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ON SUMMING SEQUENCE SPACES

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Abstract. In this paper, keeping in view the idea of difference sequence space $E(\Delta)$ of Kızmaz [20], we availed an opportunity to introduce new kind of summing squence spaces $E(\nabla)$, $E \in \{\ell_\infty, c, c_0\}$ by exploring the sum of two consecutive terms. In addition to this we computed the continuous as well Köthe-Toeplitz duals of these spaces. Like $E(\Delta)$ (the difference sequence spaces of Kızmaz) new sequence spaces $E(\nabla)$ turned out to be much wider than E.

1. Introduction

We denote the set of all sequences with complex terms by ω which is a linear space w.r.t. the coordinate wise addition and scalar multiplication. Any subspace of ω is termed as sequence space. The classical sequence spaces ℓ_∞, c and c_0 denote the spaces of all bounded, convergent and null sequences of complex numbers respectively which are normed spaces w.r.t. norm $\|x\|_\infty = \sup_k |x_k|$. By ℓ_p $(0 < p < \infty)$ we denote the space of absolutely p-summable sequences of scalars, i.e., complex numbers normed by $\|x\|_p = \|(x_k)\|_p = (\sum_k |x_k|^p)^{\frac{1}{p}}$.

We recall some definitions and notations which can be easily found in [3, 11, 17, 21, 22, 26].

A sequence space λ is said to be

(i) Normal (solid) if

$$\overline{\lambda} = \{(y_k) \in \omega : \exists\, (x_k) \in \lambda \text{ s.t. } |y_k| \leqslant |x_k| \text{ for all} k \in \mathbb{N}\} \subseteq \lambda.$$

(ii) Monotone if λ contains the canonical preimage of all its step spaces. For any subsequence J of $\mathbb{N}$ and a sequence space λ,

$$\lambda_J = \left\{x = (x_k) : \exists\, (y_k) \in \lambda \text{ with } x_k = y_{n_k} \text{ for } n_k \in J\right\}$$

is called J-stepspace or the J-sectional subspace of λ. If $x_J \in \lambda_J$, then the canonical pre image of x_J is the sequence $\overline{x_J}$ which agrees with x_J on the indices in J and is zero elsewhere.

(iii) perfect if $\lambda^{\alpha\alpha} = \lambda$.

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(iv) Convergence free if $(x_k) \in \lambda$ and $y_k = 0$ whenever $x_k = 0$ implies $(y_k) \in \lambda$.

A sequence space λ with a linear topology is called a K-space provided each of the projection maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space λ is called an *FK*-space provided λ is a complete linear metric space and an *FK*-space whose topology is normable is called a *BK* space.

Given an *FK*-space λ, we denote the n^{th} section of a sequence $x = (x_k) \in \lambda$ by $x^{[n]} = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ and we say λ has *AK* property if $x^{[n]} \to x$ as $n \to \infty$.

DEFINITION 1. A sequence (x_k) in normed linear space $(X, \|.\|)$ is called a Schauder basis for X iff for each $x \in X$, $\exists$ scalars' sequence (t_k) such that $x = \sum_{k=1}^{\infty} t_k x_k$, that is, $\|x - \sum_{k=1}^{n} t_k x_k\| \to 0$ $(n \to \infty)$. The idea of this basis was introduced by J. Schauder in 1927 and termed as Schauder basis.

It is a fundamental fact that the study of sequence space is generally associated with the computation of its duals which is required in the matrix transformations.

For a sequence space λ,

$$\lambda^{\alpha} = \left\{ (a_k) \in w : \sum_{k=1}^{\infty} |a_k x_k| < \infty \text{ for all } x = (x_k) \in \lambda \right\}$$

and

$$\lambda^{\beta} = \left\{ (a_k) \in w : \sum_{k=1}^{\infty} a_k x_k < \infty \text{ for all } x = (x_k) \in \lambda \right\}$$

are called α-dual and β-dual spaces of λ, referred as Köthe-Toeplitz and generalized Köthe-Toeplitz duals. One can easily observe that for sequence spaces λ, ν with $\lambda \subset \nu$ we have $\nu^{\Theta} \subset \lambda^{\Theta}$, $\Theta \in \{\alpha, \beta\}$.

The continuous dual λ^* of a sequence space λ is defined as the set of all bounded linear functionals on the space λ.

