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UNIVERSAL MEANS

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Abstract. A mean U is called universal if there exists a constant p such that every mean M be comparable with pU. Some known examples of means are analyzed, establishing which of them are universal.

1. Means

As an abstract definition of means (on $\mathbb{R}_+$), usually is given the following

DEFINITION 1. A *mean* is a function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, which has the property

$$\wedge(a,b) \leqslant M(a,b) \leqslant \vee(a,b),\ \forall a,b > 0$$

where

$$\wedge(a,b) = \min(a,b), \quad \vee(a,b) = \max(a,b), \quad \forall a,b > 0\ .$$

The mean M is called *symmetric* if

$$M(a,b) = M(b,a),\ \forall a,b > 0.$$

Each mean is *reflexive* that is

$$M(a,a) = a,\ \forall a > 0,$$

which is taken as definition, if it is necessary.

We use ordinary operations with means (as functions) and order relation between means. For instance, given the means M and N and the real numbers p and q, define $pM + qN$ by

$$(pM + qN)(a,b) = pM(a,b) + qN(a,b),\ \forall a,b > 0$$

and write $pM \leqslant qN$ if

$$pM(a,b) \leqslant qN(a,b),\ \forall a,b > 0.$$

Of course, $\wedge$ and $\vee$ are means. We use also the following means (most of them were studied in [1]):

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— the *weighted Gini mean* defined by

$$\mathcal{B}_{r,s;\lambda}(a,b) = \left[\frac{\lambda a^r + (1-\lambda)\, b^r}{\lambda a^s + (1-\lambda)\, b^s}\right]^{\frac{1}{r-s}}, \ r \neq s$$

and

$$\mathcal{B}_{r,r;\lambda}(a,b) = \exp\left[\frac{\lambda a^r \log a + (1-\lambda)\, b^r \log b}{\lambda a^r + (1-\lambda)\, b^r}\right],$$

with $\lambda \in (0,1)$ fixed;

— the special case $\mathcal{B}_{r,0;\lambda} = \mathcal{P}_{r;\lambda}$ called *weighted power mean;*

— another special case, $\mathcal{B}_{r,r-1;\lambda} = \mathcal{C}_{r;\lambda}$ represents the *weighted Lehmer mean;*

— the *weighted Muirhead mean*

$$\mathcal{S}_{r,s;\lambda}(a,b) = [\lambda a^r b^s + (1-\lambda) b^r a^s]^{\frac{1}{r+s}}, rs > 0, \lambda \in (0,1);$$

— for $\lambda = 1/2$ we get symmetric means which we denote by $\mathcal{B}_{r,s}, \mathcal{P}_r, \mathcal{C}_r$ respectively $\mathcal{S}_{r,s}$;

— the *Stolarsky mean* given by

$$\mathcal{E}_{r,s}(a,b) = \left(\frac{s}{r}\frac{b^r - a^r}{b^s - a^s}\right)^{\frac{1}{r-s}}, \ rs(r-s) \neq 0;$$

— the *generalized logarithmic mean*

$$\mathcal{L}_r(a,b) = \mathcal{E}_{r,0}(a,b) = \left(\frac{b^r - a^r}{r\,(\log b - \log a)}\right)^{\frac{1}{r}}, r \neq 0,$$

with the special case of *logarithmic mean* $\mathcal{L} = \mathcal{L}_1$;

— the *generalized identric mean*

$$\mathcal{I}_r(a,b) = \mathcal{E}_{r,r}(a,b) = \frac{1}{e^{1/r}}\left(\frac{b^{b^r}}{a^{a^r}}\right)^{1/(b^r - a^r)}, r \neq 0$$

and its special case of *identric mean* $\mathcal{I} = \mathcal{I}_1$;

— the *geometric mean*

$$\mathcal{G}(a,b) = \mathcal{E}_{0,0}(a,b) = \sqrt{ab};$$

— an exponential type mean (defined in [3])

$$\mathcal{T}(a,b) = \frac{ae^a - be^b}{e^a - e^b} - 1;$$

— the *Greek means* defined by the Pythagorean school: the arithmetic mean $\mathcal{A} = \mathcal{P}_1$, the geometric mean $\mathcal{G}$, the harmonic mean $\mathcal{H} = \mathcal{P}_{-1}$, the contraharmonic mean $\mathcal{C} = \mathcal{C}_2$, and six unnamed means $\mathcal{F}_i, (i = 5, ..., 10)$, given by the following expressions (see [4] for more details):

$$\mathcal{F}_5 = \frac{\vee - \wedge + \sqrt{(\vee - \wedge)^2 + 4\wedge^2}}{2}; \qquad \mathcal{F}_6 = \frac{\wedge - \vee + \sqrt{(\vee - \wedge)^2 + 4\vee^2}}{2};$$

$$\mathcal{F}_7 = \frac{\vee^2 - \vee\wedge + \wedge^2}{\vee}; \qquad \mathcal{F}_8 = \frac{\vee^2}{2\vee - \wedge};$$

$$\mathcal{F}_9 = \frac{\wedge(2\vee - \wedge)}{\vee}; \qquad \mathcal{F}_{10} = \frac{\wedge + \sqrt{\wedge(4\vee - 3\wedge)}}{2}.$$

