Journal of Classical Analysis
Volume 28, Number 1 (2026), 151–163
doi:10.7153/jca-2026-28-10

LANDAU'S THEOREM FOR BIHARMONIC MAPPINGS IN SIMPLY CONNECTED DOMAINS

K. P. BALAKRISHNAN AND S. SUNIL VARMA *

Abstract. Let $\mathscr{B}_{\mathscr{BH},\Omega}(\alpha)$ symbolize the family of biharmonic α-Bloch mappings defined on a simply connected domain $\Omega \subset \mathbb{C}$ with a parameter $\alpha > 0$. In this work, we establish a Landau-type theorem for this class by extending the results for harmonic α-Bloch mappings to the biharmonic framework. Our analysis is carried out on the shifted disk

$$\Omega_\gamma := \left\{ \zeta \in \mathbb{C} : \left| \zeta + \frac{\gamma}{1-\gamma} \right| < \frac{1}{1-\gamma} \right\}, \qquad 0 \leqslant \gamma < 1,$$

which serves as a natural generalization of the unit disk. The estimates derived here provide explicit univalence radii and image bounds for biharmonic α-Bloch mappings on Ω_γ.

1. Introduction

Landau's theorem, a cornerstone of geometric function theory, explains the behavior of holomorphic mappings within the unit disk $\mathbb{D}$. It asserts that for a bounded holomorphic function f defined on the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$, there exists a positive constant r_0, such that f is univalent in $\mathbb{D}_{r_0} = \{\zeta \in \mathbb{C} : |\zeta| < r_0\}$ and the image $f(\mathbb{D}_{r_0})$ contains a disk of fixed radius independent of f. In a complementary direction, Bloch's theorem ensures that every normalized holomorphic function maps some subdisk of $\mathbb{D}$ onto a disk by means of radius that works for every normalized holomorphic function, not depending on f, known as the *Bloch constant*.

Colonna [13] (1987) investigated bounded analytic self-maps of the unit disk and established the sharp upper bound for their Bloch seminorm, showing that

$$B(f) = \sup_{\zeta \in \mathbb{D}} (1 - |\zeta|^2) |f'(\zeta)| = 1. \tag{1}$$

The extremals are exactly disk automorphisms and certain functions with infinitely many zeros satisfying a pseudohyperbolic clustering condition, while nonvanishing inner, outer, and finite Blaschke products are not extremal.

Clunie and Sheil-Small [12] (1984) proposed harmonic mappings represented as $f = h + \overline{g}$, with both h and g holomorphic in $\mathbb{D}$. They developed criteria for univalence, obtained distortion and covering results, and analyzed geometric characteristics

Mathematics subject classification (2020): 30A10, 30B10, 30C50, 30C55, 30C62, 30F45, 30H30, 31A05, 30C20, 30C65.

Keywords and phrases: Landau's theorem, biharmonic mappings, simply connected domain.

* Corresponding author.

© ELEMENT, Zagreb
Paper JCA-28-10

of these mappings. Colonna [14] (1989) explored bounded harmonic mappings and introduced the harmonic Bloch constant

$$B(f) = \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)\left(|h'(\zeta)|+|g'(\zeta)|\right)$$

and gave sharp, best-possible bounds for harmonic self-maps of $\mathbb{D}$, clarifying how $\|h'\|$, $\|g'\|$ (equivalently, the dilatation) control geometric distortion.

Duren [16] (2004) provided an extensive treatment of harmonic mappings, addressing their analytic representations, Jacobian properties, and dilatation behavior. Grigoryan [17] investigated harmonic mappings in $\mathbb{D}$ and obtained Landau- and Bloch-type estimates under the distortion condition.

Dorff and Nowak [15] (2004) derived explicit estimates in 2004 for the univalence radius and the size of the image disk for harmonic mappings in $\mathbb{D}$, extending the classical Landau theorem.

Xinzhong [23] (2008) investigated the Bloch behaviour of planar harmonic mappings and obtained sharp two-sided estimates for the harmonic Bloch constant. His results improved previously known bounds for harmonic Bloch mappings.

Liu [18] (2009) strengthened Landau-type results for planar harmonic mappings by obtaining sharper univalence and image-radius estimates under natural normalization and distortion conditions. His work refined earlier bounds and extended the results of Dorff and Nowak [15].

Beardon and Minda [6] (2007) demonstrated that many classical results in geometric function theory can be interpreted more transparently through the hyperbolic geometry of the unit disk. Their work shows that the underlying hyperbolic structure plays a key role in the distortion and covering behaviour of analytic and harmonic mappings.

Bonk, Minda, and Yanagihara [7] (1995) investigated the hyperbolic metric on Bloch regions.The hyperbolic metric on Ω is quantitatively governed by the boundary distance. One convenient way to express this is

$$\lambda_\Omega(\zeta)\,\mathrm{dist}(\zeta,\partial\Omega)\in\left[\frac{1}{2},1\right],$$

which is equivalent to the usual two-sided estimate.

Chen, Gauthier, and Hengartner [8] (2008) studied Bloch constants for planar harmonic mappings and derived sharp two-sided estimates for the harmonic Bloch constant. Their results extend the classical Bloch theorem to the harmonic case and clarify the dependence of the Bloch constant on the dilatation parameter. Chen, Ponnusamy, and Wang [10] (2011) investigated Landau-type estimates and the Marden constant within the setting of harmonic ν-Bloch functions, a family that extends the classical Bloch space by allowing a positive parameter ν.