A large amount of research work to enrich the theory of sequence spaces is due to the notion of difference spaces, the credit of introduction of which goes to H. Kızmaz [20]. He introduced

$$\ell_\infty(\Delta) = \{x = (x_k) \in w : \Delta x = (\Delta x_k) \in \ell_\infty\}$$
$$c(\Delta) = \{x = (x_k) \in w : \Delta x = (\Delta x_k) \in c\}$$
$$c_0(\Delta) = \{x = (x_k) \in w : \Delta x = (\Delta x_k) \in c_0\}$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$. In other words

$$\lambda(\Delta) = \{x = (x_k) \in w : \Delta x = (\Delta x_k) = (x_k - x_{k+1}) \in \lambda\} \text{ for } \lambda \in \{\ell_\infty, c, c_0\}.$$

It was shown that $\lambda(\Delta)$ are *BK*-spaces w.r.t. norm $\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty$ for $x = (x_k) \in \lambda(\Delta)$.

Following Kızmaz [20], various mathematicians mainly Altay and Başar [1], Başar and Braha [5], Bhardwaj and Gupta [7], Çolak [10], Et and Esi [13], Gnanaseelan and Srivastva [14], Mursaleen and Baliarsingh [25], Tripathy and Dutta [30] and many more extended this notion of difference sequence spaces to have various extensions/ generalizations. One may refer to [2, 4, 6–9, 12, 16–19, 23, 24, 27–29, 31–33] and much more references can be found therein.

Motivating from the work of Kızmaz, who observed the differences of two successive terms of a sequence, we get an opportunity to observe the behaviour of sequence by adding two successive terms of sequence with division by corresponding positional indices which we demonstrate with an operator symbol ∇, where $\nabla x_k = \dfrac{x_k + x_{k+1}}{k+k+1}$ and introduced the following:

$$\begin{aligned} \ell_\infty(\nabla) &= \{x = (x_k) \in w : \nabla x = (\nabla x_k) \in \ell_\infty\} \\ c(\nabla) &= \{x = (x_k) \in w : \nabla x = (\nabla x_k) \in c\} \\ c_0(\nabla) &= \{x = (x_k) \in w : \nabla x = (\nabla x_k) \in c_0\} \end{aligned}$$

which will be referred as summing bounded, summing convergent and summing null sequence spaces respectively.

2. Main results

THEOREM 1. *$E(\nabla)$ are Banach spaces w.r.t. norm $\|x\|_\nabla = |x_1| + \sup_k \left|\frac{x_k+x_{k+1}}{k+k+1}\right| = |x_1| + \|\nabla x\|_\infty$ for $x = (x_k) \in E(\nabla)$; $E \in \{\ell_\infty, c, c_0\}$.*

Proof. It is a routine verification that $E(\nabla)$ are linear spaces w.r.t. coordinatewise addition and coordinatewise scalar multiplication and these turn out to be normed linear spaces with respect to the norm

$$\|x\|_\nabla = |x_1| + \sup_k \left|\frac{x_k + x_{k+1}}{k+k+1}\right|, \quad x = (x_k) \in E(\nabla).$$

Here we prove that $\ell_\infty(\nabla)$ is a Banach space. Let $\left(x^{(n)}\right)$ be a Cauchy sequence in $\ell_\infty(\nabla)$ where $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots) \in \ell_\infty(\nabla)$, $n \in \mathbb{N}$. Then $\|x^{(n)} - x^{(m)}\|_\nabla \to 0$ as $m, n \to \infty$, i.e.,

$$\left|x_1^{(n)} - x_1^{(m)}\right| + \sup_k \left|\frac{x_k^{(n)} - x_k^{(m)} + x_{k+1}^{(n)} - x_{k+1}^{(m)}}{k+k+1}\right| \to 0 \text{ as } n, m \to \infty.$$

Therefore we have $\left|x_1^{(n)} - x_1^{(m)}\right| \to 0$ and $\left|\dfrac{x_k^{(n)} - x_k^{(m)} + x_{k+1}^{(n)} - x_{k+1}^{(m)}}{k+k+1}\right| \to 0$ as $n, m \to \infty$ for each $k \in \mathbb{N}$, i.e.,

$$\left|x_1^{(n)} - x_1^{(m)}\right| \to 0 \text{ and } \left|\frac{x_k^{(n)} + x_{k+1}^{(n)}}{k+k+1} - \frac{x_k^{(m)} + x_{k+1}^{(m)}}{k+k+1}\right| \to 0 \text{ as } n, m \to \infty \text{ for each } k \in \mathbb{N}. \tag{1}$$

