Journal of Classical Analysis
Volume 24, Number 1 (2024), 25–42 doi:10.7153/jca-2024-24-02

WEAKLY WEIGHTED AND RELAXED WEIGHTED SHARING OF DIFFERENTIAL DIFFERENCE POLYNOMIALS OF ENTIRE FUNCTIONS

GURUDAS BISWAS

Abstract. In the paper we apply the idea of weakly weighted sharing and relaxed weighted sharing to trace the uniqueness problems of entire functions whose differential difference polynomials share a small function. The results of the paper improve and extend some recent results due to V. HUSNA, S. RAJESHWARI AND S. H. NAVEEN KUMAR [Electronic Journal of Mathematical Analysis and Applications, **9** (2021), 248–260].

1. Introduction, definitions and results

In this paper, by meromorphic function we shall always mean meromorphic function in the complex plane. We assume that the reader is familiar with the basic notions of Nevanlinna value distribution theory (see [7], [13] and [30]). For a nonconstant meromorphic function $f(z)$, we denote by $T(r,f)$ the Nevanlinna characteristic function of $f(z)$ and by $S(r,f)$ any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\longrightarrow\infty$ outside of an exceptional set of finite linear measure. We say that $\alpha(z)$ is a small function of $f(z)$, if $\alpha(z)$ is a meromorphic function satisfying $T(r,\alpha(z))=S(r,f)$. We denote by $S(f)$ the collection of all small functions of $f(z)$.

Let k be a positive integer or infinity and $a\in\mathbb{C}\cup\{\infty\}$. Set $E(a,f)=\{z:f(z)-a=0\}$, where a zero with multiplicity k is counted k times. If the zeros are counted only once, then we denote the set by $\overline{E}(a,f)$. Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions. If $E(a,f)=E(a,g)$, then we say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicities). If $\overline{E}(a,f)=\overline{E}(a,g)$, then we say that $f(z)$ and $g(z)$ share the value a IM (ignoring multiplicities). We denote by $E_{k)}(a,f)$ the set of all a-points of $f(z)$ with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{k)}(a,f)$ the set of all distinct a-points of $f(z)$ with multiplicities not exceeding k. Throughout the paper, we denote by $\rho(f)=\limsup\limits_{r\longrightarrow\infty}\dfrac{\log T(r,f)}{\log r}$ the order of $f(z)$ (see [7], [13] and [30]). We define difference operators by

$$\Delta_\eta f(z)=f(z+\eta)-f(z),\ \ \Delta_\eta^n f(z)=\Delta_\eta^{n-1}(\Delta_\eta f(z)),$$

where η is a nonzero complex number and $n\geqslant 2$ is a positive integer. If $\eta=1$, we denote $\Delta_\eta f(z)=\Delta f(z)$. In addition, we need the following definitions.

Mathematics subject classification (2020): 30D35.
Keywords and phrases: Weakly weighted sharing, relaxed weighted sharing, entire function, differential difference polynomial.

Paper JCA-24-02

DEFINITION 1. [12] Let $f(z)$ be a nonconstant meromorphic function. An expression of the form

$$P[f] = \sum_{i=1}^{p} a_i(z) \prod_{j=0}^{q} f^{(j)}(z)^{l_{ij}}, \tag{1}$$

where $a_i(z) \in S(f)$ for $i = 1, 2, \ldots, p$ and l_{ij} are nonnegative integers for $i = 1, 2, \ldots, p$; $j = 0, 1, 2, \ldots, q$ and $d = \sum_{j=0}^{q} l_{ij}$ for each $i = 1, 2, \ldots, p$ is called a homogeneous differential polynomial of degree d generated by $f(z)$. Also we denote the quantity $Q = \max_{1 \leqslant i \leqslant p} \sum_{j=0}^{q} j l_{ij}$.

DEFINITION 2. [10] Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,a;f \mid = 1)$ the counting function of simple a-points of $f(z)$. For a positive integer k we denote by $N(r,a;f \mid \leqslant k)$ the counting function of those a-points of $f(z)$ (counted with proper multiplicities) whose multiplicities are not greater than k. By $\overline{N}(r,a;f \mid \leqslant k)$ we denote the corresponding reduced counting function. Analogously we can define $N(r,a;f \mid \geqslant k)$ and $\overline{N}(r,a;f \mid \geqslant k)$.

DEFINITION 3. [11] Let k be a positive integer or infinity. We denote by $N_k(r,a;f)$ the counting function of all a-points of $f(z)$, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and k times if $m > k$. Then

$$N_k(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid \geqslant 2) + \ldots + \overline{N}(r,a;f \mid \geqslant k).$$

Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

DEFINITION 4. [15] Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N_E(r,a;f,g)$ $(\overline{N}_E(r,a;f,g))$ by the counting function (reduced counting function) of all common zeros of $f(z) - a$ and $g(z) - a$ with the same multiplicities and by $N_0(r,a;f,g)$ $(\overline{N}_0(r,a;f,g))$ the counting function (reduced counting function) of all common zeros of $f(z) - a$ and $g(z) - a$ ignoring multiplicities. If

$$\overline{N}(r,a;f) + \overline{N}(r,a;g) - 2\overline{N}_E(r,a;f,g) = S(r,f) + S(r,g),$$

then we say that $f(z)$ and $g(z)$ share the value a “CM”. If

$$\overline{N}(r,a;f) + \overline{N}(r,a;g) - 2\overline{N}_0(r,a;f,g) = S(r,f) + S(r,g),$$

then we say that $f(z)$ and $g(z)$ share the value a “IM”.

DEFINITION 5. [15] Let $f(z)$ and $g(z)$ share the value a “IM” and k be a positive integer or infinity. Then $\overline{N}^{E}_{k)}(r,a;f,g)$ denotes the reduced counting function of those a-points of $f(z)$ whose multiplicities are equal to the corresponding a-points of $g(z)$, and both of their multiplicities are not greater than k. $\overline{N}^{0}_{(k}(r,a;f,g)$ denotes the reduced counting function of those a-points of $f(z)$ which are a-points of $g(z)$, and both of their multiplicities are not less than k.

We now introduce the following definition of weakly weighted sharing which is a scaling between sharing IM and sharing CM.

DEFINITION 6. [15] Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or infinity. If

$$\overline{N}(r,a;f \mid \leqslant k) - \overline{N}^{E}_{k)}(r,a;f,g) = S(r,f),$$
$$\overline{N}(r,a;g \mid \leqslant k) - \overline{N}^{E}_{k)}(r,a;f,g) = S(r,g),$$
$$\overline{N}(r,a;f \mid \geqslant k+1) - \overline{N}^{0}_{(k+1}(r,a;f,g) = S(r,f),$$
$$\overline{N}(r,a;g \mid \geqslant k+1) - \overline{N}^{0}_{(k+1}(r,a;f,g) = S(r,g),$$

or if $k = 0$ and

$$\overline{N}(r,a;f) - \overline{N}_0(r,a;f,g) = S(r,f),$$
$$\overline{N}(r,a;g) - \overline{N}_0(r,a;f,g) = S(r,g),$$

then we say that $f(z)$ and $g(z)$ share the value a weakly with weight k and we write $f(z)$ and $g(z)$ share "(a,k)".

In 2007, A. Banerjee and S. Mukherjee [2] introduced a new type of sharing known as relaxed weighted sharing, weaker than weakly weighted sharing and is defined as follows.