Abdulhadi and Abu Muhanna [2] (2008) proved a Landau-type theorem for a class of biharmonic functions. They considered expressions of the form

$$F(\zeta) = |\zeta|^2 G(\zeta) + H(\zeta)$$

with both G and H harmonic in $\mathbb{D}$ and are bounded by M, and obtained an explicit radius ensuring univalence. Chen, Ponnusamy, and Wang [9] (2009) studied a biharmonic extension of Landau's theorem involving the operator

$$L = \zeta \frac{\partial}{\partial \zeta} - \overline{\zeta} \frac{\partial}{\partial \overline{\zeta}}.$$

Chen, Rasila, and Wang [11] (2014) investigated Landau-type results for a broader class of polyharmonic functions. They worked with representations of the form

$$F(\zeta) = \sum_{k=1}^{n} |\zeta|^{2(k-1)} H_k(\zeta),$$

in which each H_k is a bounded harmonic function in $\mathbb{D}$, and derived corresponding univalence estimates. Under suitable normalization and boundedness conditions, they obtained explicit estimates for the radius of univalence and the size of the image disk, thus generalizing classical Landau's constant from harmonic and biharmonic mappings to the polyharmonic mapping.

Shi-Fei Chen and M S Liu [19] (2020) developed several new Landau-type theorems for classes of bounded biharmonic mappings defined on the unit disk under normalization conditions on the Jacobian. Sharp univalence radii and covering results are obtained, improving and extending earlier results in the literature. In addition, the authors establish bi-Lipschitz properties for these biharmonic mappings on their corresponding univalent disks. Ping Li *et.al* [21] (2026) investigated Landau-type theorems for bounded polyanalytic functions on the unit disk and derives explicit univalence and covering radii. The results are further extended to obtain corresponding Landau-type theorems for log-α-analytic functions.

Allu and Halder [4] (2015) established Landau's theorem for harmonic α-Bloch functions on simply connected domains, deriving bounds in terms of analytic derivatives. In a related work [5], they examined Bloch norms and the Bohr phenomenon for harmonic α-Bloch functions, connecting geometric and holomorphic aspects of Bloch spaces.This gradual development from holomorphic to harmonic, biharmonic, and polyharmonic mappings illustrates the systematic expansion of Bloch-type spaces in geometric function theory.

Motivated by these developments, the present work investigates *biharmonic α-Bloch mappings* on simply connected domains, particularly on the shifted disk

$$\Omega_\gamma := \left\{ \zeta \in \mathbb{C} : \left| \zeta + \frac{\gamma}{1-\gamma} \right| < \frac{1}{1-\gamma} \right\}, \qquad 0 \leqslant \gamma < 1,$$

and establishes a Landau-type theorem. It may be observed that Ω_γ contains the open unit disk $\mathbb{D}$. This study extends the existing framework and unifies harmonic and biharmonic Bloch-type mappings within geometric function theory.

2. Preliminaries

Let $\alpha > 0$. A holomorphic function f on $\mathbb{D}$ is said to lie in the class $\mathscr{B}(\alpha)$ whenever it satisfies

$$\beta(\alpha) = \sup_{\zeta\in\mathbb{D}}(1-|\zeta|^2)^\alpha |f'(\zeta)| < \infty.$$

Equipped with the corresponding Bloch norm $\|f\|_\alpha = |f(0)| + \beta(\alpha)$, this class becomes a Banach space. This corresponds to the special case $\alpha = 1$ [13].

Consider a simply connected domain $\Omega \subset \mathbb{C}$ and choose a conformal mapping $f : \Omega \to \mathbb{D}$. Using this map, one may describe the hyperbolic density on Ω by

$$\lambda_\Omega(\zeta) = \lambda_\mathbb{D}(f(\zeta))\,|f'(\zeta)|,$$

and the resulting function does not depend on which conformal map is selected; see [6, 7]. An holomorphic function f is said to be an α-Bloch function in Ω if

$$\beta_\Omega(\alpha) = \sup_{\zeta\in\Omega} \frac{|f'(\zeta)|}{\lambda_\Omega^\alpha(\zeta)} < \infty.$$

A complex-valued function f, twice continuously differentiable on Ω, is harmonic if it fulfills the Laplace equation

$$\Delta f = 0, \qquad \text{where} \quad \Delta = 4\frac{\partial^2}{\partial\zeta\,\partial\overline{\zeta}}.$$

Every harmonic function defined on Ω splits into an analytic component and a conjugate-analytic component. Consequently, one may write $f(\zeta) = h(\zeta) + \overline{g(\zeta)}$, with h and g holomorphic in Ω [12, 16].

The function is locally one-to-one and orientation-preserving when its Jacobian

$$J_f(\zeta) = |h'(\zeta)|^2 - |g'(\zeta)|^2$$

is positive.

Drawing from this holomorphic foundation, Allu and Halder [4, 5] formulated the class of *harmonic α-Bloch mappings*, denoted by $\mathscr{B}_{H,\Omega}(\alpha)$, defined by

$$\beta_{\mathscr{H},\Omega}(\alpha) = \sup_{\zeta\in\Omega} \frac{|h'(\zeta)| + |g'(\zeta)|}{\lambda_\Omega^\alpha(\zeta)} < \infty.$$

This class extends both holomorphic and harmonic Bloch function spaces under a unified α-Bloch norm $||.||_{\mathscr{H},\Omega.\alpha} := |f(0)| + \beta_{\mathscr{H},\Omega}(\alpha)$. They further established the following results related to Landau's constants for harmonic α-Bloch mappings.