This implies $\left(x_1^{(n)}\right)_{n\in\mathbb{N}}$ and $\left(\dfrac{x_k^{(n)}+x_{k+1}^{(n)}}{k+k+1}\right)_{n\in\mathbb{N}}$ are Cauchy sequence of scalars. Due to completeness of $\mathbb{C}$, $\exists\,\lambda_1$ and $\lambda_{k+1}\in\mathbb{C}$ such that $\lim_{n\to\infty} x_1^{(n)}=\lambda_1$ and $\lim_{n\to\infty}\frac{x_k^{(n)}+x_{k+1}^{(n)}}{k+k+1}=\lambda_{k+1}$ for each $k\in\mathbb{N}$. For $k=1$, $\lim_{n\to\infty}\frac{x_1^{(n)}+x_2^{(n)}}{3}=\lambda_2$ and so we have $\lim_{n\to\infty} x_2^{(n)}=3\lambda_2-\lambda_1$. Similarly for $k=2$, $\lim_{n\to\infty}\frac{x_2^{(n)}+x_3^{(n)}}{5}=\lambda_3$ implies $\lim_{n\to\infty} x_3^{(n)}=5\lambda_3-3\lambda_2+\lambda_1$. Inductively, for each $k\in\mathbb{N}$, $\lim_{n\to\infty} x_k^{(n)}=(2k-1)\lambda_k-(2k-3)\lambda_{k-1}+(2k-5)\lambda_{k-2}-\ldots+(-1)^{k-1}\lambda_1$. Setting $\mu_1=\lambda_1$, $\mu_2=3\lambda_2-\lambda_1$, $\mu_3=5\lambda_3-3\lambda_2+\lambda_1$ and $\mu_k=(2k-1)\lambda_k-(2k-3)\lambda_{k-1}+(2k-5)\lambda_{k-2}-\ldots+(-1)^{k-1}\lambda_1$, for $k\geqslant 2$. Setting $\mu=(\mu_1,\mu_2,\mu_3,\ldots)$. Clearly $\frac{\mu_1+\mu_2}{3}=\lambda_1$, $\frac{\mu_2+\mu_3}{5}=\lambda_3,\ldots,\frac{\mu_k+\mu_{k+1}}{2k+1}=\lambda_{k+1}$, $k\in\mathbb{N}$. Letting $m\to\infty$ in 1, we get $\left|\dfrac{x_k^{(n)}+x_{k+1}^{(n)}}{k+k+1}-\lambda_{k+1}\right|\to 0$ as $n\to\infty$ for each $k\in\mathbb{N}$, i.e.,

$$\left|\frac{x_k^{(n)}+x_{k+1}^{(n)}}{k+k+1}-\frac{\mu_k+\mu_{k+1}}{k+k+1}\right|\to 0 \text{ as } n\to\infty$$

which leads us

$$\sup_{k\geqslant 1}\left|\frac{x_k^{(n)}-\mu_k+x_{k+1}^{(n)}-\mu_{k+1}}{k+k+1}\right|\to 0 \ \text{ as } \ n\to\infty.$$

Thus $\left|x_1^{(n)}-x_1\right|+\sup_{k\geqslant 1}\left|\dfrac{x_k^{(n)}-\mu_k+x_{k+1}^{(n)}-\mu_{k+1}}{k+k+1}\right|\to 0$ as $n\to\infty$, i.e., $\left\|x^{(n)}-x\right\|_{\nabla}\to 0$ as $n\to\infty$ implying $x^{(n)}\to\mu$ as $n\to\infty$. For sufficiently large N as $x^N-\mu\in\ell_\infty(\nabla)$, so $\mu\in\ell_\infty(\nabla)$. This proves that $\ell_\infty(\nabla)$ is a Banach space. □

THEOREM 2. *$E(\nabla)$ are BK spaces, $E\in\{\ell_\infty,c,c_0\}$.*

Proof. Let $\left(x^{(n)}\right)$ be a sequence in $E(\nabla)$ such that $x^{(n)}\to x$ as $n\to\infty$ where $x^{(n)}=\left(x_k^{(n)}\right)_{k\in\mathbb{N}}$ and $x=(x_k)\in E(\nabla)$. As $\left\|x^{(n)}-x\right\|_{\nabla}\to 0$ so we have

$$\left|x_1^{(n)}-x_1\right|+\sup_k\left|\frac{(x_k^{(n)}-x_k)+(x_{k+1}^{(n)}-x_{k+1})}{k+k+1}\right|\to 0 \text{ as } n\to\infty$$

which in turn implies for each $k\in\mathbb{N}$, $\left|x_k^{(n)}-x_k+x_{k+1}^{(n)}-x_{k+1}\right|\to 0$ and $\left|x_1^{(n)}-x_1\right|\to 0$ as $n\to\infty$. Inductively assume $\left|x_k^{(n)}-x_k\right|\to 0$ as $n\to\infty$. The result now follows from the inequality $\left|x_{k+1}^{(n)}-x_{k+1}\right|\leqslant\left|x_{k+1}^{(n)}-x_{k+1}+x_k^{(n)}-x_k\right|+\left|x_k^{(n)}-x_k\right|$. □