2. Upper universal means

Let us introduce the following

DEFINITION 2. A mean U is called *upper universal* if there exists a constant $p > 0$ such that

$$p\vee \leqslant U \leqslant \vee.$$

REMARK 3. Of course U is an *upper universal* mean if and only if the inequality

$$M \leqslant \frac{1}{p}U$$

holds for every mean M .

THEOREM 4. *The following means are upper universal: $i)$ $\mathcal{B}_{r,s;\lambda}$ for $r > s > 0$; $ii)$ $\mathcal{B}_{r,r;\lambda}$ for $r > 0$; $iii)$ $\mathcal{C}_{r;\lambda}$ for $r > 1$; $iv)$ $\mathcal{P}_{r;\lambda}$ for $r > 0$; $v)$ $\mathcal{E}_{r,s}$ for $rs(r-s) \neq 0$; $vi)$ $\mathcal{F}_5$; $vii)$ $\mathcal{F}_6$; $viii)$ $\mathcal{F}_7$; $ix)$ $\mathcal{F}_8$.*

Proof. $i)$ If $a > b$ then $\mathcal{B}_{r,s;\lambda}(a,b) \geqslant \lambda^{\frac{1}{r-s}} a$. Indeed, for $r > s$ this is equivalent with the condition

$$\lambda(1-\lambda)a^r \geqslant (1-\lambda)b^s\left(\lambda a^{r-s} - b^{r-s}\right).$$

But this is true as

$$\lambda a^r \geqslant \lambda a^{r-s}b^s \geqslant b^s\left(\lambda a^{r-s} - b^{r-s}\right), \text{ for } s > 0.$$

If $a < b$ then $\mathcal{B}_{r,s;\lambda}(a,b) \geqslant (1-\lambda)^{\frac{1}{r-s}} b$ which is equivalent with the condition

$$\lambda(1-\lambda)b^r \geqslant \lambda a^s\left[(1-\lambda)b^{r-s} - a^{r-s}\right],$$

or, for $s > 0$,

$$(1-\lambda)b^r \geqslant (1-\lambda)b^{r-s}a^s \geqslant a^s\left[(1-\lambda)b^{r-s} - a^{r-s}\right].$$

Thus $\mathcal{B}_{r,s;\lambda}$ is an upper universal mean.

ii) If $a > b$ then $\mathcal{B}_{r,r;\lambda}(a,b) \geqslant pa$ holds if and only if

$$(1-\lambda)\, b^r (\ln b - \ln a) \geqslant \ln p \cdot [\lambda a^r + (1-\lambda)\, b^r]\,.$$

Denoting $(a/b)^r = t > 1,$ we get the condition

$$f(t) = \frac{\ln t}{\lambda t + 1 - \lambda} \leqslant \frac{r}{1-\lambda} \ln \frac{1}{p}.$$

But $f(t) \leqslant m,$ for $t > 1,$ so that we can choose $p = \exp(-(1-\lambda)\cdot m/r)$. Similar computations can be done for $a < b$.

iii) It is a simple consequence of *i*) for $s = r - 1$.

iv) Is not a consequence of *i*) because $s = 0,$ but it can be done on the same way.

v) It is easy to verify that $\mathcal{E}_{r,s} \geqslant \left(\frac{s}{r}\right)^{\frac{1}{r-s}} \vee$.

vi) The inequality $\mathcal{F}_5 \geqslant p\vee$ is equivalent with the condition $\wedge^2 - p \vee \wedge + \left(p - p^2\right) \vee^2 \geqslant 0$. This holds for $p \leqslant 4/5$.

vii) Similarly, $\mathcal{F}_6 \geqslant p\vee$ if $\left(p^2 + p - 1\right) \vee \leqslant p\wedge$ thus $p \leqslant \left(\sqrt{5} - 1\right)/2.$

viii) $\mathcal{F}_7 \geqslant p\vee$ if $(1-p)\vee^2 - \vee\wedge + \wedge^2 \geqslant 0,$ hence $p \leqslant 3/4.$

ix) $\mathcal{F}_8 \geqslant p\vee$ if $(1-2p)\vee + p\wedge \geqslant 0,$ thus $p \leqslant 1/2.$ □

REMARK 5. If $U \geqslant p\vee$, then $U \geqslant p'\vee$ for every $p' < p$, thus it can be interesting to determine the greatest value of p with the given property. The value given before for p is not the best in the case of $\mathcal{C}_{r;\lambda}$. For instance,

$$\mathcal{C} \geqslant p\vee \quad \text{if} \quad (1-p)\vee^2 - p\vee\wedge + \wedge^2 \geqslant 0,$$

thus the best value of p is $2\left(\sqrt{2} - 1\right)$ not $1/2$.