LEMMA 1. ([4]) *Let $f_1 = h_1 + \overline{g_1}$ be harmonic in $\mathbb{D}$ and suppose that the analytic and co-analytic part admit the series*

$$h_1(\zeta) = \sum_{n=1}^{\infty} \alpha_n(\zeta-\gamma)^n, \qquad g_1(\zeta) = \sum_{n=1}^{\infty} \beta_n(\zeta-\gamma)^n,$$

valid for $|\zeta-\gamma|<1-\gamma$. *Assume further that* $\lambda_f(\gamma)=\beta$ *for some* $\beta\in(0,1]$, *and that*

$$\|f\|_{\mathscr{H},\mathbb{D}}(\alpha)\leqslant M(1-\gamma)^{\alpha},\qquad M>0.$$

Then for every $n\geqslant 2$,

$$|\alpha_n|+|\beta_n|\leqslant \inf_{0<r<1}\mu(r),$$

where

$$\mu(r)=\frac{m^2(r,\gamma)-\beta^2}{n\,r^{n-1}(1-\gamma)^{n-1}(1-\gamma^2)\,m(r,\gamma)},$$

and

$$m(r,\gamma)=\frac{M(1-\gamma)^{\alpha}}{\left(1-(1-\gamma)^2r^2\right)^{\alpha}}.$$

THEOREM 1. ([4]) *Let* $f=h+g$ *be a harmonic mapping in the shifted disk* Ω_γ *with*

$$h(\zeta)=\sum_{n=1}^{\infty}a_n\zeta^n,\qquad g(\zeta)=\sum_{n=1}^{\infty}b_n\zeta^n.$$

Assume that $\lambda_f(0)=\lambda$ *and* $\|f\|_{H,\Omega_\gamma}(\alpha)\leqslant L$. *Then for every integer* $n\geqslant 2$,

$$|a_n|+|b_n|\leqslant C_{n,M}(\alpha,\lambda,\gamma),\tag{2}$$

where

$$C_{n,M}(\alpha,\lambda,\gamma)=\inf_{0<r<1}\frac{m^2(r,\gamma)-\dfrac{\lambda^2}{(1-\gamma)^2}}{nr^{n-1}(1+\gamma)\,m(r,\gamma)},$$

and

$$m(r,\gamma)=\frac{L}{1-\gamma}\frac{(1-\gamma)^{\alpha}}{(1-(1-\gamma)^2r^2)^{\alpha}}.$$

By using the above results they estimated Landau's theorem as follows

THEOREM 2. ([4]) *Let* $f=h+g$ *be a harmonic mapping on* γ *satisfying* $f(0)=0$, $\lambda_f(0)=\lambda>0$, *and* $\|f\|_{H,\gamma(\alpha)}\leqslant M$. *Assume that*

$$h(\zeta)=\sum_{n=1}^{\infty}a_n\zeta^n,\qquad g(\zeta)=\sum_{n=1}^{\infty}b_n\zeta^n,\quad \zeta\in\mathbb{D},$$

and that the coefficients satisfy $|a_n|+|b_n|\leqslant C_{n,M}(\alpha,\lambda,\gamma)$ *for all* $n\geqslant 2$.

Then f *is univalent in the disk* $\mathbb{D}_{\rho_0}$, *where* $\rho_0\in(0,1)$ *is the unique solution to*

$$\sum_{n=2}^{\infty}C_{n,M}(\alpha,\lambda,\gamma)\,n\,\rho^{n-1}=\lambda.$$

Furthermore, the image $f(\mathbb{D}_{\rho_0})$ *contains the disk* $\mathbb{D}_{\rho}$, *where*

$$\rho=\lambda\,\rho_0-\sum_{n=2}^{\infty}C_{n,M}(\alpha,\lambda,\gamma)\,\rho_0^n.$$

Additional studies concerning harmonic mappings and Bloch constants are available in [8, 15, 17, 23]. While their results were confined to harmonic mappings, we aim to obtain Landau's theorem for biharmonic mappings defined on similar domains [2, 9, 11]. This generalization enhances the existing theory by bridging harmonic and biharmonic Bloch-type spaces within geometric function theory.

A function $F = u + iv$, assumed to be four times continuously differentiable on a domain $D \subset \mathbb{C}$, is said to be *biharmonic* provided that its first Laplacian ΔF turns out to be harmonic [1, 2]. Equivalently, F satisfies the biharmonic equation

$$\Delta(\Delta F) = 0.$$

A mapping F defined in a simply connected domain D is biharmonic if and only if it can be represented as

$$F(\zeta) = |\zeta|^2 G(\zeta) + K(\zeta), \qquad \zeta = re^{i\theta} \in D,$$

where G and K are harmonic in D [16]. Each of these harmonic components takes the form

$$G = \mathscr{G}_1 + \overline{\mathscr{G}_2}, \qquad K = \mathscr{K}_1 + \overline{\mathscr{K}_2},$$

where $\mathscr{G}_1, \mathscr{G}_2, \mathscr{K}_1$, and $\mathscr{K}_2$ are holomorphic in D.

For a biharmonic mapping W, the Jacobian is given by

$$J_W = |W_\zeta|^2 - |W_{\bar{\zeta}}|^2,$$

Defining

$$\lambda_W = |W_\zeta| - |W_{\bar{\zeta}}|, \qquad \Lambda_W = |W_\zeta| + |W_{\bar{\zeta}}|,$$

we obtain $J_W = \lambda_W \Lambda_W$.

