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MULITDIMENSIONAL HARDY TYPE INEQUALITIES FOR $p < 0$ AND $0 < p < 1$

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Abstract. In this paper we establish some new multidimensional Hardy type inequalities for the cases $p < 0$ and $0 < p < 1$. These inequalities complement, generalize and unify most of the existing results of this type in the literature e.g. those in [4] and [9]. Some of the results are new also for the one dimensional case.

1. Introduction

In [5] Hardy announced and proved in [6] the following integral inequality (see also [8, Chapter 9, Theorem 328]): If $p > 1$, $f(x) \geqslant 0$, and $F(x) = \int_0^x f(t)dt$, then

$$\int_0^\infty \left(\frac{F}{x}\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p dx. \tag{1.1}$$

The constant $\left(\frac{p}{p-1}\right)^p$ is the best possible.

Moreover, in 1928 Hardy [7] (see also [8, Chapter 9, Theorem 330, p. 245]) proved a generalized form of (1.1), namely that if $p > 1$, $m \neq 1$, and $F(x)$ is defined by

$$F(x) = \begin{cases} \int_0^x f(t)dt, & m > 1, \\ \int_x^\infty f(t)dt & m < 1, \end{cases} \tag{1.2}$$

then

$$\int_0^\infty x^{-m} F^p dx \leqslant \left(\frac{p}{|m-1|}\right)^p \int_0^\infty x^{-m}(xf)^p dx. \tag{1.3}$$

The constant is the best possible.

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Furthermore, Hardy [7] (see also [8, Chapter 9, Theorem 347, p. 256]) pointed out that if m and F satisfy the conditions of the above result, but $0 < p < 1,$ then

$$\int_0^\infty x^{-m} F^p dx \geqslant \left(\frac{p}{|m-1|}\right)^p \int_0^\infty x^{-m} (xf)^p dx. \tag{1.4}$$

For further remarks concerning the history, development, generalizations and applications of inequalities (1.1) and (1.3) see for instance, [1], [2], [8], [11], [12], [15] and the references cited therein.

The first result for the unweighted one dimensional Hardy's inequality in the discrete case for $p < 0$ was obtained in 1928 by Knopp [10] (see also [12] and the references cited therein). The weighted Hardy inequalities for negative powers appeared in the papers of Beesack and Heinig [1] and Heinig [2] where the cases $p, q < 0$ and $0 < p, q < 1$ are considered. They studied the reverse Hardy inequality

$$\left(\int_0^\infty [f(x)v(x)]^p \, dx\right)^{\frac{1}{p}} \leqslant C \left(\int_0^\infty \left[u(x) \int_0^x f(t)dt\right]^q dx\right)^{\frac{1}{q}} \tag{1.5}$$

and its dual version by deriving some necessary as well as some sufficient conditions for their validity.

The unweighted multidimensional Hardy-type inequalities for the cases $p < 0$ and $0 < p < 1$ were studied in [9] by using a convexity argument.

In this paper we prove and discuss some weighted multidimensional Hardy type inequalities for the cases $p < 0$ and $0 < p < 1$ of the type (1.3) for different values of m. Some results are new also for the one dimensional case. The techniques that will be used in the proofs are mainly a convexity argument, which is very different from the classical methods used e.g. by Beesack and Heinig [1], Heinig [2] and Hardy [8].

The paper is organized as follows: In order not to disturb our discussions later on we use Section 2 to present some preliminaries, including some convexity results from the paper [14]. The main results are given in Section 3, while our concluding examples and remarks are presented in Section 4.

2. Preliminaries

Throughout the paper all functions are assumed to be measurable. Here and in the sequel the notations $\mathbf{b}$, $\mathbf{x}$, $(\mathbf{0}, \mathbf{b})$, $(\mathbf{b},\infty]$, $[\mathbf{b},\boldsymbol{\infty})$ as usual means $\mathbf{b} = (b_1, ..., b_n)$, $\mathbf{x} = (x_1, ..., x_n)$, $(\mathbf{0}, \boldsymbol{b}) = \{\boldsymbol{x} \in \mathbb{R}^n : 0 < x_j < b_j, \, j = 1, 2, ..., n\}$, $(\boldsymbol{b}, \boldsymbol{\infty}] = \{\boldsymbol{x} \in \mathbb{R}^n : b_j < x_j \leqslant \infty, \, j = 1, 2, ..., n\}$, $[\boldsymbol{b}, \boldsymbol{\infty}) = \{\boldsymbol{x} \in \mathbb{R}^n : b_j \leqslant x_j < \infty, \, j = 1, 2, ..., n\}$ and $\mathbf{b} < \mathbf{x}$ means that $b_j < x_{j,} \; j = 1, 2, ..., n.$ $(n \in \mathbb{Z}_+)$.

