 1967, Joyal, Labelle, and Rahman [11] extended Theorem A to include polynomials with coefficients that are monotonic but not necessarily non-negative, as stated in the following result.

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* Corresponding author.

THEOREM B. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *such that* $a_0 \leqslant a_1 \leqslant \ldots \leqslant a_n$*, then all the zeros of* p *lie in* $|z| \leqslant \frac{1}{|a_n|}(|a_0| + a_n - a_0)$.

Of course, when $a_0 \geqslant 0$, then Theorem B reduces to Theorem A.

Since the latter half of the 19th century, researchers have delved deeply into estimating the locations of zeros of algebraic polynomials with specific coefficients. This endeavor has yielded substantial breakthroughs, notably exemplified by the Eneström–Kakeya theorem and its diverse extensions. Given the complexity of the complex domain, a natural query arises regarding the potential results in the quaternionic domain. This paper seeks to extend certain classical Eneström–Kakeya results to the realm of quaternions.

2. Preliminary and background

The historical discovery of quaternions by Sir Rowan William Hamilton is fascinating. Indeed, Hamilton's quest for a three-dimensional number system led to the development of quaternions, a four-dimensional number system, on October 16, 1843. This number system is the quaternions which we denote as $\mathbb{H}$ in honour of Hamilton. We shall use the standard notation $\mathbb{H} = \{\alpha + \beta i + \gamma j + \delta k | \alpha, \beta, \gamma, \delta \in \mathbb{R}\}$ where satisfy $i^2 = j^2 = k^2 = ijk = -1$. The quaternions are the standard example of non-commutative division ring and also forms a four dimensional vector space over $\mathbb{R}$ with $\{1, i, j, k\}$ as a basis.

For $q = \alpha + \beta i + \gamma j + \delta k \in \mathbb{H}$, the real part of q is α and β, γ, δ are the imaginary parts. The conjugate is $q^* = \alpha - \beta i - \gamma j - \delta k$ and modulus is $|q| = \sqrt{qq^*} = \sqrt{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}$. The modulus is then a norm on $\mathbb{H}$. For $r > 0$, we define the ball $B(0, r) = \{q \in \mathbb{H} \mid |q| < r\}$.

We define the indeterminate for a quaternionic polynomial as q. Without commutatively we are left with the polynomial aq^n and the polynomial $a_0 q a_1 q \cdots q a_n$, where $a = a_0 a_1 \cdots a_n$, as different. To alleviate this problem, we adopt the standard that polynomials have indeterminate on the left and coefficients on the right so that we have the quaternionic polynomial $p_1(q) = \sum\limits_{l=0}^{m} q^l a_l$. For such a p_1 and $p_2(q) = \sum\limits_{l=0}^{n} q^l b_l$, the regular product of p_1 and p_2 is defined by $\big(p_1 * p_2\big)(q) = \sum\limits_{i,j=0}^{n,m} q^{i+j} a_i b_j$. It is interesting to note that the above notation convention for quaternionic polynomials aligns with the definition of the regular product for power series of a quaternionic variable, as outlined in Definition 3.1 of the reference [6]. The $*$ multiplication coinciding with the usual point-wise multiplication when the polynomial has real coefficients is a notable characteristic. In particular, the absence of commutativity in the $*$ multiplication leads to behaviour distinct from the real or complex case. The Factor Theorem, which is a fundamental result in commutative algebra, states that if a is a zero of a polynomial $p(z)$, then $(z - a)$ is a divisor of $p(z)$. However, this theorem does not hold in a non-commutative ring (see Theorem III. 6.6 of [8]). In the context of quaternionic polynomials and non-commutative multiplication, the behaviour of zeros and divisors is different from that in commutative rings. In the Quaternion case, the second degree polynomial q^2+1 has an infinite number of zeros namely $q_0 = i\ or j\ ork$ and all those given by $w_0 = h^{-1}q_0h, h \in \mathbb{H}$.