Following the investigations of Allu and Halder [4], where Landau-type radius bounds were derived for harmonic functions belonging to the α-Bloch family on simply connected domains,in this paper we first extend *harmonic α-Bloch mappings* to *biharmonic α-Bloch mappings*. Further we determine Landau's constant for biharmonic mapping in simply connected domain defined using holomorphic function in $\mathbb{D}$.

DEFINITION 1. Let $\alpha \in (0, \infty)$. A biharmonic mapping

$$F(\zeta) = |\zeta|^2 G(\zeta) + K(\zeta)$$

defined on a simply connected domain $\Omega \subset \mathbb{C}$ is called a *biharmonic α-Bloch mapping* if

$$\beta_{\mathscr{BH},\Omega}(\alpha) := \sup_{\zeta \in \Omega} \frac{|F_\zeta(\zeta)| + |F_{\bar{\zeta}}(\zeta)|}{\lambda_\Omega^\alpha(\zeta)} < \infty,$$

where the hyperbolic metric $\lambda_\Omega(\zeta)$ is given by

$$\lambda_\Omega(\zeta) = \lambda_{\mathbb{D}}(f(\zeta))\,|f'(\zeta)|, \qquad \zeta \in \Omega,$$

for some conformal map $f:\Omega\to\mathbb{D}$, and

$$\lambda_{\mathbb{D}}(\zeta)=\frac{1}{1-|\zeta|^2},\qquad \zeta\in\mathbb{D}.$$

The class of all biharmonic α-Bloch mappings on Ω is denoted by $\mathscr{B}_{\mathscr{BH},\Omega}(\alpha)$. The corresponding norm is defined as

$$\|F\|_{\mathscr{BH},\Omega,(\alpha)}:=|F(0)|+\beta_{\mathscr{BH},\Omega}(\alpha).$$

3. Main results

THEOREM 3. *Let $F(\zeta)=|\zeta|^2G(\zeta)+K(\zeta)$ be a biharmonic mapping in Ω_γ with $G=g_1+\overline{g_2}$, $K=k_1+\overline{k_2}$, $F(0)=\lambda_K(0)-\lambda=0$ and $\|G\|_{\mathscr{H},\Omega_\gamma}(\alpha)\leqslant M_1$ and $\|K\|_{\mathscr{H},\Omega_\gamma}(\alpha)\leqslant M_2$ with*

$$g_1(\zeta)=\sum_{n=1}^{\infty}a_n\zeta^n,\quad g_2(\zeta)=\sum_{n=1}^{\infty}b_n\zeta^n,\quad k_1(\zeta)=\sum_{n=1}^{\infty}c_n\zeta^n,\quad k_2(\zeta)=\sum_{n=1}^{\infty}d_n\zeta^n \text{ in } \mathbb{D}.$$

Then F is univalent in $\mathbb{D}_{\rho_0}:=\{\zeta:|\zeta|<\rho_0\}$, where ρ_0 is the unique root of

$$\sum_{n=1}^{\infty}C_{n,M_1}(\alpha,\lambda,\gamma)(n+2)r^{n+1}+\sum_{n=2}^{\infty}C_{n,M_2}(\alpha,\lambda,\gamma)nr^{n-1}=\lambda$$

in $(0,1)$. Further, $F(\mathbb{D}_{\rho_0})$ contains the disk $\mathbb{D}_{\rho_1}$, where

$$\rho_1=\lambda\rho_0-\sum_{n=1}^{\infty}C_{n,M_1}(\alpha,\lambda,\gamma)\rho_0^{n+2}-\sum_{n=2}^{\infty}C_{n,M_2}(\alpha,\lambda,\gamma)\rho_0^{n}.$$

Proof. For fixed $0<r<1$, choose distinct ζ_1,ζ_2, with $|\zeta_1|<r$ and $|\zeta_2|<r$. Then

$$\begin{aligned}F(\zeta_1)-F(\zeta_2)&=\int_{[\zeta_1,\zeta_2]}F_\zeta(\zeta)\,d\zeta+F_{\overline{\zeta}}(\zeta)\,d\overline{\zeta}\\&=\int_{[\zeta_1,\zeta_2]}\left(\zeta G(\zeta)+r^2G_\zeta(\zeta)+K_\zeta(\zeta)\right)d\zeta\\&\quad+\left(\overline{\zeta}G(\zeta)+r^2G_{\overline{\zeta}}(\zeta)+K_{\overline{\zeta}}(\zeta)\right)d\overline{\zeta},\end{aligned}$$

where $[\zeta_1,\zeta_2]$ is the line segment from ζ_1 to ζ_2. Since

$$J_F(0)=\lambda_K(0)=\lambda,$$

we have

$$
\begin{aligned}
&|F(\zeta_1)-F(\zeta_2)|\\
&\geqslant \left|\int_{[\zeta_1,\zeta_2]}\left(K_{\zeta}(0)\,d\zeta+K_{\overline{\zeta}}(0)\,d\overline{\zeta}\right)\right|\\
&\quad-\left|\int_{[\zeta_1,\zeta_2]}G(\zeta)(\zeta\,d\zeta+\overline{\zeta}\,d\overline{\zeta})+\int_{[\zeta_1,\zeta_2]}r^2\left(G_{\zeta}(\zeta)\,d\zeta+G_{\overline{\zeta}}(\zeta)\,d\overline{\zeta}\right)\right.\\
&\qquad\left.+\int_{[\zeta_1,\zeta_2]}\left(K_{\zeta}(\zeta)-K_{\zeta}(0)\right)d\zeta+\int_{[\zeta_1,\zeta_2]}\left(K_{\overline{\zeta}}(\zeta)-K_{\overline{\zeta}}(0)\right)d\overline{\zeta}\right|\\
&\geqslant |\zeta_2-\zeta_1|\left(\lambda-2r|G(\zeta)|-\sum_{n=1}^{\infty}(|a_n|+|b_n|)nr^{n+1}-\sum_{n=2}^{\infty}(|c_n|+|d_n|)nr^{n-1}\right)\\
&=|\zeta_2-\zeta_1|\left(\lambda-\sum_{n=1}^{\infty}C_{n,M_1}(\alpha,\lambda,\gamma)(n+2)r^{n+1}-\sum_{n=2}^{\infty}C_{n,M_2}(\alpha,\lambda,\gamma)nr^{n-1}\right)\\
&=:|\zeta_2-\zeta_1|\Phi_1(r).
\end{aligned}
$$