THEOREM 3. *$c(\nabla)$ has Schauder basis namely $\{\overline{e}, e_1, e_2, \dots\}$ where $\overline{e} = (1,2,3,\dots)$ and $e_k = (0,0,\dots,1,0,\dots)$ with 1 in k^{th} place and zero elsewhere, $k \in \mathbb{N}$.*

Proof. Let $x = (x_k) \in c(\nabla)$ with $\lim_{k\to\infty} \frac{x_k + x_{k+1}}{k+k+1} = l$. Now

$$\left\| x - l\,\overline{e} - \sum_{k=1}^{n} (x_k - lk)e_k \right\|_{\nabla} = \sup_{k>n} \left| \frac{x_k + x_{k+1}}{k+k+1} - l \right| \to 0 \text{ as } n \to \infty$$

which implies that $x = l\,\overline{e} + \sum_k (x_k - lk)e_k$. □

COROLLARY 1. *$c(\nabla)$ and $c_0(\nabla)$ are separable spaces.*

Proof. The result follows from the fact that if a normed linear space has Schauder basis, then it is separable. □

COROLLARY 2. *$c_0(\nabla)$ has Schauder basis as $\{e_1, e_2, \dots, e_k, \dots\}$.*

THEOREM 4. *The continuous dual of $c(\nabla)$ is ℓ_1.*

Proof. By Theorem 3, $\{\overline{e}, e_1, e_2, \dots\}$ is a Schauder basis for $c(\nabla)$ and every $x = (x_k) \in c(\nabla)$ has a unique representation $x = l\,\overline{e} + \sum_k (x_k - lk)e_k$, where $l = \lim_{k\to\infty} \frac{x_k + x_{k+1}}{k+k+1}$. We define a map $T : c^*(\nabla) \longrightarrow \ell_1$ as follow:

Let $f \in c^*(\nabla)$. Then $f(x) = lf(\overline{e}) + \sum_k (x_k - lk) f(e_k)$ for any $x = (x_k) \in c(\nabla)$ with $\lim_{k\to\infty} \frac{x_k + x_{k+1}}{k+k+1} = l$. Setting

$$x_k = \begin{cases} k\, sgn\, f(e_k) & 3 < k \leqslant r \\ 0 & k > r \text{ or } k = 1 \\ k & \text{if } 1 < k \leqslant 3 \end{cases} \qquad \text{for any } r > 3.$$

Then $x = (x_k) \in c_0 \subseteq c(\nabla)$ and $\|x\|_{\nabla} = 1$. For this particular choice of $x = (x_k)$, we have $f(x) = 0f(\overline{e}) + \sum_k x_k f(e_k) = 2f(e_2) + 3f(e_3) + \sum_{k>3}^{r} k|f(e_k)|$. As f is bounded so $|f(x)| \leqslant \|f\| \|x\|_{\nabla}$ on $c(\nabla)$. From this we get $\left|2f(e_2) + 3f(e_3) + \sum_{k>3}^{r} k|f(e_k)|\right| \leqslant \|f\|$ for any $r > 3$. Since $r > 3$ is arbitrary, so $\sum_k k|f(e_k)| < \infty$. Now rewriting $f(x)$ as we have

$$f(x) = l\left[f(\overline{e}) - \sum_k k f(e_k) \right] + \sum_k x_k f(e_k) = la + \sum_k a_k x_k$$

where $a = f(\overline{e}) - \sum_k f(e_k)k$; $a_k = f(e_k)$ where the sequence (ka_k) is in ℓ_1. We are now in a position to define $T(f) = (a, a_1, 2a_2, 3a_3, \dots)$, where a, a_n appears in the representation of f. It is easy to show T is surjective linear map and $\|T(f)\| = \|f\|$. □

THEOREM 5. *$c(\nabla)$ does not have AK property.*

Proof. Let $x=(x_k)=(k)\in c(\nabla)$. Consider n^{th} section of the sequence $x=(k)$ as $x^{[n]}=(1,2,3,\ldots,n,0,0,0,\ldots)$. Then

$$\begin{aligned}\left\|x-x^{[n]}\right\|_{\nabla} &= \|(0,0,\ldots,n+1,n+2,\ldots)\|_{\nabla}\\ &= \sup_{k>n}\left|\frac{k+k+1}{k+k+1}\right| = 1 \nrightarrow 0 \text{ as } n\to\infty. \quad \square\end{aligned}$$