DEFINITION 7. [2] We denote by $\overline{N}(r,a;f \mid = p; g \mid = q)$ the reduced counting function of common a-points of $f(z)$ and $g(z)$ with multiplicities p and q respectively.

DEFINITION 8. [2] Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or infinity. Suppose that $f(z)$ and $g(z)$ share the value a "IM". If for $p \neq q$,

$$\sum_{p,q \leqslant k} \overline{N}(r,a;f \mid = p; g \mid = q) = S(r),$$

then we say that $f(z)$ and $g(z)$ share the value a with weight k in a relaxed manner and in that case we write $f(z)$ and $g(z)$ share $(a,k)^*$.

Many research works on weakly weighted sharing and relaxed weighted sharing of differential difference polynomials of entire and meromorphic functions have been done by many mathematicians in the world (see [1], [3], [9], [20], [21], [23], [24], [25]). Recently, value distribution and uniqueness in differential difference analogue has become a subject of great interest among the researchers.

In 2001, M. L. Fang and W. Hong [6] investigated the uniqueness problem of differential polynomial of the form $f^n(z)(f(z)-1)f'(z)$ and proved the following theorem.

THEOREM A. *Let $f(z)$ and $g(z)$ be two transcendental entire functions and $n \geqslant 11$ be an integer. If $f^n(z)(f(z)-1)f'(z)$ and $g^n(z)(g(z)-1)g'(z)$ share the value 1 CM, then $f(z) \equiv g(z)$.*

In 2004, W. C. Lin and H. X. Yi [16] improved the above theorem by considering the fixed point sharing and obtained the following theorem.

THEOREM B. *Let $f(z)$ and $g(z)$ be two transcendental entire functions and $n \geqslant 7$ be an integer. If $f^n(z)(f(z)-1)f'(z)$ and $g^n(z)(g(z)-1)g'(z)$ share the value z CM, then $f(z) \equiv g(z)$.*

In 2007, I. Laine and C. C. Yang [14] studied the difference polynomial of the form $f^n(z)f(z+\eta)$ and proved the following result.

THEOREM C. *Let $f(z)$ be a transcendental entire function of finite order and η be a nonzero complex constant. Then for $n \geqslant 2$, $f^n(z)f(z+\eta)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

The following example shows that the above theorem does not hold for $n = 1$.

EXAMPLE 1. [14] Let $f(z) = 1 + e^z$. Then $f(z)f(z+\pi i) - 1 = -e^{2z}$ has no zeros.

Also we give an example below which shows that the Theorem C is not valid if $f(z)$ is of infinite order.

EXAMPLE 2. [17] Let $f(z) = e^{-e^z}$. Then $f^2(z)f(z+\eta) - 2 = -1$ and $\rho(f) = \infty$, where η is the nonzero constant satisfying $e^\eta = -2$. Evidently, $f^2(z)f(z+\eta) - 2$ has no zeros.

In 2010, X. G. Qi, L. Z. Yang and K. Liu [26] proved the following uniqueness result corresponding to Theorem C.

THEOREM D. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and η be a nonzero complex constant, and let $n \geqslant 6$ be an integer. If $f^n(z)f(z+\eta)$ and $g^n(z)g(z+\eta)$ share 1 CM, then either $fg = t_1$ or $f \equiv t_2 g$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = t_2^{n+1} = 1$.*

In the same year J. L. Zhang [31] considered the zeros of one certain type of difference polynomial and proved the following result.

THEOREM E. *Let $f(z)$ be a transcendental entire function of finite order, $\alpha(z)(\not\equiv 0)$ be a small function with respect to $f(z)$ and η be a nonzero complex constant. If $n \geqslant 2$ is an integer then $f^n(z)(f(z)-1)f(z+\eta) - \alpha(z)$ has infinitely many zeros.*

In the same paper the author also proved the following uniqueness result.

THEOREM F. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that η is a nonzero complex constant and $n \geqslant 7$ is an integer. If $f^n(z)(f(z)-1)f(z+\eta)$ and $g^n(z)(g(z)-1)g(z+\eta)$ share $\alpha(z)$ CM, then $f(z) \equiv g(z)$.*

Naturally one may ask the following question.

QUESTION 1. Is it possible to relax the nature of sharing the small function in Theorem F?

In 2014, using the idea of weakly weighted sharing and relaxed weighted sharing, C. Meng [22] proved the following results which improve and supplement Theorem F in different directions.

THEOREM G. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that η is a nonzero complex constant, and $n \geqslant 7$ is an integer. If $f^n(z)(f(z)-1)f(z+\eta)$ and $g^n(z)(g(z)-1)g(z+\eta)$ share "$(\alpha,2)$", then $f(z)\equiv g(z)$.*

THEOREM H. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that η is a nonzero complex constant and $n \geqslant 10$ is an integer. If $f^n(z)(f(z)-1)f(z+\eta)$ and $g^n(z)(g(z)-1)g(z+\eta)$ share $(\alpha,2)^*$, then $f(z)\equiv g(z)$.*

THEOREM I. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that η is a nonzero complex constant and $n \geqslant 16$ is an integer. If $\overline{E}_{2)}(\alpha(z), f^n(z)(f(z)-1)f(z+\eta)) = \overline{E}_{2)}(\alpha(z), g^n(z)(g(z)-1)g(z+\eta))$, then $f(z)\equiv g(z)$.*

Observing the above results the following question is inevitable.

QUESTION 2. What can be said about the relationship between two entire functions $f(z)$ and $g(z)$ if one replace $f^n(z)(f(z)-1)f(z+\eta)$ by $f^n(z)(f^m(z)-1)f(z+\eta)$ in Theorems G–I where $m\ (\geqslant 1)$ is any integer?

In 2015, P. Sahoo [27] answered the above question and proved the following results which generalize Theorems G–I.

THEOREM J. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function of both $f(z)$ and $g(z)$. Suppose that η is a nonzero complex constant, n and $m(\geqslant 1)$ are integers such that $n \geqslant m+6$. If $f^n(z)(f^m(z)-1)f(z+\eta)$ and $g^n(z)(g^m(z)-1)g(z+\eta)$ share "$(\alpha(z),2)$", then $f(z)\equiv tg(z)$ where $t^m = 1$.*

THEOREM K. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function of both $f(z)$ and $g(z)$. Suppose that η is a nonzero complex constant, n and $m(\geqslant 1)$ are integers such that $n \geqslant 2m+8$. If $f^n(z)(f^m(z)-1)f(z+\eta)$ and $g^n(z)(g^m(z)-1)g(z+\eta)$ share $(\alpha(z),2)^*$, then $f(z)\equiv tg(z)$ where $t^m = 1$.*

THEOREM L. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function of both $f(z)$ and $g(z)$. Suppose that η is a nonzero complex constant, n and $m(\geqslant 1)$ are integers such that $n \geqslant 4m+12$. If $\overline{E}_{2)}(\alpha(z), f^n(z)(f^m(z)-1)f(z+\eta)) = \overline{E}_{2)}(\alpha(z), g^n(z)(g^m(z)-1)g(z+\eta))$, then $f(z)\equiv tg(z)$ where $t^m = 1$.*

Regarding Theorems J–L, one may ask the following question.