We can prove also general results of following types.

THEOREM 6. *If $f : \mathbb{R}_+ \to \mathbb{R}$ is a bijective function and f^{-1} is concave then the quasi arithmetic mean $\mathcal{A}_{f;\lambda}$ defined by*

$$\mathcal{A}_{f;\lambda}(a,b) = f^{-1}(\lambda f(a) + (1-\lambda) f(b))$$

is upper universal.

Proof. As f^{-1} is concave then

$$f^{-1}(\lambda f(a) + (1-\lambda) f(b)) \geqslant \lambda f^{-1} \circ f(a) + (1-\lambda) f^{-1} \circ f(b) = \lambda a + (1-\lambda) b,$$

thus $\mathcal{A}_{f;\lambda} \geqslant p\vee$, where $p = \min(\lambda, 1-\lambda)$. □

EXAMPLE 7. Take $f = \exp$.

THEOREM 8. *If* $f : \mathbb{R}_+ \to \mathbb{R}$ *is an increasing convex function, then*

$$\mathcal{M}_f(a,b) = f^{-1}\left(\frac{1}{b-a}\int_a^b f(x)dx\right), \forall a,b>0,$$

defines an upper universal mean $\mathcal{M}_f$.

Proof. The Hadamard inequality for the convex function f gives

$$\frac{1}{b-a}\int_a^b f(x)dx \geqslant f\left(\frac{a+b}{2}\right), \forall a,b>0.$$

As f is increasing, we get $\mathcal{M}_f(a,b) \geqslant (a+b)/2$ thus $\mathcal{M}_f \geqslant \vee/2$. □

EXAMPLE 9. Take again $f = \exp$.

3. Lower universal means

DEFINITION 10. A mean U is called *lower universal* if there exists a constant p such that

$$\wedge \leqslant U \leqslant p\wedge .$$

REMARK 11. The mean U is *lower universal* if and only if the inequality

$$M \geqslant \frac{1}{p}U$$

holds for every mean M .

THEOREM 12. *The following means are lower universal:* *i)* $\mathcal{B}_{r,s;\lambda}$ *for* $r<s<0$; *ii)* $\mathcal{B}_{r,r;\lambda}$ *for* $r<0$; *iii)* $\mathcal{P}_{r;\lambda}$ *for* $r<0$; *iv)* $\mathcal{F}_9$.

Proof. *i)* If $a>b$ then $\mathcal{B}_{r,s;\lambda}(a,b) \leqslant (1-\lambda)^{\frac{1}{r-s}} b$. Indeed, for $r<s$ this is equivalent with the condition

$$\lambda(1-\lambda)b^r \geqslant \lambda a^s\left[(1-\lambda)b^{r-s} - a^{r-s}\right].$$

But this is true because

$$(1-\lambda)b^r \geqslant (1-\lambda)b^{r-s}a^s \geqslant a^s\left[(1-\lambda)b^{r-s} - a^{r-s}\right], \text{ for } s<0.$$

If $a<b$ we have $\mathcal{B}_{r,s;\lambda}(a,b) \leqslant \lambda^{\frac{1}{r-s}} a$ which is equivalent with the condition

$$\lambda(1-\lambda)a^r \geqslant (1-\lambda)b^s\left(\lambda a^{r-s} - b^{r-s}\right),$$

or

$$\lambda a^r \geqslant \lambda a^{r-s}b^s \geqslant b^s\left(\lambda a^{r-s} - b^{r-s}\right), \text{ for } s<0.$$

Thus $\mathcal{B}_{r,s;\lambda} \leqslant p\wedge$, for $p = \max\left\{\lambda^{\frac{1}{r-s}}, (1-\lambda)^{\frac{1}{r-s}}\right\}$.

ii) If $a > b$ then $\mathcal{B}_{r,r;\lambda}(a,b) \geqslant pa$ holds if and only if

$$\lambda a^r (\ln a - \ln b) \leqslant \ln p \cdot [\lambda a^r + (1-\lambda) b^r].$$

Denoting $(b/a)^r = t > 1$, we get the condition

$$f(t) = \frac{\ln t}{\lambda t + (1-\lambda) t} \leqslant \frac{r}{\lambda} \ln \frac{1}{p}.$$