where

$$\Phi_1(r)=\lambda-\sum_{n=1}^{\infty}C_{n,M_1}(\alpha,\lambda,\gamma)(n+2)r^{n+1}-\sum_{n=2}^{\infty}C_{n,M_2}(\alpha,\lambda,\gamma)nr^{n-1}$$

It is clear that Φ_1 is continuous and differentiable in $(0,1)$ and

$$\Phi_1'(r)=-\left[\sum_{n=1}^{\infty}C_{n,M_1}(\alpha,\lambda,\gamma)(n+2)(n+1)r^{n}+\sum_{n=2}^{\infty}C_{n,M_2}(\alpha,\lambda,\gamma)n(n-1)r^{n-2}\right]<0,$$

for $r\in(0,1)$, which shows that Φ_1 is decreasing. Since $\Phi_1(0)=\lambda>0$ and $\lim_{r\to 1^-}\Phi_1(r)=-\infty$, it follows that Φ_1 possesses a unique root in $(0,1)$. Choose ρ_0 so that $|F(\zeta_1)-F(\zeta_2)|>0$ for any $r\in(0,\rho_0)$. This shows that F is univalent in $\mathbb{D}_{\rho_0}$.

For $|\zeta|=\rho_0$,

$$
\begin{aligned}
|F(\zeta)|&\geqslant |c_1\zeta+d_1\overline{\zeta}|-\rho_0^2\left|\sum_{n=1}^{\infty}(a_n\zeta^n+b_n\overline{\zeta}^n)\right|-\left|\sum_{n=2}^{\infty}(c_n\zeta^n+d_n\overline{\zeta}^n)\right|\\
&\geqslant \lambda\rho_0-\sum_{n=1}^{\infty}(|a_n|+|b_n|)\rho_0^{n+2}-\sum_{n=2}^{\infty}(|c_n|+|d_n|)\rho_0^{n}\\
&\geqslant \lambda\rho_0-\sum_{n=1}^{\infty}C_{n,M_1}(\alpha,\lambda,\gamma)\rho_0^{n+2}-\sum_{n=2}^{\infty}C_{n,M_2}(\alpha,\lambda,\gamma)\rho_0^{n}=\rho_1.
\end{aligned}
$$

Thus, the theorem establishes the univalence of biharmonic mappings in $\mathbb{D}_{\rho_0}$ and guarantees the existence of a schlicht disk $\mathbb{D}_{\rho_1}\subset F(\mathbb{D}_{\rho_0})$. $\square$

REMARK 1. If $G = 0$ the above theorem reduces to Theorem 2.

THEOREM 4. *Let $F(\zeta) = |\zeta|^2 G(\zeta)$ be a biharmonic mapping in Ω_γ with $G = h + \overline{g}$, $\lambda_G(0) = \lambda$ and $\|G\|_{\mathcal{H},\Omega_\gamma}(\alpha) \leqslant M$. Then*

$$h(\zeta) = \sum_{n=1}^{\infty} a_n \zeta^n, \quad g(\zeta) = \sum_{n=1}^{\infty} b_n \zeta^n \text{ in } \mathbb{D}.$$

Then F is univalent in $\mathbb{D}_{\rho_2} := \{\zeta : |\zeta| < \rho_2\}$, where ρ_2 is the unique root of

$$\sum_{n=2}^{\infty} (n+2) C_{n,M}(\alpha, \lambda, \gamma) r^{n-1} = \lambda, \qquad r \in (0,1).$$

Moreover, $F(\mathbb{D}_{\rho_2})$ contains a disk $\mathbb{D}_{\rho_3}$, where

$$\rho_3 = \lambda \rho_2^3 - \sum_{n=2}^{\infty} C_{n,M}(\alpha, \lambda, \gamma) \rho_2^{n+2}.$$

Proof. Fix $0 < \rho < 1$ and choose ζ_1, ζ_2 with $\zeta_1 \neq \zeta_2$, $|\zeta_1| < r$ and $|\zeta_2| < r$. Then

$$F(\zeta_1) - F(\zeta_2) = \int_{[\zeta_1,\zeta_2]} F_\zeta(\zeta)\, d\zeta + F_{\overline{\zeta}}(\zeta)\, d\overline{\zeta}.$$

with

$$F(\zeta_1) - F(\zeta_2) = \int_{[\zeta_1,\zeta_2]} \left(\zeta G + r^2 h'\right) d\zeta + \int_{[\zeta_1,\zeta_2]} \left(\overline{\zeta} G + r^2 g'\right) d\overline{\zeta},$$

where $[\zeta_1, \zeta_2]$ is the line-segment from ζ_1 to ζ_2, $\zeta = t\zeta_2 + (1-t)\zeta_1$ and $0 \leqslant t \leqslant 1$, and $G = h + \overline{g}$.