QUESTION 3. What can be said about the entire functions $f(z)$ and $g(z)$ if we consider the difference polynomials of the form $(f^n(z)(f^m(z)-1)f(z+\eta))^{(k)}$ where $k(\geqslant 0)$ is an integer?

To find out the possible answer of the above question, in 2018 P. Sahoo and present author [28] proved following three theorems which improved and extended Theorems J–L respectively.

THEOREM M. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function of both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that η is a nonzero complex constant, n, $k(\geqslant 0)$ and $m(\geqslant 1)$ are integers such that $n\geqslant 2k+m+6$. If $(f^n(z)(f^m(z)-1)f(z+\eta))^{(k)}$ and $(g^n(z)(g^m(z)-1)g(z+\eta))^{(k)}$ share "$(\alpha(z),2)$", then $f(z)\equiv tg(z)$ where $t^m=1$.*

THEOREM N. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function of both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that η is a nonzero complex constant, n, $k(\geqslant 0)$ and $m(\geqslant 1)$ are integers such that $n\geqslant 3k+2m+8$. If $(f^n(z)(f^m(z)-1)f(z+\eta))^{(k)}$ and $(g^n(z)(g^m(z)-1)g(z+\eta))^{(k)}$ share $(\alpha(z),2)^*$, then $f(z)\equiv tg(z)$ where $t^m=1$.*

THEOREM O. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function with respect to both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that η is a nonzero complex constant, n, $k(\geqslant 0)$ and $m(\geqslant 1)$ are integers such that $n\geqslant 5k+4m+12$. If $\overline{E}_{2)}(\alpha(z),(f^n(z)(f^m(z)-1)f(z+\eta))^{(k)})=\overline{E}_{2)}(\alpha(z),(g^n(z)(g^m(z)-1)g(z+\eta))^{(k)})$, then $f(z)\equiv tg(z)$ where $t^m=1$.*

REMARK 1. For $k=0$, we get the Theorems J–L respectively.

Recently, V. Husna, S. Rajeshwari and S. H. Naveen Kumar [8] considered uniqueness problems of more general differential difference polynomials of the form $(f^n(z)(f^m(z)-1)\prod_{i=1}^{p}f(z+\eta_i)^{\nu_i})^{(k)}$ where $f(z)$ is a transcendental entire function of finite order, n, m, p, k and ν_i $(i=1,2,\ldots,p)$ are nonnegative integers and η_i $(i=1,2,\ldots,p)$ are distinct finite complex numbers. They proved the following uniqueness results which extend and improve many previous results in this direction.

THEOREM P. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function of both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that η_i $(i=1,2,\ldots,p)$ are nonzero complex constants, n, $k(\geqslant 0)$ and $m(\geqslant 1)$ are integers such that $n\geqslant 2k+m+\sigma+5$ where $\sigma=\sum_{i=1}^{p}\nu_i$. If $(f^n(z)(f^m(z)-1)\prod_{i=1}^{p}f(z+\eta_i)^{\nu_i})^{(k)}$ and $(g^n(z)(g^m(z)-1)\prod_{i=1}^{p}g(z+\eta_i)^{\nu_i})^{(k)}$ share "$(\alpha(z),2)$", then $f(z)\equiv tg(z)$ where $t^m=t^{n+\sigma}=1$.*

THEOREM Q. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function of both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that η_i $(i=1,2,\ldots,p)$ are nonzero complex constants, n, $k(\geqslant 0)$ and $m(\geqslant 1)$ are integers such that $n\geqslant 3k+2m+2\sigma+6$ where $\sigma=\sum_{i=1}^{p}\nu_i$.*

If $(f^n(z)(f^m(z)-1)\prod_{i=1}^{p} f(z+\eta_i)^{\nu_i})^{(k)}$ *and* $(g^n(z)(g^m(z)-1)\prod_{i=1}^{p} g(z+\eta_i)^{\nu_i})^{(k)}$ *share* $(\alpha(z),2)^*$*, then* $f(z)\equiv tg(z)$ *where* $t^m=t^{n+\sigma}=1$.

THEOREM R. *Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions of finite order, and* $\alpha(z)(\not\equiv 0,\infty)$ *be a small function of both* $f(z)$ *and* $g(z)$ *with finitely many zeros. Suppose that* η_i $(i=1,\,2,\ldots,p)$ *are nonzero complex constants,* n*,* $k(\geqslant 0)$ *and* $m(\geqslant 1)$ *are integers such that* $n\geqslant 5k+4m+4\sigma+8$ *where* $\sigma=\sum_{i=1}^{p}\nu_i$*. If* $\overline{E}_{2)}(\alpha(z),(f^n(z)(f^m(z)-1)\prod_{i=1}^{p} f(z+\eta_i)^{\nu_i})^{(k)})=\overline{E}_{2)}(\alpha(z),(g^n(z)(g^m(z)-1)\prod_{i=1}^{p} g(z+\eta_i)^{\nu_i})^{(k)})$*, then* $f(z)\equiv tg(z)$ *where* $t^m=t^{n+\sigma}=1$.

Due to the questions of V. Husna, S. Rajeshwari and S. H. Naveen Kumar, we study the following Differential difference polynomials of the form $(f^n(z)(f^m(z)-1)\prod_{i=1}^{p} f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q} f^{(j)}(z)^{\mu_j})^{(k)}$ where μ_j $(j=1,\,2,\ldots,q)$ are nonnegative integers, which is the motivation of the present paper.

We prove following three theorems which improve and extend Theorems P–R respectively. The following theorems are the main results of the paper.

THEOREM 1. *Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions of finite order, and* $\alpha(z)(\not\equiv 0,\infty)$ *be a small function of both* $f(z)$ *and* $g(z)$ *with finitely many zeros. Suppose that* η_i $(i=1,\,2,\ldots,p)$ *are nonzero complex constants,* n*,* $k(\geqslant 0)$ *and* $m(\geqslant 1)$ *are integers such that* $n\geqslant 2k+m+\sigma+\tau+5$ *where* $\sigma=\sum_{i=1}^{p}\nu_i$*, and* $\tau=\sum_{j=1}^{q}\mu_j$*. If* $(f^n(z)(f^m(z)-1)\prod_{i=1}^{p} f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q} f^{(j)}(z)^{\mu_j})^{(k)}$ *and* $(g^n(z)(g^m(z)-1)\prod_{i=1}^{p} g(z+\eta_i)^{\nu_i}\prod_{j=1}^{q} g^{(j)}(z)^{\mu_j})^{(k)}$ *share "*$(\alpha(z),2)$*", then either* $f(z)\equiv tg(z)$ *where* $t^m=t^{n+\sigma+\tau}=1$ *or* $f(z)$ *and* $g(z)$ *satisfy the algebraic equation* $R(f,g)=0$*, where* $R(f,g)$ *is given by*

$$R(f,g)=f^n(z)(f^m(z)-1)\prod_{i=1}^{p} f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q} f^{(j)}(z)^{\mu_j}-g^n(z)(g^m(z)-1)\times\prod_{i=1}^{p} g(z+\eta_i)^{\nu_i}\prod_{j=1}^{q} g^{(j)}(z)^{\mu_j}.$$