We now present some results in the recent paper [14], which are crucial to the proofs of our main results.

LEMMA 2.1. *Let* $\mathbf{b} \in (\mathbf{0}, \infty]$, $-\infty \leqslant a < c \leqslant \infty$ *and let* Φ *be a positive function on* $[a,c]$. *Suppose that the weight function* u *defined on* $(\mathbf{0}, \mathbf{b})$ *is nonnegative such that* $\frac{u(x_1,\dots,x_n)}{x_1^2\dots x_n^2}$ *is locally integrable on* $(\mathbf{0}, \mathbf{b})$ *and the weight function* v *is defined by*

$$v(t_1,\dots,t_n) = t_1\dots t_n \int_{t_1}^{b_1}\dots\int_{t_n}^{b_n} \frac{u(x_1,\dots,x_n)}{x_1^2\dots x_n^2} dx_1\dots dx_n, \quad \boldsymbol{t} \in (\mathbf{0}, \mathbf{b}).$$

(i) If Φ *is convex, then*

$$\int_0^{b_1}\dots\int_0^{b_n} u(x_1,\dots,x_n)\Phi\left(\frac{1}{x_1\dots x_n}\int_0^{x_1}\dots\int_0^{x_n} f\left(t_1,\dots,t_n\right)dt_1\dots dt_n\right)\frac{dx_1\dots dx_n}{x_1\dots x_n}$$

$$\leqslant \int_0^{b_1}\dots\int_0^{b_n} v\left(x_1,\dots,x_n\right)\Phi\left(f\left(x_1,\dots,x_n\right)\right)\frac{dx_1\dots dx_n}{x_1\dots x_n} \tag{2.1}$$

holds for every function f on $(\mathbf{0}, \mathbf{b})$ *such that* $a < f(x_1,\dots,x_n) < c$.

(ii) If Φ *is concave, then*

$$\int_0^{b_1}\dots\int_0^{b_n} u(x_1,\dots,x_n)\Phi\left(\frac{1}{x_1\dots x_n}\int_0^{x_1}\dots\int_0^{x_n} f\left(t_1,\dots,t_n\right)dt_1\dots t_n\right)\frac{dx_1\dots dx_n}{x_1\dots x_n}$$

$$\geqslant \int_0^{b_1}\dots\int_0^{b_n} v\left(x_1,\dots,x_n\right)\Phi\left(f\left(x_1,\dots,x_n\right)\right)\frac{dx_1\dots dx_n}{x_1\dots x_n} \tag{2.2}$$

holds for every function f on $(\mathbf{0}, \mathbf{b})$ *such that* $a < f(x_1,\dots,x_n) < c$.

Proof. The proof is easy and just a consequence of Jensen's inequality and Fubini's theorem (for details see [14]). □

LEMMA 2.2. *Let* $\mathbf{b} \in [\mathbf{0}, \infty)$, $-\infty \leqslant a < c \leqslant \infty$ *and* Φ *be a positive function on* $[a,c]$. *Assume that the weight function* u *defined on* $[\mathbf{b}, \infty)$ *is nonnegative such that* $\frac{u(x_1,\dots,x_n)}{x_1^2\dots x_n^2}$ *is locally integrable on* $[\mathbf{b}, \infty)$ *and the weight function* v *is defined by*

$$v(t_1,\dots,t_n) = \frac{1}{t_1\dots t_n}\int_{b_1}^{t_1}\dots\int_{b_n}^{t_n} u(x_1,\dots,x_n)dx_1\dots dx_n < \infty, \quad \boldsymbol{t} \in (\mathbf{b}, \infty).$$