Let $|a_1|^2 - |b_1|^2 = \lambda$ and proceeding as before, one obtains

$$\begin{aligned}
&|F(\zeta_1) - F(\zeta_2)| \\
&= \left| \int_{[\zeta_1,\zeta_2]} \left(\zeta G + r^2 h'\right) d\zeta + \int_{[\zeta_1,\zeta_2]} \left(\overline{\zeta} G + r^2 g'\right) d\overline{\zeta} \right| \\
&= \left| \int_{[\zeta_1,\zeta_2]} \left(h'(0)\zeta + g'(0)\overline{\zeta}\right)(\zeta\, d\zeta + \overline{\zeta}\, d\overline{\zeta}) \right. \\
&\quad + \int_{[\zeta_1,\zeta_2]} \left((h(\zeta) - h'(0)\zeta) + (g(\zeta) - g'(0)\zeta)\right)(\zeta\, d\zeta + \overline{\zeta}\, d\overline{\zeta}) \\
&\quad + \int_{[\zeta_1,\zeta_2]} r^2 \left(h'(0)\, d\zeta + g'(0)\, d\overline{\zeta}\right) \\
&\quad \left. + \int_{[\zeta_1,\zeta_2]} r^2 \left((h'(\zeta) - h'(0))\, d\zeta + (g'(\zeta) - g'(0))\, d\overline{\zeta}\right) \right| \\
&\geqslant \left| \int_{[\zeta_1,\zeta_2]} \left(2r^2\, d\zeta + \zeta^2\, d\overline{\zeta}\right)\left(|h'(0)| - |g'(0)|\right) \right| \\
&\quad - 2|\zeta_2 - \zeta_1| \sum_{n=2}^{\infty} (|a_n| + |b_n|) \int_0^1 r^{n+1}\, dt - |\zeta_2 - \zeta_1| \sum_{n=2}^{\infty} (|a_n| + |b_n|)\, n \int_0^1 r^{n+1}\, dt.
\end{aligned}$$

$$
\begin{aligned}
&|F(\zeta_1)-F(\zeta_2)|\\
&\geqslant |\zeta_2-\zeta_1|\Bigg(\Big(\Big|\int_0^1 r^2\,dt\Big|\Big)(|h'(0)|-|g'(0)|)\\
&\qquad -2\sum_{n=2}^{\infty}(|a_n|+|b_n|)\int_0^1 r^{n+1}\,dt-\sum_{n=2}^{\infty}(|a_n|+|b_n|)n\int_0^1 r^{n+1}\,dt\Bigg)\\
&\geqslant |\zeta_2-\zeta_1|\Big(\Big|\int_0^1 r^2\,dt\Big|\Big)\left(\lambda-2\sum_{n=2}^{\infty}C_{n,M}(\alpha,\lambda,\gamma)r^{n-1}-\sum_{n=2}^{\infty}C_{n,M}(\alpha,\lambda,\gamma)nr^{n-1}\right).\\
&\geqslant |\zeta_2-\zeta_1|\Big(\Big|\int_0^1 r^2\,dt\Big|\Big)\left(\lambda-2\sum_{n=2}^{\infty}C_{n,M}(\alpha,\lambda,\gamma)r^{n-1}-\sum_{n=2}^{\infty}C_{n,M}(\alpha,\lambda,\gamma)nr^{n-1}\right)\\
&=: |\zeta_2-\zeta_1|\Phi_2(r).
\end{aligned}
$$

The function Φ_2 is easily seen to be continuous and differentiable on the interval $(0,1)$, and

$$\Phi_2'(r)=-\left[\sum_{n=2}^{\infty}(n+2)C_{n,M}(\alpha,\lambda,\gamma)(n-1)r^{n-2}\right]<0.$$

Thus Φ_2 is decreasing, and since $\Phi_2(0)=\lambda>0$ and $\lim_{r\to 1^-}\Phi_2(r)=-\infty$, Φ_2 has a unique root $\rho_2\in(0,1)$. Hence F is univalent in $\mathbb{D}_{\rho_2}$.

For $|\zeta|=\rho_2$,

$$
\begin{aligned}
|F(\zeta)|=|\zeta|^2|G(\zeta)|&=|\zeta|^2|h(\zeta)+\overline{g}(\zeta)|\\
&\geqslant \rho_2^2\Big(|a_1|-|b_1|-\sum_{n=2}^{\infty}(|a_n|+|b_n|)\rho_2^{n-1}\Big)\\
&\geqslant \rho_2^2\Big(\lambda-\sum_{n=2}^{\infty}C_{n,M}(\alpha,\lambda,\gamma)\rho_2^{n-1}\Big)=\lambda\rho_2^3-\sum_{n=2}^{\infty}C_{n,M}(\alpha,\lambda,\gamma)\rho_2^{n+2}=\rho_3.
\end{aligned}
$$

Hence, the result confirms that F is univalent in $\mathbb{D}_{\rho_2}$ and that the image $F(\mathbb{D}_{\rho_2})$ contains a schlicht disk $\mathbb{D}_{\rho_3}$. □

In the following section, we focus on a class of biharmonic mappings generated by a harmonic transformation operator, which was introduced by Muir [22].