The development of a new theory of regularity for functions, particularly focusing on polynomials of a quaternionic variable, represents a significant advancement in quaternionic analysis. The recent studies, such as [2] and [6]–[8], indicates an active area of research in understanding the properties of quaternionic functions and polynomials. In particular, the discreteness of zero sets as a fundamental property of holomorphic functions of a complex variable. The preliminary steps involving the structure of the zero sets of quaternionic regular functions and the factorization property of zeros provide foundational insights into the behaviour of these functions. The work by Gentili and Stoppato [6], where they establish a necessary and sufficient condition for a quaternionic regular function to have a zero at a point in terms of the coefficients of the power series expansion, represents a significant contribution to understanding the relationship between coefficients and zeros in quaternionic functions. In 2020, Carney et al. [2] proved the following extension of Theorem A for the quaternionic polynomial $p(q)$:

THEOREM C. *If $p(q) = \sum_{v=0}^{n} q^v a_v$ is a quaternionic polynomial of degree n with real coefficients satisfying $0 < a_0 \leqslant a_1 \leqslant \ldots \leqslant a_n$, then all the zeros of p lie in $|q| \leqslant 1$.*

In the same paper, they proved the following result which replaces the condition of monotonicity on the real coefficients by monotonicity in the real and imaginary parts of the quaternion coefficients:

THEOREM D. *If $p(q) = \sum_{v=0}^{n} q^l a_l$ is a quaternionic polynomial of degree n where $a_l = \alpha_l + \beta_l i + \gamma_l j + \delta_l k \in \mathbb{H}$; $0 \leqslant l \leqslant n$ and*

$$\alpha_n \geqslant \alpha_{n-1} \geqslant \cdots \geqslant \alpha_0; \quad \beta_n \geqslant \beta_{n-1} \geqslant \cdots \geqslant \beta_0$$
$$\gamma_n \geqslant \gamma_{n-1} \geqslant \cdots \geqslant \gamma_0; \quad \delta_n \geqslant \delta_{n-1} \geqslant \cdots \geqslant \delta_0;$$

then all the zeros of p lie in

$$|q| \leqslant \frac{\left(|\alpha_0| - \alpha_0 + a_n\right) + \left(|\beta_0| - \beta_0 + \beta_n\right) + \left(|\gamma_0| - \gamma_0 + \gamma_n\right) + \left(|\delta_0| - \delta_0 + \delta_n\right)}{|a_n|}.$$

As a generalization of above results, D. Tripathi [23] recently proved the following result:

THEOREM E. *If $p(q) = \sum_{v=0}^{n} q^l a_l$ is a quaternionic polynomial of degree n where $a_l = \alpha_l + \beta_l i + \gamma_l j + \delta_l k \in \mathbb{H}$; $0 \leqslant l \leqslant n$ and*

$$\alpha_n \geqslant \alpha_{n-1} \geqslant \cdots \geqslant \alpha_l; \quad \beta_n \geqslant \beta_{n-1} \geqslant \cdots \geqslant \beta_l$$
$$\gamma_n \geqslant \gamma_{n-1} \geqslant \cdots \geqslant \gamma_l; \quad \delta_n \geqslant \delta_{n-1} \geqslant \cdots \geqslant \delta_l; \quad 0 \leqslant l \leqslant n,$$

then all the zeros of p lie in

$$|q| \leqslant \frac{|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + \left(\alpha_n - \alpha_l\right) + \left(\beta_n - \beta_l\right) + \left(\gamma_n - \gamma_l\right) + \left(\delta_n - \delta_l\right) + M_l}{|a_n|},$$

where

$$M_l=\sum_{s=1}^{l}\Big[|\alpha_s-\alpha_{s-1}|+|\beta_s-\beta_{s-1}|+|\gamma_s-\gamma_{s-1}|+|\delta_s-\delta_{s-1}|\Big].$$

Recognizing the importance of determining the roots of regular functions, particularly polynomials, in numerical mathematics, it is natural to study the geometric properties of these functions and the distribution of their roots. Recent literature has seen several works focusing on generalizations and refinements of these results. Interested readers can find more details in references [14, 15, 16, 17, 18, 20, 21]. The main goal of this paper is to generalize Theorem D and Theorem E and also produce some results related to the Eneström-Kakeya theorem by making use of recently established result on structure of the zero sets of regular functions (Lemma 1) for a quaternion variable.