(i) *If* Φ *is convex, then*

$$\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty} u(x_1,\ldots,x_n)\Phi\left(x_1\ldots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f\ (t_1,\ldots,t_n)\,\frac{dt_1\ldots dt_n}{t_1^2\ldots t_n^2}\right)\frac{dx_1\ldots dx_n}{x_1\ldots x_n}$$

$$\leqslant \int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty} v\,(x_1,\ldots,x_n)\,\Phi\,(f\ (x_1,\ldots,x_n))\,\frac{dx_1\ldots dx_n}{x_1\ldots x_n} \tag{2.3}$$

holds for every function f on $[\mathbf{b},\infty)$ *such that* $a<f\,(x_1,\ldots,x_n)<c.$

(ii) *If* Φ *is concave, then*

$$\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty} u(x_1,\ldots,x_n)\Phi\left(x_1\ldots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f\ (t_1,\ldots,t_n)\,\frac{dt_1\ldots dt_n}{t_1^2\ldots t_n^2}\right)\frac{dx_1\ldots dx_n}{x_1\ldots x_n}$$

$$\geqslant \int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty} v\,(x_1,\ldots,x_n)\,\Phi\,(f\ (x_1,\ldots,x_n))\,\frac{dx_1\ldots dx_n}{x_1\ldots x_n} \tag{2.4}$$

holds for every function f on $[\mathbf{b},\infty)$ *such that* $a<f\,(x_1,\ldots,x_n)<c.$

Proof. The proof follows by applying Jensen's inequality and Fubini's theorem (for details see [14]). $\square$

3. Main results

Our first result reads:

THEOREM 3.1. *Let* $p<0,\ \mathbf{b}\in(\mathbf{0},\infty],$ *and let f be a nontrival and nonnegative function on* $(\mathbf{0},\mathbf{b})$ *and assume that*

$$0<\int_0^{b_1}\cdots\int_0^{b_n} x_1^{p-m}\ldots x_n^{p-m} f^p(x_1,\ldots,x_n)dx_1\ldots dx_n<\infty.$$

If $m<1,$ then

$$\int_0^{b_1}\cdots\int_0^{b_n} x_1^{-m}\ldots x_n^{-m}\left(\int_0^{x_1}\cdots\int_0^{x_n} f\ (t_1,\ldots,t_n)\,dt_1\ldots dt_n\right)^p dx_1\ldots dx_n$$

$$\leqslant\left(\frac{p}{m-1}\right)^{pn}\int_0^{b_1}\cdots\int_0^{b_n}\left[1-\left(\frac{x_1}{b_1}\right)^{\frac{m-1}{p}}\right]\cdots\left[1-\left(\frac{x_n}{b_n}\right)^{\frac{m-1}{p}}\right]\times$$

$$x_1^{p-m}\ldots x_n^{p-m} f^p(x_1,\ldots,x_n)dx_1\ldots dx_n. \tag{3.1}$$

Proof. We use Lemma 2.1 (i) with the convex function $\Phi(x) = x^p$ and the weight function $u(x_1,...,x_n) \equiv 1$ (so that $v(x_1,...,x_n) = \left(1-\frac{x_1}{b_1}\right)...\left(1-\frac{x_n}{b_n}\right)$). Then inequality (2.1) yields

$$\int_0^{b_1}...\int_0^{b_n}\left(\frac{1}{x_1...x_n}\int_0^{x_1}...\int_0^{x_n} f\,(t_1,...,t_n)\,dt_1...dt_n\right)^p \frac{dx_1...dx_n}{x_1...x_n}$$
$$\leqslant \int_0^{b_1}...\int_0^{b_n}\left(1-\frac{x_1}{b_1}\right)...\left(1-\frac{x_n}{b_n}\right) f^p(x_1,...,x_n)\frac{dx_1...dx_n}{x_1...x_n}. \qquad (3.2)$$