THEOREM 6. *$c_0(\nabla)$ has AK property.*

Proof. Let $x=(x_k)\in c(\nabla)$ and its n^{th} section is

$$x^{[n]}=(x_1,x_2,x_3,\ldots,x_n,0,0,0,\ldots).$$

Now

$$\begin{aligned}\left\|x-x^{[n]}\right\|_{\nabla} &= \|(0,0,\ldots,x_{n+1},x_{n+2},\ldots)\|_{\nabla}\\ &= \sup_{k>n}\left|\frac{x_k+x_{k+1}}{k+k+1}\right| \to 0 \text{ as } n\to\infty. \quad \square\end{aligned}$$

THEOREM 7. *$c(\nabla)$ is not monotone.*

Proof. For $(x_k)=(k)\in\ell_\infty(\nabla)$, take $y_k=(1,0,0,0,5,0,0,0,9,0,0,0,13,0,0,\ldots)$. Clearly $\nabla y_{4k+1}=\frac{4k+1+0}{(4k+1)+(4k+2)}=\frac{4k+1}{8k+3}\to\frac{1}{2}\neq 0$ and $\nabla y_{4k+2}=\frac{0+0}{(4k+2)+(4k+3)}=0\nrightarrow\frac{1}{2}$ as $k\to\infty$, i.e., subsequences (∇y_{4k+1}) and (∇y_{4k+2}) of (∇y_k) does not converge to same limit, hence $(y_k)\notin c(\nabla)$. $\square$

COROLLARY 3. *$c(\nabla)$ is neither normal nor convergence free space.*

Proof. The proof follows from the fact that for a sequence space X, convergence free $\Rightarrow$ normal $\Rightarrow$ monotone. $\square$

COROLLARY 4. *$c(\nabla)$ is neither perfect.*

Proof. The proof follows from fact that every perfect space is normal. $\square$

THEOREM 8. *$c_0(\nabla)$ is not a monotone space.*

Proof. Let $(x_k)=(1,-2,3,-4,5,-6,7,-8,\ldots)\in c_0(\nabla)$ but $(y_k)=(1,0,0,0,5,0,0,0,9,\ldots)\notin c_0(\nabla)$. $\square$

In view of Corollary 3 and Corollary 4, $c_0(\nabla)$ possess none of the property of normality, convergence free and perfectness.

THEOREM 9. *$\ell_\infty(\nabla)$ is not monotone.*

Proof. Let $(x_k) = (1^2, -2^2, 3^2, -4^2, \dots) \in \ell_\infty(\nabla)$.
Then $(y_k) = (1, 0, 3^2, 0, 5^2, 0, 7^2, \dots) \notin \ell_\infty(\nabla)$. □

THEOREM 10.

(i) $c \subset c(\nabla)$

(ii) $c_0 \subset c_0(\nabla)$

(iii) $\ell_\infty \subset \ell_\infty(\nabla)$

Proof.

(i) Let $(x_k) \in c$ with $\lim_k x_k = l$. As $\lim_{k\to\infty} \dfrac{x_k + x_{k+1}}{k+k+1} = 0$ so $(x_k) \in c(\nabla)$.

(ii) The proof is similar to (i).

(iii) Let $(x_k) \in \ell_\infty$. Then, there exists $M > 0$ such that $|x_k| \leqslant M$ for all $k \geqslant 1$. Now,

$$\left|\frac{x_k + x_{k+1}}{k+k+1}\right| \leqslant \frac{2M}{2k+1} \leqslant M \text{ for all } k \geqslant 1$$

and hence $(x_k) \in \ell_\infty(\nabla)$. □

REMARK 1. Inclusion in (i) and (iii) is proper in view of sequence $(x_k) = (1, 2, 3, \dots)$ and inclusion in (ii) is proper in view of the sequence $(x_k) = \left((-1)^k\right)$.

In view of Theorem 10, we have the following inclusion figure

$$\begin{array}{ccccc} c_0(\nabla) & \subset & c(\nabla) & \subset & \ell_\infty(\nabla) \\ \cup & & \cup & & \cup \\ c_0 & \subset & c & \subset & \ell_\infty \end{array}$$

REMARK 2. By definition of summing sequence spaces $E(\nabla)$, it is clear that $c_0(\nabla) \subset c(\nabla) \subset \ell_\infty(\nabla)$. For the first proper inclusion, we consider the sequence $(x_k) = (k)$ and for the second proper inclusion, we have $(x_k) = \left((-1)^k k^2\right)$.