3. Main results

In this section, we mention our main result about the location of zeros of quaternionic polynomials with restricted coefficients. More precisely, we prove the following result:

THEOREM 1. *If*

$$p(q)=q^na_n+q^{n-1}a_{n-1}+\cdots+q^ua_u+\cdots+q^va_v+\cdots+qa_1++a_0$$

is a polynomial of degree n *where* $v\leqslant u$ *and* $a_l=\alpha_l+\beta_li+\gamma_lj+\delta_lk\in\mathbb{H};\ 0\leqslant l\leqslant n$ *such that*

$$\begin{aligned}&\alpha_u\geqslant\alpha_{u-1}\geqslant\cdots\geqslant\alpha_v;\quad \beta_u\geqslant\beta_{u-1}\geqslant\cdots\geqslant\beta_v\\ &\gamma_u\geqslant\gamma_{u-1}\geqslant\cdots\geqslant\gamma_v;\quad \delta_u\geqslant\delta_{u-1}\geqslant\cdots\geqslant\delta_v,\end{aligned}\tag{1}$$

then all the zeros of $p(q)$ *lie in*

$$|q|\leqslant\frac{|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|+\big(\alpha_u-\alpha_v\big)+\big(\beta_u-\beta_v\big)+\big(\gamma_u-\gamma_v\big)+\big(\delta_u-\delta_v\big)+M_u+M_v}{|a_n|}.$$

Here

$$M_u=\sum_{s=u+1}^{n}\Big[|\alpha_s-\alpha_{s-1}|+|\beta_s-\beta_{s-1}|+|\gamma_s-\gamma_{s-1}|+|\delta_s-\delta_{s-1}|\Big].$$

and

$$M_v=\sum_{s=1}^{v}\Big[|\alpha_s-\alpha_{s-1}|+|\beta_s-\beta_{s-1}|+|\gamma_s-\gamma_{s-1}|+|\delta_s-\delta_{s-1}|\Big].$$

For $u=n$ and $v=0$ Theorem 1 reduces to Theorem D. Also for $u=n$, the number $M_u=0$ therefore Theorem 1 reduces to Theorem E. If we take $\beta_l=\gamma_l=\delta_l=0$, $l=0,1,2,\cdots,n$ so that $a_l=\alpha_l\in\mathbb{R}$, we get for $u=n$ and $v=0$ in Theorem 1 the following quaternionic analogue of Theorem B:

COROLLARY 1. *If* $p(q) = \sum_{l=0}^{n} q^l a_l$ *is a polynomial of degree* n*, where* q *is quaternionic variable, with real coefficients satisfying*

$$a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_o$$

then all the zeros of $p(q)$ *lie in*

$$|q| \leqslant \frac{|a_0| + a_n - a_0}{|a_n|}.$$

For $l = 0, 1, 2, \cdots, n$, if we take $\beta_l = \gamma_l = \delta_l = 0$ in Theorem 1 so that $a_l = \alpha_l \in \mathbb{R}$, we obtain the following result:

COROLLARY 2. *If*

$$p(q) = q^n a_n + q^{n-1} a_{n-1} + \cdots + q^u a_u + \cdots + q^v a_v + \cdots + q a_1 + +a_0,$$

where $v \leqslant u$*, is a polynomial of degree* n *with real coefficients satisfying*

$$a_u \geqslant a_{u-1} \geqslant \cdots \geqslant a_v$$

then all the zeros of $p(q)$ *lie in*

$$|q| \leqslant \frac{|a_0| + \left(a_u - a_v\right) + M'_u + M'_v}{|a_n|},$$

where

$$M'_u = \sum_{s=u+1}^{n} |a_s - a_{s-1}| \quad \textit{and} \quad M'_v = \sum_{s=1}^{v} |a_s - a_{s-1}|.$$

Taking $u = n$ in Corollary 2, we get a recently proved result [Corollary 3.2 of [23]].