Now, replace b_j by $a_j = b_j^{\frac{m-1}{p}}$ and choose for f the function $x \mapsto f\left(x_1^{\frac{p}{m-1}},...,x_n^{\frac{p}{m-1}}\right)x_1^{\frac{p}{m-1}-1}...x_n^{\frac{p}{m-1}-1}$. Thereafter, by using the substitutions $s_j = t_j^{\frac{p}{m-1}}$ and $y_j = x_j^{\frac{p}{m-1}}$, respectively, the left hand side of (3.2) becomes

$$\int_0^{a_1}...\int_0^{a_n}\left(\frac{1}{x_1...x_n}\int_0^{x_1}...\int_0^{x_n} f\left(t_1^{\frac{p}{m-1}},...,t_n^{\frac{p}{m-1}}\right)t_1^{\frac{p}{m-1}-1}...t_n^{\frac{p}{m-1}-1}dt_1...dt_n\right)^p \frac{dx_1...dx_n}{x_1...x_n}$$
$$= \left(\frac{m-1}{p}\right)^{pn}\int_0^{a_1}...\int_0^{a_n}\left(\frac{1}{x_1...x_n}\int_0^{x_1^{\frac{p}{m-1}}}...\int_0^{x_n^{\frac{p}{m-1}}} f\,(s_1,...,s_n)\,ds_1...ds_n\right)^p\frac{dx_1...dx_n}{x_1...x_n}$$
$$= \left(\frac{m-1}{p}\right)^{pn+n}\int_0^{b_1}...\int_0^{b_n}\left(\int_0^{y_1}...\int_0^{y_n} f\,(s_1,...,s_n)\,ds_1...ds_n\right)^p y_1^{-\left(\frac{m-1}{p}\right)p}...y_n^{-\left(\frac{m-1}{p}\right)p}\frac{dy_1...dy_n}{y_1...y_n}$$
$$= \left(\frac{m-1}{p}\right)^{pn+n}\int_0^{b_1}...\int_0^{b_n}\left(\int_0^{y_1}...\int_0^{y_n} g\,(s_1,...,s_n)\,ds_1...ds_n\right)^p y_1^{-m}...y_n^{-m}dy_1...dy_n.$$
$$(3.3)$$

Similarly, the right hand side of (3.2) yields

$$\int_0^{a_1}...\int_0^{a_n}\left(1-\frac{x_1}{a_1}\right)...\left(1-\frac{x_n}{a_n}\right)f^p(x_1^{\frac{p}{m-1}},...,x_n^{\frac{p}{m-1}})x_1^{p\left(\frac{p}{m-1}-1\right)}...x_n^{p\left(\frac{p}{m-1}-1\right)}\frac{dx_1...dx_n}{x_1...x_n}$$
$$= \left(\frac{m-1}{p}\right)^n\int_0^{b_1}...\int_0^{b_n}\left[1-\left(\frac{y_1}{b_1}\right)^{\frac{m-1}{p}}\right]...\left[1-\left(\frac{y_n}{b_n}\right)^{\frac{m-1}{p}}\right]\times$$
$$f^p(y_1,...,y_n)y_1^{p-m+1}...y_n^{p-m+1}\frac{dy_1...dy_n}{y_1,...,y_n}$$

$$= \left(\frac{m-1}{p}\right)^n \int_0^{b_1} \cdots \int_0^{b_n} \left[1-\left(\frac{y_1}{b_1}\right)^{\frac{m-1}{p}}\right] \cdots \left[1-\left(\frac{y_n}{b_n}\right)^{\frac{m-1}{p}}\right] \times$$
$$y_1^{p-m} \dots y_n^{p-m} f^p(y_1,\dots,y_n) dy_1 \dots dy_n. \tag{3.4}$$

(3.1) follows by just combining (3.3) and (3.4). The proof is complete. □

In the next theorem we state the dual of Theorem 3.1.

THEOREM 3.2. *Let* $p < 0, \mathbf{b} \in [\mathbf{0},\infty)$, *let* f *be a nontrival and nonnegative function on* $(\mathbf{b},\infty)$. *If* $m > 1$ *and*

$$0 < \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} x_1^{p-m} \dots x_n^{p-m} f^p(x_1,\dots,x_n) dx_1 \dots dx_n < \infty,$$

then

$$\int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} x_1^{-m} \dots x_n^{-m} \left(\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f\,(t_1,\dots,t_n)\, dt_1 \dots dt_n\right)^p dx_1 \dots dx_n$$
$$\leqslant \left(\frac{p}{1-m}\right)^{pn} \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \left[1-\left(\frac{b_1}{x_1}\right)^{\frac{1-m}{p}}\right] \cdots \left[1-\left(\frac{b_n}{x_n}\right)^{\frac{1-m}{p}}\right] \times$$
$$x_1^{p-m} \dots x_n^{p-m} f^p(x_1,\dots,x_n) dx_1 \dots dx_n. \tag{3.5}$$