THEOREM 11. *Let E be a Banach sequence space and F is a closed subspace of E. Then $\nabla(F)$ is closed in $\nabla(E)$.*

Proof. As $F \subseteq E$ so $\nabla(F) \subseteq \nabla(E)$. Let $a = (a_1, a_2, \dots) \in \overline{\nabla(F)}$. Then there exists a sequence $\left(a^{(n)}\right)$ in $\nabla(F)$ such that $\left\|a^{(n)} - a\right\|_\nabla \to 0$ as $n \to \infty$ where $a^{(n)} = \left(a_1^{(n)}, a_2^{(n)}, \dots\right)$ for all $n \in \mathbb{N}$, i.e., $\left|a_1^{(n)} - a_1\right| + \left\|\nabla a^{(n)} - \nabla a\right\|_\infty \to 0$ as $n \to \infty$ and so $\left\|\nabla a^{(n)} - \nabla a\right\|_\infty \to 0$ as $n \to \infty$. As $\left(\nabla a^{(n)}\right)$ is a sequence in F so, $\nabla a \in \overline{F}$. This implies $a \in \nabla(\overline{F})$. Hence $\overline{\nabla(F)} \subset \nabla(\overline{F})$.

Conversely, following similar lines we have $\nabla\overline{F} \subset \overline{\nabla(F)}$. Therefore, $\overline{\nabla(F)} = \nabla(\overline{F})$ and since F is closed, $\nabla(F) = \nabla(\overline{F})$. □

COROLLARY 5. *If E is a separable Banach space, then so is $\nabla(E)$.*

Proof. Let E be a separable Banach space. Then E has a countable dense subset say F, i.e., $\overline{F}=E$ and F is countable. By Theorem 11, $\overline{\nabla(F)}=\nabla(\overline{F})$ and so $\overline{\nabla(F)}=\nabla(E)$. Thus $\nabla(F)$ is dense in $\nabla(E)$. Define a map $\varphi:\nabla(F)\longrightarrow F$ by $\varphi(((x_k))=(\nabla x_k)$ for all $(x_k)\in\nabla(F)$. Then it is clear that φ is bijective. Therefore $\nabla(F)$ is countable as F is countable. Thus $\nabla(F)$ is countable dense subset of $\nabla(E)$. □

COROLLARY 6. *$c_0(\nabla)$ and $c(\nabla)$ are separable spaces.*

Proof. The proof follows in view of Corollary 5. □

THEOREM 12. *Let E be a sequence space. If F is convex subset of E, then $\nabla(F)$ is a convex set in $\nabla(E)$.*

Proof. Let $(x_k),(y_k)\in\nabla(F)$, then $(\nabla x_k),(\nabla y_k)\in F$. Now

$$\alpha\nabla x_k+\beta\nabla y_k=\nabla(\alpha x_k+\beta y_k)\ \text{ for }\ \alpha,\beta\geqslant 0,\ \alpha+\beta=1.$$

As F is convex, $(\alpha\nabla x_k+\beta\nabla y_k)\in F$ and so $(\nabla(\alpha x_k+\beta y_k))$, i.e., $(\alpha x_k+\beta y_k)\in\nabla(F)$ for $\alpha,\beta\geqslant 0$, $\alpha+\beta=1$. □

THEOREM 13. $\ell_\infty\cap\nabla(c)=\ell_\infty\cap\nabla(c_0)$.

Proof. Trivially $\ell_\infty\cap\nabla(c_0)\subseteq\ell_\infty\cap\nabla(c)$. For reverse inclusion, let $x=(x_k)\in\ell_\infty\cap\nabla(c)$. Then $x=(x_k)\in\ell_\infty$ and $\dfrac{x_k+x_{k+1}}{k+k+1}\to l$ for some l as $k\to\infty$. Since (x_k) is a bounded sequence so $\lim\limits_{k\to\infty}\dfrac{x_k+x_{k+1}}{k+k+1}=0$. This implies $l=0$ and $x\in\ell_\infty\cap\nabla(c_0)$. □

The following theorem characterizes the structure of $\ell_\infty(\nabla)$.

THEOREM 14. *$\langle x_k\rangle\in\ell_\infty(\nabla)$ iff*

(i) $\sup_k k^{-2}|x_k|<\infty$

(ii) $\sup_k k^{-2}|kx_{k+1}+(k+1)x_k|<\infty$.