For $u = v = n$, Theorem 1 gives the following result which gives a new bound for the zeros of quaternionic polynomials without any restriction on the coefficients:

COROLLARY 3. *All the zeros of polynomial* $p(q) = \sum_{l=0}^{n} q^l a_l$ *where* q *is quaternionic variable, with quaternionic coefficients* $a_l = \alpha_l + \beta_l i + \gamma_l j + \delta_l k \in \mathbb{H}$ *lie in*

$$|q| \leqslant \frac{|\alpha_0| + |\beta_0| + |\gamma_0| + |\delta_0| + \sum_{s=1}^{n} \left(|\alpha_l - \alpha_{l-1}| + |\beta_l - \beta_{l-1}| + +|\gamma_l - \gamma_{l-1}| + |\delta_l - \delta_{l-1}|\right)}{|a_n|}.$$

If we take all coefficients a_l*;* $l = 0, 1, 2, \cdots, n$ *real satisfying* $a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_0 > 0$ *then Corollary 3 reduces to Theorem C.*

4. Lemmas

In order to prove the Theorem 1, we need the following lemma due to Gentili et al [7]:

LEMMA 1. *Let f and g be given quaternionic power series with radii of convergence greater than R and let $q_0 \in B(0, R)$. Then $\big(f * g\big)(q_0) = 0$ if and only if $f(q_0) = 0$ or $f(q_0) \neq 0$ implies $g\big(f(q_0)^{-1} q_0 f(q_0)\big) = 0$.*

5. Proofs of Theorems

Proof of Theorem 1. For $v \leqslant u$, given quaternionic polynomial is

$$p(q) = q^n a_n + q^{n-1} a_{n-1} + \cdots + q^u a_u + \cdots + q^v a_v + \cdots + q a_1 + a_0.$$

We have

$$\begin{aligned}
\big|p(q) * (1-q)\big| =& \Big| -q^{n+1} a_n + q^n (a_n - a_{n-1}) + q^{n-1}(a_{n-1} - a_{n-2}) + \cdots + q^u (a_u - a_{u-1}) \\
& + q^{u-1}(a_{u-1} - a_{u-2}) + \cdots + q^v (a_v - a_{v-1}) + \cdots + q(a_1 - a_0) + a_0 \Big| \\
\geqslant & |q|^{n+1}|a_n| - \Big(|q|^n |a_n - a_{n-1}| + |q|^{n-1}|a_{n-1} - a_{n-2}| + \cdots \\
& + |q|^u |a_u - a_{u-1}| + \cdots + |q|^v |a_v - a_{v-1}| + \cdots + |q||a_1 - a_0| + |a_0| \Big) \\
\geqslant & |q|^n \bigg\{ |q||a_n| - \bigg[|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|q|} + \cdots \\
& + \frac{|a_u - a_{u-1}|}{|q|^{n-u}} + \cdots + \frac{|a_v - a_{v-1}|}{|q|^{n-v}|} + \cdots + \frac{|a_1 - a_0|}{|q|^{n-1}} + \frac{|a_0|}{|q|^n} \bigg] \bigg\}.
\end{aligned}$$

Let $|q| > 1$ so that $\frac{1}{|q|^{n-t}} < 1,\ t = 0, 1, 2, \cdots, n$, therefore we obtain from above

$$\begin{aligned}
\big|p(q) * (1-q)\big| > |q|^n \bigg\{ & |q||a_n| - \Big[|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots \\
& + |a_u - a_{u-1}| + \cdots + |a_v - a_{v-1}| + \cdots + |a_1 - a_0| + |a_0| \Big] \bigg\}. \qquad (2)
\end{aligned}$$