Proof. We use Lemma 2.2 (i) with the convex function $\Phi(x) = x^p$ and the weight function $u(x_1,\dots,x_n) \equiv 1$ (so that $v(x_1,\dots,x_n) = \left(1-\frac{b_1}{x_1}\right) \dots \left(1-\frac{b_n}{x_n}\right)$). Then inequality (2.3) becomes

$$\int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \left(x_1 \dots x_n \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f\,(t_1,\dots,t_n)\, \frac{dt_1 \dots dt_n}{t_1^2 \dots t_n^2}\right)^p \frac{dx_1 \dots dx_n}{x_1 \dots x_n}$$
$$\leqslant \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \left(1-\frac{b_1}{x_1}\right) \cdots \left(1-\frac{b_n}{x_n}\right) f^p(x_1,\dots,x_n) \frac{dx_1 \dots dx_n}{x_1 \dots x_n}. \tag{3.6}$$

Now, replace b_j by $a_j = b_j^{\frac{1-m}{p}}$ and choose for f the function $x \mapsto f\left(x_1^{\frac{p}{1-m}},\dots,x_n^{\frac{p}{1-m}}\right) x_1^{\frac{p}{1-m}+1} \dots x_n^{\frac{p}{1-m}+1}$. Thereafter, use the substitutions $s_j = t_j^{\frac{p}{1-m}}$

and $y_j = x_j^{\frac{p}{1-m}}$, respectively. Then the left hand side of (3.6) yields

$$\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\left(x_1\ldots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f\left(t_1^{\frac{p}{1-m}},\ldots,t_n^{\frac{p}{1-m}}\right) t_1^{\frac{p}{1-m}+1}\ldots t_n^{\frac{p}{1-m}+1}\frac{dt_1\ldots dt_n}{t_1^2\ldots t_n^2}\right)^p \frac{dx_1\ldots dx_n}{x_1\ldots x_n}$$

$$= \left(\frac{1-m}{p}\right)^{pn}\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\left(\int_{x_1^{\frac{p}{1-m}}}^{\infty}\cdots\int_{x_n^{\frac{p}{1-m}}}^{\infty} f\left(s_1,\ldots,s_n\right) ds_1\ldots ds_n\right)^p x_1^p\ldots x_n^p\frac{dx_1\ldots dx_n}{x_1\ldots x_n}$$

$$= \left(\frac{1-m}{p}\right)^{pn+n}\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\left(\int_{y_1}^{\infty}\cdots\int_{y_n}^{\infty} f\left(s_1,\ldots,s_n\right) ds_1\ldots ds_n\right)^p \times$$
$$y_1^{p\left(\frac{1-m}{p}\right)-1}\ldots y_n^{p\left(\frac{1-m}{p}\right)-1} dy_1\ldots dy_n$$

$$= \left(\frac{1-m}{p}\right)^{pn+n}\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\left(\int_{y_1}^{\infty}\cdots\int_{y_n}^{\infty} f\left(s_1,\ldots,s_n\right) ds_1\ldots ds_n\right)^p y_1^{-m}\ldots y_n^{-m} dy_1\ldots dy_n. \tag{3.7}$$

Similarly, the right hand side of (3.6) reads

$$\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}\left(1-\frac{a_1}{x_1}\right)\cdots\left(1-\frac{a_n}{x_n}\right) f^p(x_1^{\frac{p}{1-m}},\ldots,x_n^{\frac{p}{1-m}}) x_1^{p\left(\frac{p}{1-m}+1\right)}\ldots x_n^{p\left(\frac{p}{1-m}+1\right)}\frac{dx_1\ldots dx_n}{x_1\ldots x_n}$$

$$= \left(\frac{1-m}{p}\right)^{n}\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\left[1-\left(\frac{b_1}{y_1}\right)^{\frac{1-m}{p}}\right]\cdots\left[1-\left(\frac{b_n}{y_n}\right)^{\frac{1-m}{p}}\right]\times$$
$$y_1^{p-m}\ldots y_n^{p-m} f^p(y_1,\ldots,y_n) dy_1\ldots dy_n. \tag{3.8}$$