THEOREM 2. *Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions of finite order, and* $\alpha(z)(\not\equiv 0,\infty)$ *be a small function of both* $f(z)$ *and* $g(z)$ *with finitely many zeros. Suppose that* η_i $(i=1,\,2,\ldots,p)$ *are nonzero complex constants,* n*,* $k(\geqslant 0)$ *and* $m(\geqslant 1)$ *are integers such that* $n\geqslant 3k+2m+2\sigma+2\tau+6$ *where* $\sigma=\sum_{i=1}^{p}\nu_i$*, and* $\tau=\sum_{j=1}^{q}\mu_j$*. If* $(f^n(z)(f^m(z)-1)\prod_{i=1}^{p} f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q} f^{(j)}(z)^{\mu_j})^{(k)}$ *and* $(g^n(z)(g^m(z)-1)\prod_{i=1}^{p} g(z+\eta_i)^{\nu_i}\prod_{j=1}^{q} g^{(j)}(z)^{\mu_j})^{(k)}$ *share* $(\alpha(z),2)^*$*, then either* $f(z)\equiv tg(z)$ *where* $t^m=t^{n+\sigma+\tau}=1$ *or* $f(z)$ *and* $g(z)$ *satisfy the algebraic equation* $R(f,g)=0$*, where* $R(f,g)$ *is given by*

$$R(f,g)=f^n(z)(f^m(z)-1)\prod_{i=1}^{p} f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q} f^{(j)}(z)^{\mu_j}-g^n(z)(g^m(z)-1)\times\prod_{i=1}^{p} g(z+\eta_i)^{\nu_i}\prod_{j=1}^{q} g^{(j)}(z)^{\mu_j}.$$

THEOREM 3. *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0,\infty)$ be a small function of both $f(z)$ and $g(z)$ with finitely many zeros. Suppose that η_i $(i=1,\,2,\ldots,p)$ are nonzero complex constants, n, $k(\geqslant 0)$ and $m(\geqslant 1)$ are integers such that $n\geqslant 5k+4m+4\sigma+4\tau+8$ where $\sigma=\sum_{i=1}^{p}\nu_i$, and $\tau=\sum_{j=1}^{q}\mu_j$. If $\overline{E}_{2)}(\alpha(z),(f^n(z)(f^m(z)-1)\prod_{i=1}^{p}f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}f^{(j)}(z)^{\mu_j})^{(k)})=\overline{E}_{2)}(\alpha(z),(g^n(z)(g^m(z)-1)\prod_{i=1}^{p}g(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}g^{(j)}(z)^{\mu_j})^{(k)})$, then either $f(z)\equiv tg(z)$ where $t^m=t^{n+\sigma+\tau}=1$ or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g)=0$, where $R(f,g)$ is given by*

$$\begin{aligned}R(f,g)\,&=f^n(z)(f^m(z)-1)\prod_{i=1}^{p}f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}f^{(j)}(z)^{\mu_j}-g^n(z)(g^m(z)-1)\\&\quad\times\prod_{i=1}^{p}g(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}g^{(j)}(z)^{\mu_j}.\end{aligned}$$

REMARK 2. Since Theorems P–R are the special cases of Theorems 1–3 respectively for $\tau=0$, Theorems 1–3 improve and extend Theorems P–R respectively.

The following example indicates the support of Theorem 1.

EXAMPLE 3. Let $g(z)=e^z$ and $f=te^z$, where t is a constant such that $t^{n+2}=t=1$. Let $k=0$, $m=1$, $p=1$, $q=1$, $\nu_1=1$, $\mu_1=1$, and $\eta_1=2\pi i$. It immediately yields that $n\geqslant 2k+m+\sigma+\tau+5$ and $(f^n(z)(f^m(z)-1)\prod_{i=1}^{p}f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}f^{(j)}(z)^{\mu_j})^{(k)}$ and $(g^n(z)(g^m(z)-1)\prod_{i=1}^{p}g(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}g^{(j)}(z)^{\mu_j})^{(k)}$ share $\alpha(z)$ CM and hence they share "$(\alpha(z),2)$". Here $f(z)$ and $g(z)$ satisfy the relation $f(z)\equiv tg(z)$ where $t^m=t^{n+\sigma+\tau}=1$.

If we take $g(z)=e^z$, $f=te^z$, where t is a constant such that $t^{n+2}=t=1$ and $k=0$, $m=1$, $p=1$, $q=1$, $\nu_1=1$, $\mu_1=1$, & $\eta_1=2\pi i$, it can be easily verified that the conclusion of Theorems 2 and 3 are also satisfied.

2. Lemmas

In this section, we state some lemmas which will be needed in the sequel. We denote by H the following function:

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right),$$

where F and G are nonconstant meromorphic functions defined in the complex plane $\mathbb{C}$.

LEMMA 1. [7, 13, 30] *Let $f(z)$ be a transcendental meromorphic function and k be a positive integer. Then*

$$m\left(r,\frac{f^{(k)}}{f}\right)=S(r,f).$$

LEMMA 2. [4, 5] *Let $f(z)$ be a nonconstant meromorphic function of finite order $\rho(f)$ and $\eta \in \mathbb{C}\setminus\{0\}$. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O(r^{\rho(f)-1+\varepsilon}).$$

LEMMA 3. [18] *Let $f(z)$ be a meromorphic function of finite order $\rho(f)$ and η be a fixed nonzero complex constant. Then*

$$\begin{aligned}
N(r,0;f(z+\eta)) &\leqslant N(r,0;f)+S(r,f),\\
sN(r,\infty;f(z+\eta)) &\leqslant N(r,\infty;f)+S(r,f),\\
\overline{N}(r,0;f(z+\eta)) &\leqslant \overline{N}(r,0;f)+S(r,f),\\
\overline{N}(r,\infty;f(z+\eta)) &\leqslant \overline{N}(r,\infty;f)+S(r,f),
\end{aligned}$$

outside of possible exceptional set with finite logarithmic measure.

LEMMA 4. [5] *Let $f(z)$ be a meromorphic function of order $\rho(f) < \infty$, and let η be a nonzero complex constant. Then for each $\varepsilon > 0$, we have*

$$T(r,f(z+\eta)) = T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + O(\log r).$$

LEMMA 5. [29] *Let $f(z)$ be a nonconstant meromorphic function and let*

$$P(f) = a_m f^m + a_{m-1} f^{m-1} + \ldots + a_1 f + a_0,$$

where $a_i \in S(f)$ for $i = 0,\ 1, \ldots, m$, $a_m \neq 0$ be a polynomial of degree m. Then

$$T(r,P(f)) = mT(r,f) + S(r,f).$$

LEMMA 6. [12] *Let $f(z)$ be a nonconstant meromorphic function and P[f] be defined by (1). Then*

$$T(r,P) \leqslant dT(r,f) + Q\overline{N}(r,\infty;f) + S(r,f)$$

and

$$\begin{aligned}
N(r,0;P) &\leqslant T(r,P) - dT(r,f) + dN(r,0;f) + S(r,f)\\
&\leqslant Q\overline{N}(r,\infty;f) + dN(r,0;f) + S(r,f).
\end{aligned}$$

LEMMA 7. [32] *Let $f(z)$ be a nonconstant meromorphic function, and p, k be positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right) \leqslant T\left(r,f^{(k)}\right) - T(r,f) + N_{p+k}(r,0;f) + S(r,f), \tag{2}$$

$$N_p\left(r,0;f^{(k)}\right) \leqslant k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f). \tag{3}$$

LEMMA 8. [2] *Let F and G be two nonconstant meromorphic functions that share "$(1,2)$" and $H \not\equiv 0$. Then*

$$T(r,F) \leqslant N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) - \sum_{p=3}^{\infty} \overline{N}\left(r,0;\frac{G'}{G}\,|\geqslant p\right) + S(r,F) + S(r,G),$$

and the same inequality holds for $T(r,G)$.