4. Biharmonic mappings associated with the operator T_c

Muir [22] (2012) extended the framework established by Clunie and Sheil-Small [12], proposing a generalized harmonic operator T_c. In 2012, generalized the work of Clunie and Sheil-Small for harmonic mappings of the form, where h and g are holomorphic in the unit disk $\mathbb{D}$, by defining the operator

$$T_c[f](\zeta)=\frac{f(\zeta)+c\,\zeta f'(\zeta)}{1+c}+\overline{\frac{f(\zeta)-c\,\zeta f'(\zeta)}{1+c}},\qquad \zeta\in\mathbb{D},\quad c>0. \tag{3}$$

The operator T_c retains harmonicity and plays a key role in studying harmonic functions that are convex in one or multiple directions. It also serves as a transformation tool for generating biharmonic functions with desirable geometric characteristics and mapping behaviors.

The next theorem presents the univalence criteria and image-domain characteristics associated with the operator T_c.

THEOREM 5. *Let $f(\zeta)=\sum_{n=0}^{\infty}a_n\zeta^n$ be holomorphic in $\mathbb{D}$, and let*

$$F(\zeta)=|\zeta|^2T_c[f](\zeta)$$

be a biharmonic mapping in Ω_γ with $\lambda_{T_c}(0)=1$, where T_c is defined as in (3), and assume

$$\|T_c\|_{\mathscr{H},\Omega_\gamma}(\alpha)\leqslant M.$$

Then F is univalent in $\mathbb{D}_{\rho_4}:=\{\zeta:|\zeta|<\rho_4\}$, where ρ_4 is the unique root in $(0,1)$ of

$$\sum_{n=2}^{\infty}(2+n)B_{n,M}(\alpha,\lambda,\gamma)\rho^{n-1}=c.$$

Moreover, $F(\mathbb{D}_{\rho_4})$ contains a disk $\mathbb{D}_{\rho_5}$, where

$$\rho_5=\frac{2\rho_4^2}{1+c}-2\sum_{n=2}^{\infty}\frac{(2+n)B_{n,M}(\alpha,\lambda,\gamma)\rho_4^n}{1+c},$$

and the coefficients $B_{n,M}(\alpha,\lambda,\gamma)$ are given by

$$B_{n,M}(\alpha,\lambda,\gamma):=\inf_{0<r<1}\left[\frac{(1+c)\left(m^2(r,\gamma)-\lambda^2\right)}{2nr^{n-1}(1-\gamma)^2(1+\gamma)m(r,\gamma)}\right],$$

with $m(r,\gamma)$ as in Theorem A.

Proof. Select ρ such that $0<\rho<1$. and distinct ζ_1,ζ_2 with $|\zeta_1|<\rho$ and $|\zeta_2|<\rho$. Then

$$F(\zeta_1)-F(\zeta_2)=\int_{[\zeta_1,\zeta_2]}F_\zeta(\zeta)\,d\zeta+F_{\overline{\zeta}}(\zeta)\,d\overline{\zeta},$$

and, using $T_c[f](\zeta)=G(\zeta)=h(\zeta)+\overline{g(\zeta)}$ (with h,g as in the definition of T_c), we write

$$F(\zeta_1)-F(\zeta_2)=\int_{[\zeta_1,\zeta_2]}\left(\zeta G+r^2h'\right)d\zeta+\int_{[\zeta_1,\zeta_2]}\left(\overline{\zeta}G+r^2g'\right)d\overline{\zeta}.$$

Following the standard decomposition used earlier and using the normalization and coefficient estimates for $T_c[f]$, we obtain

$$\begin{aligned}|F(\zeta_1)-F(\zeta_2)|&\geqslant|\zeta_2-\zeta_1|\left(\frac{2c}{1+c}-\frac{4}{1+c}\sum_{n=2}^{\infty}B_{n,M}(\alpha,\lambda,\gamma)\rho^{n-1}\right.\\&\qquad\left.-\frac{2}{1+c}\sum_{n=2}^{\infty}B_{n,M}(\alpha,\lambda,\gamma)n\rho^{n-1}\right)\\&=:|\zeta_2-\zeta_1|\Phi_3(\rho).\end{aligned}$$

The function $\Phi_3(\rho)$ is continuous on $(0,1)$ and differentiable with

$$\Phi_3'(\rho) = -\frac{2}{1+c}\left[\sum_{n=2}^{\infty}(n+2)B_{n,M}(\alpha,\lambda,\gamma)(n-1)\rho^{n-2}\right] < 0,$$

so Φ_3 is strictly decreasing. Since $\Phi_3(0) = \dfrac{2c}{1+c} > 0$ and $\lim_{\rho\to 1^-}\Phi_3(\rho) = -\infty$, Φ_3 has a unique root $\rho_4 \in (0,1)$. Choosing ρ_4 yields $|F(\zeta_1) - F(\zeta_2)| > 0$ for all $|\zeta_1|, |\zeta_2| < \rho_4$ with $\zeta_1 \neq \zeta_2$, hence F is univalent in $\mathbb{D}_{\rho_4}$.

To obtain the schlicht disk radius, evaluate on $|\zeta| = \rho_4$. Writing the first-order term explicitly and using the coefficient bounds, we get

$$\begin{aligned}|F(\zeta)| &= \rho_4^2\left|\sum_{n=1}^{\infty}\left(\frac{1+cn}{1+c}a_n\zeta^n + \frac{1-cn}{1+c}a_n\overline{\zeta}^n\right)\right| \\ &\geqslant \frac{2}{1+c}\rho_4^3 - \sum_{n=2}^{\infty}B_{n,M}(\alpha,\lambda,\gamma)\rho_4^{n+2} = \frac{2\rho_4^2}{1+c} - 2\sum_{n=2}^{\infty}\frac{(2+n)B_{n,M}(\alpha,\lambda,\gamma)\rho_4^n}{1+c},\end{aligned}$$

which yields the stated lower bound ρ_5 for a disk contained in $F(\mathbb{D}_{\rho_4})$. This completes the proof. □

5. Conclusion

This paper extends Landau's theorem to the family of biharmonic α-Bloch mappings defined over simply connected regions, especially the shifted disk Ω_γ we derive explicit univalence and image-inclusion bounds by transferring coefficient estimates from the harmonic α-Bloch class, thereby establishing a unified biharmonic framework that generalizes the corresponding holomorphic and harmonic results.