Proof. Let $(x_k)\in\ell_\infty(\nabla)$. Then there exists $M>0$ such that

$$|x_k+x_{k+1}|\leqslant M(2k+1)\ \text{ for all }\ k\in\mathbb{N}.$$

Now

$$\begin{aligned}
&|x_k+x_1|\\
&=\begin{cases}|(x_k+x_{k-1})-(x_{k-1}+x_{k-2})+(x_{k-2}+x_{k-3})+\ldots-(x_2+x_1)+2x_1| & \text{if } k \text{ odd}\\ |(x_k+x_{k-1})-(x_{k-1}+x_{k-2})+(x_{k-2}+x_{k-3})+\ldots+(x_2+x_1)| & \text{if } k \text{ is even}\end{cases}\\
&\leqslant\begin{cases}M[(2k-1)+(2k-3)+(2k-5)+\ldots+3]+2|x_1| & \text{if } k \text{ odd}\\ M[(2k-1)+(2k-3)+(2k-5)+\ldots+3] & \text{if } k \text{ is even,}\end{cases}
\end{aligned}$$

i.e., $|x_k+x_1| \leqslant M[(k-1)(k+1)]+2|x_1|$ for all $k\in\mathbb{N}$. Now for $k\in\mathbb{N}$, $|x_k| \leqslant |x_k+x_1|+|x_1| \leqslant M(k^2-1)+3|x_1|$ which implies $\sup_k k^{-2}|x_k|<\infty$. On the other hand

$$\begin{aligned}|k(x_{k+1})+(k+1)x_k| &\leqslant k|x_k+x_{k+1}|+|x_k|\\ &\leqslant Mk(2k+1)+O(k^2) \text{ by } (i).\end{aligned}$$

This implies $\sup_k k^{-2}|kx_{k+1}+(k+1)x_k|<\infty$.

Conversely, $|kx_{k+1}+(k+1)x_k| \geqslant k|x_k+x_{k+1}|-|x_k|$ for all $k\in\mathbb{N}$. This yields

$$\begin{aligned}|x_k+x_{k+1}| &\leqslant \frac{1}{k}|kx_{k+1}+(k+1)x_k|+\frac{1}{k}|x_k|\\ &\leqslant \frac{1}{k}O(k^2)+\frac{1}{k}O(k^2)\end{aligned}$$

this implies $\sup_k\left|\frac{x_k+x_{k+1}}{k+k+1}\right|<\infty$ where $(x_k)\in\ell_\infty(\nabla)$. □

Before proceeding to have dual spaces of $E(\nabla)$, $E\in\{\ell_\infty,c,c_0\}$, we observe the following points by defining a map

(I) $s:\ell_\infty(\nabla)\longrightarrow\ell_\infty(\nabla)$ as $s(x)=(0,x_2,x_3,\dots)$ for all $x=(x_k)\in\ell_\infty(\nabla)$.

It is easy to verify s is a linear operator and for all $x\in\ell_\infty(\nabla)$,

$$\|s(x)\|_\nabla=\sup_{k\geqslant1}\left|\frac{x_k+x_{k+1}}{k+k+1}\right| \leqslant |x_1|+\left\|\frac{x_k+x_{k+1}}{k+k+1}\right\|_\infty=1.\|\nabla x\|_\infty$$

i.e., $\|s(x)\|_\nabla\leqslant 1.\|\nabla x\|_\infty$ for all $x\in\ell_\infty(\nabla)$ which implies $\|s\|_\nabla\leqslant1$. As

$$\begin{aligned}\|sx\|_\nabla&=\|(0,2,3,\dots)\|_\nabla \text{ for } x=(k)\in\ell_\infty(\nabla)\\ &=1=\|x\|_\nabla\end{aligned}$$

implies that $\|s\|_\nabla=1$. Here range space of s is

$$s(\ell_\infty(\nabla))=\{(x_1,x_2,\dots):(x_k)\in\ell_\infty(\nabla) \text{ with } x_1=0\}\subseteq\ell_\infty(\nabla)$$

is a subspace of $\ell_\infty(\nabla)$. For $x\in s(\ell_\infty(\nabla))$, $\|x\|_\nabla=\|\nabla x\|_\infty$.