LEMMA 9. [2] *Let F and G be two nonconstant meromorphic functions that share $(1,2)^*$ and $H \not\equiv 0$. Then*

$$T(r,F) \leqslant N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + \overline{N}(r,0;F) + \overline{N}(r,\infty;F) - m(r,1;G) + S(r,F) + S(r,G),$$

and the same inequality is true for $T(r,G)$.

LEMMA 10. [19] *Let F and G be two nonconstant entire functions, and $p \geqslant 2$ be an integer. If $\overline{E}_{p)}(1,F) = \overline{E}_{p)}(1,G)$ and $H \not\equiv 0$, then*

$$T(r,F) \leqslant N_2(r,0;F) + N_2(r,0;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F) + S(r,G),$$

and the same inequality holds for $T(r,G)$.

LEMMA 11. *Let $f(z)$ be an entire function of order $\rho(f) < \infty$, and*

$$F_1 = f^n(z)(f^m(z)-1)\prod_{i=1}^{p} f(z+\eta_i)^{\nu_i} \prod_{j=1}^{q} f^{(j)}(z)^{\mu_j}.$$

Then

$$T(r,F_1) = (n+m+\sigma+\tau)T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f).$$

Proof. Using Lemmas 1, 2 and 5, we get

$$\begin{aligned}(n+m+\sigma+\tau)T(r,f) &= T(r,f^{n+\sigma+\tau}(f^m-1)) + S(r,f)\\ &= m(r,f^{n+\sigma+\tau}(f^m-1)) + S(r,f)\\ &\leqslant m\left(\frac{f^{n+\sigma+\tau}(f^m-1)}{F_1}\right) + m(r,F_1) + S(r,f)\\ &\leqslant m\left(\frac{f^{\sigma+\tau}}{\prod_{i=1}^{p} f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q} f^{(j)}(z)^{\mu_j}}\right) + m(r,F_1) + S(r,f)\\ &\leqslant T(r,F_1) + O(r^{\rho(f)-1+\varepsilon}) + S(r,f). \qquad (4)\end{aligned}$$

Again from Lemmas 1, 2 and 5, we obtain

$$\begin{aligned} T(r,F_1) &\leqslant m(r,f^n)+m(r,f^m-1)+m\left(f^{\sigma+\tau}\cdot\prod_{i=1}^{p}\frac{f(z+\eta_i)^{\nu_i}}{f(z)^{\nu_i}}\cdot\prod_{j=1}^{q}\frac{f^{(j)}(z)^{\mu_j}}{f(z)^{\mu_j}}\right)+S(r,f)\\ &\leqslant (n+m+\sigma+\tau)T(r,f)+O(r^{\rho(f)-1+\varepsilon})+S(r,f). \end{aligned} \quad (5)$$

From (4) and (5), we get the Lemma. □

LEMMA 12. *Let $f(z)$ and $g(z)$ be two entire functions, $n(\geqslant 1)$, $m(\geqslant 1)$, $k(\geqslant 0)$ be integers, and let*

$$F=(f^n(z)(f^m(z)-1)\prod_{i=1}^{p}f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}f^{(j)}(z)^{\mu_j})^{(k)},$$

$$G=(g^n(z)(g^m(z)-1)\prod_{i=1}^{p}g(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}g^{(j)}(z)^{\mu_j})^{(k)}.$$

If there exists nonzero constants c_1 and c_2 such that $\overline{N}(r,c_1;F)=\overline{N}(r,0;G)$ and $\overline{N}(r,c_2;G)=\overline{N}(r,0;F)$, then $n\leqslant 2k+m+\sigma+\tau+2$.

Proof. We put $F_1=f^n(z)(f^m(z)-1)\prod_{i=1}^{p}f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}f^{(j)}(z)^{\mu_j}$ and $G_1=g^n(z)(g^m(z)-1)\prod_{i=1}^{p}g(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}g^{(j)}(z)^{\mu_j}$. By the second fundamental theorem of Nevanlinna we have

$$\begin{aligned} T(r,F) &\leqslant \overline{N}(r,0;F)+\overline{N}(r,c_1;F)+S(r,F)\\ &\leqslant \overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F). \end{aligned} \quad (6)$$

Using (6) and Lemmas 3, 4, 6, 7 and 11 we obtain

$$\begin{aligned} &(n+m+\sigma+\tau)T(r,f)\\ &\leqslant T(r,F)-\overline{N}(r,0;F)+N_{k+1}(r,0;F_1)+O(r^{\rho(f)-1+\varepsilon})+S(r,f)\\ &\leqslant \overline{N}(r,0;G)+N_{k+1}(r,0;F_1)+O(r^{\rho(f)-1+\varepsilon})+S(r,f)\\ &\leqslant N_{k+1}(r,0;F_1)+N_{k+1}(r,0;G_1)+O(r^{\rho(f)-1+\varepsilon})+S(r,f)+S(r,g)\\ &\leqslant (k+1)(\overline{N}(r,0;f)+\overline{N}(r,0;g))+N(r,1;f^m)+N(r,1;g^m)\\ &\quad +N(r,0;\prod_{i=1}^{p}f(z+\eta_i)^{\nu_i})+N(r,0;\prod_{i=1}^{p}g(z+\eta_i)^{\nu_i})+N(r,0;\prod_{j=1}^{q}f^{(j)}(z)^{\mu_j})\\ &\quad +N(r,0;\prod_{j=1}^{q}g^{(j)}(z)^{\mu_j})+O(r^{\rho(f)-1+\varepsilon})+S(r,f)+S(r,g)\\ &\leqslant (k+m+\sigma+\tau+1)(T(r,f)+T(r,g))+O(r^{\rho(f)-1+\varepsilon})+S(r,f)+S(r,g). \end{aligned} \quad (7)$$

Similarly

$$\begin{aligned} &(n+m+\sigma+\tau)T(r,g)\\ &\leqslant (k+m+\sigma+\tau+1)(T(r,f)+T(r,g))+O(r^{\rho(g)-1+\varepsilon})+S(r,f)+S(r,g). \end{aligned} \quad (8)$$

Combining (7) and (8) we obtain

$$(n-2k-m-\sigma-\tau-2)(T(r,f)+T(r,g))$$
$$\leqslant O(r^{\rho(f)-1+\varepsilon})+O(r^{\rho(g)-1+\varepsilon})+S(r,f)+S(r,g),$$

which gives $n \leqslant 2k+m+\sigma+\tau+2$. This proves the Lemma. □

3. Proof of the Theorems

Proof of Theorem 1. Let $F=\frac{F_1^{(k)}}{\alpha(z)}$ and $G=\frac{G_1^{(k)}}{\alpha(z)}$ where $F_1=f^n(z)(f^m(z)-1)\prod_{i=1}^{p}f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}f^{(j)}(z)^{\mu_j}$ and $G_1=g^n(z)(g^m(z)-1)\prod_{i=1}^{p}g(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}g^{(j)}(z)^{\mu_j}$. Then F and G are transcendental meromorphic functions that share "$(1,2)$" except the zeros and poles of $\alpha(z)$. If possible, we may assume that $H\not\equiv 0$. Then using Lemmas 7, 11 we obtain from Lemma 8

$$\begin{aligned}T(r,F)&\leqslant N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+S(r,F)+S(r,G)\\&\leqslant T(r,F)-(n+m+\sigma+\tau)T(r,f)+N_{k+2}(r,0;F_1)+N_{k+2}(r,0;G_1)\\&\quad+O(r^{\rho(f)-1+\varepsilon})+S(r,f)+S(r,g).\end{aligned}\tag{9}$$