Inequality (3.5) follows by combining (3.7) and (3.8). The proof is complete. □

Our next result, which deals with the case $0 < p < 1$, is the following:

THEOREM 3.3. *Let* $0 < p < 1$, $\mathbf{b} \in (\mathbf{0},\infty]$, *and let* f *be a nontrival and nonnegative function on* $(\mathbf{0},\mathbf{b})$ *and assume that*

$$0 < \int_0^{b_1}\cdots\int_0^{b_n} x_1^{p-m}\ldots x_n^{p-m} f^p(x_1,\ldots,x_n) dx_1\ldots dx_n < \infty.$$

If $m > 1$, *then*

$$\int_0^{b_1} \cdots \int_0^{b_n} x_1^{-m} \ldots x_n^{-m} \left(\int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \ldots, t_n)\, dt_1 \ldots dt_n \right)^p dx_1 \ldots dx_n$$
$$\geqslant \left(\frac{p}{m-1} \right)^{pn} \int_0^{b_1} \cdots \int_0^{b_n} \left[1 - \left(\frac{x_1}{b_1} \right)^{\frac{m-1}{p}} \right] \cdots \left[1 - \left(\frac{x_n}{b_n} \right)^{\frac{m-1}{p}} \right] \times$$
$$x_1^{p-m} \ldots x_n^{p-m} f^p(x_1, \ldots, x_n) dx_1 \ldots dx_n. \tag{3.9}$$

Proof. The proof is completely similar to that of Theorem 3.1 and, hence, the details are omitted. In this case we note that the function $\Phi(x) = x^p$, for $0 < p < 1$ is concave and thus only the inequality signs are reversed. □

Our final result in this section is the following:

THEOREM 3.4. *Let* $0 < p < 1$, $\mathbf{b} \in [\mathbf{0}, \infty)$, *let* f *be a nontrival and nonnegative function on* $(\mathbf{b}, \infty)$, *and let* $m < 1$, *and*

$$0 < \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} x_1^{p-m} \ldots x_n^{p-m} f^p(x_1, \ldots, x_n) dx_1 \ldots dx_n < \infty.$$

Then

$$\int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} x_1^{-m} \ldots x_n^{-m} \left(\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f(t_1, \ldots, t_n)\, dt_1 \ldots dt_n \right)^p dx_1 \ldots dx_n$$
$$\geqslant \left(\frac{p}{1-m} \right)^{pn} \int_{b_1}^{\infty} \cdots \int_{b_n}^{\infty} \left[1 - \left(\frac{b_1}{x_1} \right)^{\frac{1-m}{p}} \right] \cdots \left[1 - \left(\frac{b_n}{x_n} \right)^{\frac{1-m}{p}} \right] \times$$
$$x_1^{p-m} \ldots x_n^{p-m} f^p(x_1, \ldots, x_n) dx_1 \ldots dx_n. \tag{3.10}$$

Proof. The proof is completely similar to the proof of Theorem 3.2. In this case we note that the function $\Phi(x) = x^p$, for $0 < p < 1$ is concave and, hence, only the inequality signs are reversed. □

4. Concluding examples and remarks

By using our Theorems 3.1 and Corollary 2.1 (i) in [14] for the cases $b_j = \infty$, $j = 1, 2, \ldots, n$, and $m = p$, respectively, we obtain the following multidimensional Hardy type inequalities:

EXAMPLE 4.1. *If* $n \in \mathbb{Z}_+$ *and* $p < 0,\ m < 1$ *or* $p > 1,\ m > 1,$ *then*

$$\int_0^\infty \cdots \int_0^\infty x_1^{-m} \ldots x_n^{-m} \left(\int_0^{x_1} \cdots \int_0^{x_n} f\,(t_1, \ldots, t_n)\, dt_1 \ldots dt_n \right)^p dx_1 \ldots dx_n$$
$$\leqslant \left(\frac{p}{m-1} \right)^{pn} \int_0^\infty \cdots \int_0^\infty x_1^{p-m} \ldots x_n^{p-m} f^p(x_1, \ldots, x_n) dx_1 \ldots dx_n. \tag{4.1}$$