(II) Here we prove that $s(\ell_\infty(\nabla))$ and ℓ_∞ are topologically equivalent. Let us define a linear operator $T:s(\ell_\infty(\nabla))\longrightarrow\ell_\infty$ as

$$\begin{aligned}T(x)&=\nabla x \text{ for all } x=(x_k)\in s(\ell_\infty(\nabla))\\ &=\left(\frac{x_k+x_{k+1}}{k+k+1}\right) \text{ which is one-one.}\end{aligned}$$

In order to to prove T is onto, let $y=(y_k)\in\ell_\infty$ then there exists $x=(0,x_2,x_3,\dots)$ $\in s(\ell_\infty(\nabla))$ where $x_2=3y_1$, $x_3=5y_2-3y_1$, $x_4=7y_3-5y_2+3y_1,\dots,x_k=(2k-1)y_{k-1}-(2k-3)y_{k-2}+(2k-5)y_{k-3}-\dots+(-1)^k3y_1,\dots$ such that $Tx=y$.

T is bounded: for $x \in s[\ell_\infty(\nabla)]$ we have $(\nabla x) \in \ell_\infty$ with $x_1 = 0$

$$\|Tx\|_\infty = \|\nabla x\|_\infty = |x_1| + \|\nabla x\|_\infty = \|x\|_\nabla$$

i.e., $\|Tx\|_\infty = 1.\|x\|_\nabla$ implies T is bounded, hence a continuous linear operator. As T is one-one and onto, so $T^{-1} : \ell_\infty \longrightarrow s(\ell_\infty(\nabla))$ defined as

$$T^{-1}(y_1, y_2, \ldots) = \left(0, 3y_1, 5y_2 - 3y_1, \ldots, \sum_{j=1}^{k}(2j-1)y_{j-1}(-1)^{k-j}, \ldots\right)$$

and

$$\begin{aligned}\left\|T^{-1}y\right\|_\nabla &= |0| + \sup_{k\geqslant 1}\left|\frac{\sum_{j=1}^{k}(2j-1)y_{j-1}(-1)^{k-j} + \sum_{j=1}^{k+1}(2j-1)y_{j-1}(-1)^{k-j+1}}{k+k+1}\right| \\ &= \sup_{k\geqslant 1}|y_k| = \|y\|_\infty \text{ for all } y \in \ell_\infty\end{aligned}$$

which yields boundedness of T^{-1}. Thus $T : s(\ell_\infty(\nabla)) \longrightarrow \ell_\infty$ is a homeomorphism, i.e., $s[\ell_\infty(\nabla)] \cong \ell_\infty$.

(III) Similarly, we may have $sc(\nabla) \cong c$, $sc_0(\nabla) \cong c_0$ and so

$$[sc(\nabla)]^* = [sc_0(\nabla)]^* = [s\ell_\infty(\nabla)]^* = \ell_1.$$

THEOREM 15. $[s\ell_\infty(\nabla)]^\alpha = \left\{(a_k) : \sum k^2|a_k| < \infty\right\} = D$.

Proof. Let $(a_k) \in D$ so $\sum_k k^2|a_k| < \infty$. Now for all $x = (x_k) \in [s\ell_\infty(\nabla)]$,

$$\sup_k \left|\frac{x_k + x_{k+1}}{k+k+1}\right| < \infty$$

and so we have $\sup_k k^{-2}|x_k| < \infty$ (in view of Theorem 14). Say $k^{-2}|x_k| \leqslant M$ for all $k \geqslant 1$. The result follows from the fact

$$\sum_k |a_k x_k| = \sum_k (k^2|a_k|)(k^{-2}|x_k|).$$

Conversely, let $(a_k) \in [s\ell_\infty(\nabla)]^\alpha$ so $\sum_k |a_k x_k| < \infty$ for all $x = (x_k) \in \ell_\infty(\nabla)$. Take $x_k = (k-1)^2(-1)^2$ $k \geqslant 1$, i.e., $x = (x_k) = (0, 1^2, -2^2, 3^2, -4^2, \ldots)$. Then

$$\begin{aligned}\sup_k \left|\frac{x_k + x_{k+1}}{k+k+1}\right| &= \sup_k \left|\frac{(k-1)^2(-1)^k + k^2(-1)^{k+1}}{2k+1}\right| \\ &= 1 \text{ and so } (x_k) \in s\ell_\infty(\nabla).\end{aligned}$$

$\Rightarrow \sum_k (k-1)^2|a_k| < \infty$, i.e., $\sum_k k^2|a_k| < \infty$. $\square$

REMARK 3. It is an open problem to have β-dual spaces of $s[c(\nabla)]$ and $s[c_0(\nabla)]$.

3. Conclusion

The present paper mainly concerns with the introduction of some new kind of sequence spaces along with the determination of their continuous as well as Köthe-Toeplitz duals. In our opinion this is just a start of peep into theory of sequence space via this work. One may steped into further for higher ∇^2, ∇^m and various generalizations as for the case of difference sequence spaces and can be achived a lot.

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