Therefore using Lemmas 3, 4, 6 and 11, we get from (9)

$$\begin{aligned}&(n+m+\sigma+\tau)T(r,f)\\&\leqslant(k+2)(\overline{N}(r,0;f)+\overline{N}(r,0;g))+N(r,1;f^m)+N(r,1;g^m)\\&\quad+N(r,0;\prod_{i=1}^{p}f(z+\eta_i)^{\nu_i})+N(r,0;\prod_{i=1}^{p}g(z+\eta_i)^{\nu_i})+N(r,0;\prod_{j=1}^{q}f^{(j)}(z)^{\mu_j})\\&\quad+N(r,0;\prod_{j=1}^{q}g^{(j)}(z)^{\mu_j})+O(r^{\rho(f)-1+\varepsilon})+S(r,f)+S(r,g)\\&\leqslant(k+m+\sigma+\tau+2)(T(r,f)+T(r,g))+O(r^{\rho(f)-1+\varepsilon})+S(r,f)+S(r,g).\end{aligned}\tag{10}$$

In a similar manner we obtain

$$\begin{aligned}&(n+m+\sigma+\tau)T(r,g)\\&\leqslant(k+m+\sigma+\tau+2)\{T(r,f)+T(r,g)\}+O(r^{\rho(g)-1+\varepsilon})+S(r,f)+S(r,g).\end{aligned}\tag{11}$$

(10) and (11) together yields

$$(n-2k-m-\sigma-\tau-4)\{T(r,f)+T(r,g)\}$$
$$\leqslant O(r^{\rho(f)-1+\varepsilon})+O(r^{\rho(g)-1+\varepsilon})+S(r,f)+S(r,g),$$

a contradiction with the assumption that $n\geqslant 2k+m+\sigma+\tau+5$. Therefore we must have $H=0$. Then

$$\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right)=0.$$

Integrating both side twice we get from above

$$\frac{1}{F-1}=\frac{A}{G-1}+B, \tag{12}$$

where $A(\neq 0)$ and B are constants. From (12) it is clear that F, G share 1 CM and hence they share "$(1,2)$". Therefore $n \geqslant 2k+m+\sigma+\tau+5$. We now discuss the following three cases separately.

Case 1. Suppose that $B\neq 0$ and $A=B$. Then from (12) we obtain

$$\frac{1}{F-1}=\frac{BG}{G-1}. \tag{13}$$

If $B=-1$, then from (13) we obtain $FG=1$. Then

$$(f^n(z)(f^m(z)-1)\prod_{i=1}^{p} f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q} f^{(j)}(z)^{\mu_j})^{(k)}(g^n(z)(g^m(z)-1)$$
$$\times\prod_{i=1}^{p} g(z+\eta_i)^{\nu_i}\prod_{j=1}^{q} g^{(j)}(z)^{\mu_j})^{(k)}=\alpha^2(z).$$

Since the number of zeros of $\alpha(z)$ is finite, it follows that $f(z)$ as well as $g(z)$ have finitely many zeros. We put $f(z)=h(z)e^{\beta(z)}$, where $h(z)$ is a nonzero polynomial and $\beta(z)$ is a nonconstant polynomial. Now replacing $\prod_{i=1}^{p} h(z+\eta_i)^{\nu_i}$ by $\lambda(z)$ and $\sum_{i=1}^{p}\nu_i\beta(z+\eta_i)$ by $\gamma(z)$ we deduce that

$$\begin{aligned}
&(f^n(z)(f^m(z)-1)\prod_{i=1}^{p} f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q} f^{(j)}(z)^{\mu_j})^{(k)}\\
&=\left(h^n(z)e^{n\beta(z)}(h^m(z)e^{m\beta(z)}-1)\prod_{i=1}^{p} h(z+\eta_i)^{\nu_i}e^{\nu_i\beta(z+\eta_i)}\prod_{j=1}^{q}((h(z)e^{\beta(z)})^{(j)})^{\mu_j}\right)^{(k)}\\
&=\left(h^n(z)\lambda(z)e^{(n+\tau)\beta(z)+\gamma(z)}(h^m(z)e^{m\beta(z)}-1)\omega(z)\right)^{(k)}\\
&\qquad\text{[where } \omega(z) \text{ is a function of } h(z),h'(z),\ldots,h^{(q)}(z),\beta'(z),\beta''(z),\ldots,\beta^{(q)}(z)]\\
&=\left(h^{n+m}(z)\lambda(z)\omega(z)e^{(n+m+\tau)\beta(z)+\gamma(z)}-h^n(z)\lambda(z)\omega(z)e^{(n+\tau)\beta(z)+\gamma(z)}\right)^{(k)}\\
&=e^{(n+m+\tau)\beta(z)+\gamma(z)}P_1\Big(h(z),h'(z),\ldots,h^{(k)}(z),\beta'(z),\beta''(z),\ldots,\beta^{(k)}(z),\\
&\quad\gamma'(z),\gamma''(z),\ldots,\gamma^{(k)}(z),\lambda(z),\lambda'(z),\ldots,\lambda^{(k)}(z),\omega(z),\omega'(z),\ldots,\omega^{(k)}(z)\Big)\\
&\quad-e^{(n+\tau)\beta(z)+\gamma(z)}P_2\Big(h(z),h'(z),\ldots,h^{(k)}(z),\ldots,\beta'(z),\beta''(z),\ldots,\beta^{(k)}(z),\\
&\quad\gamma'(z),\gamma''(z),\ldots,\gamma^{(k)}(z),\lambda(z),\lambda'(z),\ldots,\lambda^{(k)}(z),\omega(z),\omega'(z),\ldots,\omega^{(k)}(z)\Big)\\
&=e^{(n+\tau)\beta(z)+\gamma(z)}(P_1e^{m\beta(z)}-P_2).
\end{aligned}$$

Obviously $P_1e^{m\beta(z)}-P_2$ has infinite number of zeros, which contradicts with the fact that $g(z)$ is an entire function.