Acknowledgement. The authors sincerely thank the Referees for their valuable inputs and suggestions which has helped us in improving the manuscript.

REFERENCES

[1] Z. ABDULHADI, Y. ABU MUHANNA AND S. KHOURI, *On univalent solutions of the biharmonic equations*, J. Inequal. Appl., (5) (2005), 469–478.

[2] Z. ABDULHADI AND Y. ABU MUHANNA, *Landau's theorem for biharmonic mappings*, J. Math. Anal. Appl., **338** (2008), 705–709, doi:10.1016/j.jmaa.2007.05.065.

[3] Y. ABU MUHANNA AND G. SCHOBER, *Harmonic mappings onto convex mapping domains*, Canad. J. Math., **39** (6) (1987), 1489–1530.

[4] V. ALLU AND H. HALDER, *Landau's theorem for harmonic Bloch functions on simply connected domains*, Complex Variables and Elliptic Equations, **70** (2) (2025), 213–227, doi:10.1080/17476933.2023.2298834.

[5] V. ALLU AND H. HALDER, *On Bloch norm and Bohr phenomenon for harmonic Bloch functions on simply connected domains*, available at https://arxiv.org/pdf/2108.05899.pdf.

[6] A. F. BEARDON AND D. MINDA, *The hyperbolic metric and geometric function theory*, Proc. Int. Workshop on Quasiconformal Mappings and their Applications (IWQCMA05), 2007.

[7] M. BONK, D. MINDA AND H. YANAGIHARA, *The hyperbolic metric on Bloch regions*, Comput. Methods Funct. Theory (Penang) (1994), 89-100. Ser. Approx. Decompos., vol. 5; World Sci. Publ., River Edge (NJ), 1995.
[8] H. CHEN, P. GAUTHIER AND W. HENGARTNER, *Bloch constants for planar harmonic mappings*, Proc. Amer. Math. Soc., **128** (2000), 3231–3240, doi:10.1090/proc/2000-128-11.
[9] S. CHEN, S. PONNUSAMY AND X. WANG, *Landau's theorem for certain biharmonic mappings*, Appl. Math. Comput., **208** (2009), 427–433.
[10] S. CHEN, S. PONNUSAMY AND X. WANG, *Landau's theorem and Marden constant for harmonic ν-Bloch mappings*, Bull. Aust. Math. Soc., **84** (1) (2011), 19–32, doi:10.1017/S0004972711002140.
[11] J. CHEN, A. RASILA AND X. WANG, *Landau's theorem for polyharmonic mappings*, J. Math. Anal. Appl., **409** (2014), 934–945, doi:10.1016/j.jmaa.2013.07.061.
[12] J. CLUNIE AND T. SHEIL-SMALL, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I. Math., **9** (1984), 3–25.
[13] F. COLONNA, *The Bloch constant of bounded analytic functions*, J. London Math. Soc., **36** (1987), 95–101, doi:10.1112/jlms/s2-36.1.95.
[14] F. COLONNA, *The Bloch constant of bounded harmonic mappings*, Indiana Univ. Math. J., **38** (4) (1989), 829–840, available at http://www.jstor.org/stable/24895370.
[15] M. DORFF AND M. NOWAK, *Landau's theorem for planar harmonic mappings*, Comput. Methods Funct. Theory, **4** (1) (2004), 151–158, doi:10.1007/BF03321062.
[16] P. DUREN, *Harmonic Mappings in the Plane*, Cambridge University Press, Cambridge, 2004.
[17] A. GRIGORYAN, *Landau and Bloch theorems for harmonic mappings*, Complex Var. Theory Appl., **51** (1) (2006), 81–87.
[18] M.-S. LIU, *Landau's theorem for planar harmonic mappings*, Comput. Math. Appl., **57** (7) (2009), 1142–1146, doi:10.1016/j.camwa.2009.01.009.
[19] S.-F. CHEN AND M.-S. LIU, *Landau-type theorems and bi-Lipschitz theorems for bounded biharmonic mappings*, Monatsh. Math., **193** (2020), 1–33.
[20] G. LIU AND S. PONNUSAMY, *On harmonic ν-Bloch and ν-Bloch type mappings*, available at https://arxiv.org/pdf/1707.01572.
[21] P. LI, M.-S. LIU, S. PONNUSAMY AND H. ZHAO, *Landau-type theorems for polyanalytic and log-α-analytic functions*, J. Math. Anal. Appl., **556** (2026), 130109.
[22] S. MUIR, *Harmonic mappings convex in one or every direction*, Comput. Methods Funct. Theory, **12** (1) (2012), 221–239.
[23] H. XINZHONG, *Estimates on Bloch constants for planar harmonic mappings*, J. Math. Anal. Appl., **337** (2008), 880–887, doi:10.1016/j.jmaa.2007.02.087.

(Received December 2, 2025)

K. P. Balakrishnan
Department of Mathematics
Madras Christian College
Chennai, India
e-mail: kpbalu2014@gmail.com

S. Sunil Varma
Department of Mathematics
Madras Christian College
Chennai, India
e-mail: sunilvarma@mcc.edu.in