EXAMPLE 4.2. *If* $n \in \mathbb{Z}_+$ *and* $p < 0$ *or* $p > 1,$ *then*

$$\int_0^{b_1} \cdots \int_0^{b_n} \left(\frac{1}{x_1 \ldots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f\,(t_1, \ldots, t_n)\, dt_1 \ldots dt_n \right)^p dx_1 \ldots dx_n$$
$$\leqslant \left(\frac{p}{p-1} \right)^{pn} \int_0^{b_1} \cdots \int_0^{b_n} \left[1 - \left(\frac{x_1}{b_1} \right)^{\frac{p-1}{p}} \right] \cdots \left[1 - \left(\frac{x_n}{b_n} \right)^{\frac{p-1}{p}} \right] \times$$
$$f^p(x_1, \ldots, x_n) dx_1 \ldots dx_n. \tag{4.2}$$

REMARK 4.1. *By using Theorem 3.3 we see that if* $0 < p < 1,\ m > 1,$ *then (4.1) holds in the reversed direction and if* $0 < p < 1,$ *then (4.2) holds in the reversed direction.*

By using our Theorem 3.2 and Corollary 2.1 (ii) in [14] for the cases $b_j = \infty,$ $j = 1, 2, \ldots, n,$ we obtain the following dual version of Example 4.1:

EXAMPLE 4.3. *If* $n \in \mathbb{Z}_+$ *and* $p < 0,\ m > 1$ *or* $p > 1,\ m < 1,$ *then*

$$\int_0^\infty \cdots \int_0^\infty x_1^{-m} \ldots x_n^{-m} \left(\int_{x_1}^\infty \cdots \int_{x_n}^\infty f\,(t_1, \ldots, t_n)\, dt_1 \ldots dt_n \right)^p dx_1 \ldots dx_n$$
$$\leqslant \left(\frac{p}{1-m} \right)^{pn} \int_0^\infty \cdots \int_0^\infty x_1^{p-m} \ldots x_n^{p-m} f^p(x_1, \ldots, x_n) dx_1 \ldots dx_n. \tag{4.3}$$

REMARK 4.2. *By using Theorem 3.4 we find that (4.3) holds in the reversed direction if* $0 < p < 1, m < 1.$

REMARK 4.3. *We remark that for the case* $n = 1,\ p > 1$ *(4.1)-(4.2) coincides with the weighted Hardy inequality (1.3). Moreover, according to Remarks 4.1 and 4.2, we obtain the reversed inequality (1.4) for the case* $n = 1,\ 0 < p < 1.$

REMARK 4.4. *For the special case* $m = p,\ 0 < p < 1$ *in Theorem 3.4, the inequality (3.10) coincides with inequality (2.4) in [9, Corollary 2.2 (b)]. In particular, for* $n = 1,\ b_1 = 0$ *(3.10) reduces to [8, Theorem 337, p. 251].*

We believe that the result in Theorem 3.4 is new also in the case $n = 1$:

EXAMPLE 4.4. *If* $0 \leqslant b < \infty,\ 0 < p < 1$ *and* $m < 1,$ *then*

$$\int_b^\infty x^{-m}\left(\int_x^\infty f(t)\,dt\right)^p dx \geqslant \left(\frac{p}{1-m}\right)^p \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1-m}{p}}\right] x^{p-m} f^p(x)dx. \qquad (4.4)$$

In particular, if $m = 0$ *(4.4) reads:*

$$\int_b^\infty \left(\int_x^\infty f(t)\,dt\right)^p dx \geqslant p^p \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1}{p}}\right] x^p f^p(x)dx.$$

REMARK 4.5. *Some complementary results to those obtained in this paper for the case* $p > 1$ *are recently proved and discussed in [14].*

REMARK 4.6. *For the case* $p > 1$ *some multidimensional Hardy type inequalities were also proved in [3]. They used another mixed mean inequality technique and it may very well be possible to prove some of the results in this paper by using this technique.*

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