If $B\neq -1$, from (13) we have $\frac{1}{F}=\frac{BG}{(1+B)G-1}$ and so $\overline{N}(r,\frac{1}{1+B};G)=\overline{N}(r,0;F)$. Using Lemmas 3, 4, 6, 7, 11 and the second fundamental theorem of Nevanlinna, we deduce that

$$\begin{aligned}
T(r,G) &\leqslant \overline{N}(r,0;G)+\overline{N}\left(r,\frac{1}{1+B};G\right)+\overline{N}(r,\infty;G)+S(r,G)\\
&\leqslant \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+S(r,G)\\
&\leqslant N_{k+1}(r,0;F_1)+T(r,G)+N_{k+1}(r,0;G_1)\\
&\quad -(n+m+\sigma+\tau)T(r,g)+O(r^{\rho(g)-1+\varepsilon})+S(r,g).
\end{aligned}$$

This gives

$$\begin{aligned}
&(n+m+\sigma+\tau)T(r,g)\\
\leqslant\ & N_{k+1}(r,0;F_1)+N_{k+1}(r,0;G_1)+O(r^{\rho(g)-1+\varepsilon})+S(r,g)\\
\leqslant\ & (k+1)(\overline{N}(r,0;f)+\overline{N}(r,0;g))+N(r,1;f^m)+N(r,1;g^m)\\
&+N(r,0;\prod_{i=1}^{p}f(z+\eta_i)^{\nu_i})+N(r,0;\prod_{i=1}^{p}g(z+\eta_i)^{\nu_i})+N(r,0;\prod_{j=1}^{q}f^{(j)}(z)^{\mu_j})\\
&+N(r,0;\prod_{j=1}^{q}g^{(j)}(z)^{\mu_j})+O(r^{\rho(g)-1+\varepsilon})+S(r,f)+S(r,g)\\
\leqslant\ & (k+m+\sigma+\tau+1)(T(r,f)+T(r,g))+O(r^{\rho(g)-1+\varepsilon})+S(r,f)+S(r,g).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&(n-2k-m-\sigma-\tau-2)(T(r,f)+T(r,g))\\
&\leqslant O(r^{\rho(f)-1+\varepsilon})+O(r^{\rho(g)-1+\varepsilon})+S(r,f)+S(r,g),
\end{aligned}$$

a contradiction since $n\geqslant 2k+m+\sigma+\tau+5$.

Case 2. Let $B\neq 0$ and $A\neq B$. Then from (12) we get $F=\frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and so $\overline{N}(r,\frac{B-A+1}{B+1};G)=\overline{N}(r,0;F)$. Arguing similarly as in Case 1 we arrive at a contradiction.

Case 3. Let $B=0$ and $A\neq 0$. Then from (12) we get $F=\frac{G+A-1}{A}$ and $G=AF-(A-1)$. If $A\neq 1$, it follows that $\overline{N}(r,\frac{A-1}{A};F)=\overline{N}(r,0;G)$ and $\overline{N}(r,1-A;G)=\overline{N}(r,0;F)$. Now applying Lemma 12 it can be shown that $n\leqslant 2k+m+\sigma+\tau+2$, a contradiction. Thus $A=1$ and then $F=G$. Thus

$$\begin{aligned}
&(f^n(z)(f^m(z)-1)\prod_{i=1}^{p}f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}f^{(j)}(z)^{\mu_j})^{(k)}\\
&=(g^n(z)(g^m(z)-1)\prod_{i=1}^{p}g(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}g^{(j)}(z)^{\mu_j})^{(k)}.
\end{aligned}$$

Integrating once we obtain

$$\begin{aligned}&(f^n(z)(f^m(z)-1)\prod_{i=1}^{p}f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}f^{(j)}(z)^{\mu_j})^{(k-1)}\\&=(g^n(z)(g^m(z)-1)\prod_{i=1}^{p}g(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}g^{(j)}(z)^{\mu_j})^{(k-1)}+c_{k-1},\end{aligned}$$

where c_{k-1} is a constant. If $c_{k-1}\neq 0$, using Lemma 12 it follows that $n\leqslant 2k+m+\sigma+\tau$, a contradiction. Hence $c_{k-1}=0$. Repeating the process k-times, we deduce that

$$\begin{aligned}&f^n(z)(f^m(z)-1)\prod_{i=1}^{p}f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}f^{(j)}(z)^{\mu_j}\\&=g^n(z)(g^m(z)-1)\prod_{i=1}^{p}g(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}g^{(j)}(z)^{\mu_j}.\end{aligned}\tag{14}$$

Set $t(z)=\frac{f(z)}{g(z)}$. If $t(z)$ is constant, then substituting $f(z)=tg(z)$ in (14), we deduce that

$$g^m(z)(t^{n+m+\sigma+\tau}-1)-(t^{n+\sigma+\tau}-1)=0,$$

as $g(z)$ is a transcendental entire function. From above we get $t^{n+\sigma+\tau}=t^m=1$. If $t(z)$ is not constant then from (14), $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g)=0$, where $R(f,g)$ is given by

$$\begin{aligned}R(f,g)\,=\,&f^n(z)(f^m(z)-1)\prod_{i=1}^{p}f(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}f^{(j)}(z)^{\mu_j}-g^n(z)(g^m(z)-1)\\&\times\prod_{i=1}^{p}g(z+\eta_i)^{\nu_i}\prod_{j=1}^{q}g^{(j)}(z)^{\mu_j}.\end{aligned}$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. Let F, G, F_1 and G_1 be defined as in Theorem 1. Then F and G are transcendental meromorphic functions that share $(1,2)^*$ except the zeros and poles of $\alpha(z)$. Let $H\not\equiv 0$. In a similar manner of Theorem 1, using Lemma 9, we obtain

$$\begin{aligned}&(n+m+\sigma+\tau)T(r,f)\\&\leqslant N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+\overline{N}(r,0;F)+\overline{N}(r,\infty;F)\\&\quad+N_{k+2}(r,0;F_1)+O(r^{\rho(f)-1+\varepsilon})+S(r,f)+S(r,g)\\&\leqslant N_{k+2}(r,0;F_1)+N_{k+2}(r,0;G_1)+N_{k+1}(r,0;F_1)\\&\quad+O\{r^{\rho(f)-1+\varepsilon}\}+S(r,f)+S(r,g)\end{aligned}$$

$$\leqslant (2k+2m+2\sigma+2\tau+3)T(r,f)+(k+m+\sigma+\tau+2)T(r,g) \\ +O(r^{\rho(f)-1+\varepsilon})+S(r,f)+S(r,g). \quad (15)$$

In a similar manner we obtain

$$(n+m+\sigma+\tau)T(r,g) \\ \leqslant (2k+2m+2\sigma+2\tau+3)T(r,g)+(k+m+\sigma+\tau+2)T(r,f) \\ +O(r^{\rho(g)-1+\varepsilon})+S(r,f)+S(r,g). \quad (16)$$

From (15) and (16) we get

$$(n-3k-2m-2\sigma-2\tau-5)(T(r,f)+T(r,g)) \\ \leqslant O(r^{\rho(f)-1+\varepsilon})+O(r^{\rho(g)-1+\varepsilon})+S(r,f)+S(r,g),$$

contradicting with the fact that $n \geqslant 3k+2m+2\sigma+2\tau+6$. Thus we must have $H=0$. Then the result follows from the proof of Theorem 1. This completes the proof of Theorem 2. □

Proof of Theorem 3. Let F, G, F_1 and G_1 be similar as in Theorem 1. Then F and G are transcendental meromorphic functions such that $\overline{E}_{2)}(1,F)=\overline{E}_{2)}(1,G)$ except the zeros and poles of $\alpha(z)$. Let $H \not\equiv 0$. In a similar manner of Theorem 1, using Lemma 10, we obtain

$$(n+m+\sigma+\tau)T(r,f) \\ \leqslant N_2(r,0;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+N_{k+2}(r,0;F_1) \\ +O(r^{\rho(f)-1+\varepsilon})+S(r,f)+S(r,g) \\ \leqslant N_{k+2}(r,0;F_1)+N_{k+2}(r,0;G_1)+2N_{k+1}(r,0;F_1)+N_{k+1}(r,0;G_1) \\ +O(r^{\rho(f)-1+\varepsilon})+S(r,f)+S(r,g) \\ \leqslant (3k+3m+3\sigma+3\tau+4)T(r,f)+(2k+2m+2\sigma+2\tau+3)T(r,g) \\ +O(r^{\rho(f)-1+\varepsilon})+S(r,f)+S(r,g). \quad (17)$$