Noting that for all $s = 1, 2, \cdots, n$

$$\begin{aligned}
|a_s - a_{s-1}| &= |(\alpha_s - \alpha_{s-1}) + i(\beta_s - \beta_{s-1}) + j(\gamma_s - \gamma_{s-1}) + k(\delta_s - \delta_{s-1})| \\
&\leqslant |\alpha_s - \alpha_{s-1}| + |\beta_s - \beta_{s-1}| + |\gamma_s - \gamma_{s-1}| + |\delta_s - \delta_{s-1}|,
\end{aligned}$$

we obtain from inequality (2) by using the hypothesis (1) that for $|q|>1$

$$\begin{aligned}
&\left|p(q)*(1-q)\right| \\
&> |q|^n\Big\{|q||a_n| - \Big[\sum_{s=u+1}^{n}\Big(|\alpha_s-\alpha_{s-1}|+|\beta_s-\beta_{s-1}|+|\gamma_s-\gamma_{s-1}|+|\delta_s-\delta_{s-1}|\Big) \\
&\quad +(\alpha_u-\alpha_v)+(\beta_u-\beta_v)+(\gamma_u-\gamma_v)+(\delta_u-\delta_v) \\
&\quad +\sum_{s=1}^{v}\Big(|\alpha_s-\alpha_{s-1}|+|\beta_s-\beta_{s-1}|+|\gamma_s-\gamma_{s-1}|+|\delta_s-\delta_{s-1}|\Big) \\
&\quad +|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|\Big]\Big\}.
\end{aligned}$$

That is for $|q|>1$, we have

$$\begin{aligned}
\left|p(q)*(1-q)\right| > |q|^n\Big\{|q||a_n| - \Big[&M_u+M_v+(\alpha_u-\alpha_v)+(\beta_u-\beta_v) \\
&+(\gamma_u-\gamma_v)+(\delta_u-\delta_v)+|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|\Big]\Big\}.
\end{aligned}$$

Hence, if

$$\begin{aligned}
|q| > \frac{1}{|a_n|}\Big(&M_u+M_v+(\alpha_u-\alpha_v)+(\beta_u-\beta_v)+(\gamma_u-\gamma_v)+(\delta_u-\delta_v) \\
&+|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|\Big),
\end{aligned} \tag{3}$$

then for $|q|>1$, we have $|p(q)*(1-q)|>0$, that is $p(q)*(1-q)\neq 0$.

Since by Lemma 1, $p(q)*(1-q)=0$ if and only if either $p(q)=0$ or $p(q)\neq 0$ implies $1-p(q)^{-1}qp(q)=0$. Notice that $1-p(q)^{-1}qp(q)=0$ is equivalent to $p(q)^{-1}qp(q)=1$ and if $p(q)\neq 0$, this implies that $q=1$. So the only zeros of the product $p(q)*(1-q)$ are zeros of $p(q)$ and $q=1$. Therefore, it follows from (3) that $p(q)\neq 0$ for

$$|q| > \frac{M_u+M_v+(\alpha_u-\alpha_v)+(\beta_u-\beta_v)+(\gamma_u-\gamma_v)+(\delta_u-\delta_v)+|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|}{|a_n|}.$$

In other words, all the zeros of $p(q)$ lie in

$$|q| \leqslant \frac{M_u+M_v+(\alpha_u-\alpha_v)+(\beta_u-\beta_v)+(\gamma_u-\gamma_v)+(\delta_u-\delta_v)+|\alpha_0|+|\beta_0|+|\gamma_0|+|\delta_0|}{|a_n|}.$$

This completes the proof of Theorem 1. $\square$

6. Conclusions

Some fresh findings on Eneström-Kakeya Theorem for quaternionic polynomials with quaternion coefficients has been discovered that not only generalizes already proved results but are also useful in determining the regions containing all the zeros of the polynomial.

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Bilal Dar
Department of Mathematics
National Institute of Srinagar
India-190006
e-mail: darbilal67@gmail.com

Abdul Liman
Department of Mathematics
National Institute of Srinagar
India-190006
e-mail: abliman@rediffmail.com