Similarly,

$$(n+m+\sigma+\tau)T(r,g) \\ \leqslant (3k+3m+3\sigma+3\tau+4)T(r,g)+(2k+2m+2\sigma+2\tau+3)T(r,f) \\ +O(r^{\rho(g)-1+\varepsilon})+S(r,f)+S(r,g). \quad (18)$$

Combining (17) and (18) we obtain

$$(n-5k-4m-4\sigma-4\tau-7)(T(r,f)+T(r,g)) \\ \leqslant O(r^{\rho(f)-1+\varepsilon})+O(r^{\rho(g)-1+\varepsilon})+S(r,f)+S(r,g),$$

a contradiction with the assumption that $n \geqslant 5k+4m+4\sigma+4\tau+8$. Thus $H=0$ and the rest of the theorem follows from the proof of Theorem 1. This completes the proof of Theorem 3. □

OPEN PROBLEM. *In the paper we give a question for further research.*

QUESTION 4. What about the problems 1–3, if we replace the differential difference polynomial in the form $f^n(z)(f^m(z)-1)\prod_{i=1}^{p} f(z+\eta_i)^{\nu_i}\sum_{h=1}^{r} a_h(z)\prod_{j=1}^{q} f^{(j)}(z)^{\mu_{hj}}$?

Acknowledgement. The author is grateful to the referee for his/her valuable suggestions and comments towards the improvement of the paper.

REFERENCES

[1] T. ALZAHARY, *Uniqueness of meromorphic functions with weakly weighted sharing*, Bull. Aust. Math. Soc., **88**, (2013), 26–43.

[2] A. BANERJEE AND S. MUKHERJEE, *Uniqueness of meromorphic functions concerning differential monomials sharing the same value*, Bull. Math. Soc. Sci. Math. Roum., **50**, (2007), 191–206.

[3] A. BANERJEE AND S. MUKHERJEE, *Nonlinear differential polynomials sharing a small function*, Archivum Mathematicum, **44**, (2008), 41–56.

[4] W. BERGWEILER AND J. K. LANGLEY, *Zeros of differences of meromorphic functions*, Math. Proc. Camb. Phil. Soc., **142**, (2007), 133–147.

[5] Y. M. CHIANG AND S. J. FENG, *On the Nevanlinna Characteristic $f(z+\eta)$ and difference equations in complex plane*, Ramanujan J., **16** (2008), 105–129.

[6] M. L. FANG AND W. HONG, *A unicity theorem for entire functions concerning differential polynomials*, Indian J. Pure Appl. Math., **32** (2001), 1343–1348.

[7] W. K. HAYMAN, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).

[8] V. HUSNA, S. RAJESHWARI AND S. H. NAVEEN KUMAR, *A note on uniqueness of transcendental entire functions concerning differential difference polynomials of finite order*, Electronic Journal of Mathematical Analysis and Applications, **9** (2021), 248–260.

[9] S. KARAGUPPI, V. NAGARJUN AND G. SHARMA, *Uniqueness results related to value distribution of entire and meromorphic functions concerning difference polynomials*, International Journal of Mathematics Trends and Technology, **67** (2021), 112–124.

[10] I. LAHIRI, *Value distribution of certain differential polynomials*, Int. J. Math. Math. Sc., **28** (2001), 83–91.

[11] I. LAHIRI, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl., **46** (2001), 241–253.

[12] I. LAHIRI AND B. PAL, *Uniqueness of meromorphic functions with their homogeneous and linear differential polynomials sharing a small function*, Bull. Korean Math. Soc., **54** (2017), 825–838.

[13] I. LAINE, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin/Newyork (1993).

[14] I. LAINE AND C. C. YANG, *Value distribution of difference polynomials*, Proc. Japan Acad., Ser. A, Math. Sci., **83** (2007), 148–151.

[15] S. H. LIN AND W. C. LIN, *Uniqueness of meromorphic functions concerning weakly weighted sharing*, Kodai Math. J., **29** (2006), 269–280.

[16] W. C. LIN AND H. X. YI, *Uniqueness theorems for meromorphic functions concerning fixed points*, Complex Variables Theory Appl., **49** (2004), 793–806.

[17] K. LIU AND L. Z. YANG, *Value distribution of the difference operator*, Arch. Math. Basel, **92** (2009), 270–278.

[18] X. LUO AND W. C. LIN, *Value sharing results for shifts of meromorphic functions*, J. Math. Anal. Appl., **377** (2011), 441–449.

[19] X. Q. LIN AND W. C. LIN, *Uniqueness of entire functions sharing one value*, Acta Math. Sci., Ser. B, Engl. Ed., **31** (2011), 1062–1076.

[20] C. MENG, *Uniqueness for meromorphic functions and differential polynomials*, Turk. J. Math., **33** (2009), 331–340.

[21] C. MENG, *On unicity of meromorphic function and its kth order derivative*, Journal of Mathematical Inequalities, **4** (2010), 151–159.

[22] C. MENG, *Uniqueness of entire functions concerning difference polynomials*, Math. Bohemica, **139** (2014), 89–97.
[23] D. C. PRAMANIK AND J. ROY, *Weakly weighted-sharing and uniqueness of homogeneous differential polynomials*, Mat. Stud., **51** (2019), 41–49.
[24] D. C. PRAMANIK AND J. ROY, *Uniqueness of homogeneous differential polynomials of meromorphic functions concerning weakly weighted sharing*, Journal of Classical Analysis, **16** (2020), 23–35.
[25] D. C. PRAMANIK AND J. ROY, *Uniqueness of p(f) and P[f] concerning weakly weighted-sharing*, General Mathematics, **28** (2020), 11–24.
[26] X. G. QI, L. Z. YANG AND K. LIU, *Uniqueness and periodicity of meromorphic functions concerning the difference operator*, Computers and Mathematics with Applications, **60** (2010), 1739–1746.
[27] P. SAHOO, *Uniqueness of entire functions related to difference polynomials*, Commu. Math. Stat., **3** (2015), 227–238.
[28] P. SAHOO AND G. BISWAS, *Some results on uniqueness of entire functions concerning difference polynomials*, Tamkang journal of Mathematics, **49** (2018), 85–97.
[29] C. C. YANG, *On deficiencies of differential polynomials II*, Math. Z., **125** (1972), 107–112.
[30] C. C. YANG AND H. X. YI, *Uniqueness Theory of Meromorphic Functions*, Kluwer, Dordrecht (2003).
[31] J. L. ZHANG, *Value distribution and shared sets of difference of meromorphic functions*, J. Math. Anal. Appl., **367** (2010), 401–408.
[32] J. L. ZHANG AND L. Z. YANG, *Some results related to a conjecture of R. Bruck*, J. Inequal. Pure Appl. Math., **8** (2007), Art. 18.

(Received December 14, 2023)

Gurudas Biswas
Department of Mathematics
Hooghly Women's College
West Bengal-712103, India
e-mail: gdb.math@gmail.com