But $f(t) \leqslant m$, for $t > 1$, so that we can choose $p = \exp(-\lambda \cdot m/r)$. Similar computations can be done for $a < b$.

iii) If $a > b$ then $\mathcal{P}_{r;\lambda}(a,b) \leqslant (1-\lambda)^{1/r} b$, while for $a < b$ the inequality $\mathcal{P}_{r;\lambda}(a,b) \leqslant \lambda^{1/r} a$ is valid. Thus $\mathcal{C}_{r;\lambda} \leqslant p\wedge$, for $p = \max\left\{\lambda^{\frac{1}{r}}, (1-\lambda)^{\frac{1}{r}}\right\}$.

iv) We have $\mathcal{F}_9 \leqslant p\wedge$ if and only if $(2-p)\vee \leqslant \wedge$ thus $p = 2$. □

4. Universal means

DEFINITION 13. A mean U is called *universal* if it is upper universal or lower universal.

REMARK 14. If U is an universal mean then there exists a constant p such that each mean is comparable with pU.

THEOREM 15. *The following means are not universal: i) the Gini mean $\mathcal{B}_{r,s;\lambda}$ for $s < 0 < r$; ii) the logarithmic mean $\mathcal{L}_r$, $r \neq 0$; iii) the Muirhead mean $\mathcal{S}_{r,s;\lambda}$; iv) the geometric mean $\mathcal{G}$; v) the Greek mean $\mathcal{F}_{10}$.*

Proof. In the first four cases, we consider the functions:

i)

$$\frac{\mathcal{B}_{r,s;\lambda}(1,x)}{x} = \left[\frac{\lambda x^{-r} + 1 - \lambda}{\lambda x^{-s} + 1 - \lambda}\right]^{\frac{1}{r-s}},$$

ii)

$$\frac{\mathcal{L}_{r;\lambda}(1,x)}{x} = \left(\frac{x^r - 1}{r \cdot x^r \cdot \ln x}\right)^{\frac{1}{r}},$$

iii)

$$\frac{\mathcal{S}_{r,s;\lambda}(1,x)}{x} = \left[\lambda x^{-r} + (1-\lambda) x^{-s}\right]^{\frac{1}{r+s}},$$

respectively

iv)

$$\frac{G(1,x)}{x} = \frac{1}{\sqrt{x}}.$$

Each of them tends to 0 for $x \to \infty$ and to ∞ for $x \to 0$. In the case v), we have

$$\frac{\mathcal{F}_{10}}{\vee} = \frac{1}{2}\left[\frac{\wedge}{\vee} + \sqrt{\frac{\wedge}{\vee}\left(4 - 3\frac{\wedge}{\vee}\right)}\right],$$

which tends to 0 if $\vee \to \infty$ and $\wedge = 1$. Also

$$\frac{\mathcal{F}_{10}}{\wedge} = \frac{1}{2}\left(1 + \sqrt{4\frac{\vee}{\wedge} - 3}\right),$$

which tends to ∞ if $\wedge \to 0$ and $\vee = 1$. In each case, the first limit proves that the corresponding mean is not upper universal, while the second limit proves that the mean is not lower universal. □

REMARK 16. As $\mathcal{B}_{s,r;\lambda} = \mathcal{B}_{r,s;\lambda}$, we can decide about the universality property of each Gini mean using one of the previous results.

REMARK 17. If U is an upper (lower) universal mean and $W \geqslant qU$ (respectively $W \leqslant qU$), then W is also an upper (respectively lower) universal mean.

EXAMPLE 18. The following inequalities

$$\mathcal{H} < \mathcal{G} < \mathcal{L} < \mathcal{P}_{1/3} < \mathcal{I} < \mathcal{A},$$

are well known (see for instance [1]). They imply that the identric mean $\mathcal{I}$ is upper universal.

REMARK 19. In the previous sequence of inequalities the first mean $\mathcal{H}$ is lower universal, then $\mathcal{G}$ and $\mathcal{L}$ are not universal, while the last means are upper universal. In fact this order cannot be changed. Moreover, if U is a lower universal mean, W is an upper universal mean and Z is a non universal mean, then there exist the positive numbers p and q such that

$$pU \leqslant Z \leqslant qW.$$

REMARK 20. It is not easy to prove directly that $\mathcal{I}$ is upper universal. On the other hand, the inequality $\mathcal{I} \geqslant \frac{1}{8}\vee$, which follows from $\mathcal{P}_{1/3} < \mathcal{I}$ is not the best. For instance, in [2] is proved that $I > \frac{2}{e}\mathcal{A}$ which implies that

$$\mathcal{I} \geqslant \frac{1}{e}\vee.$$

COROLLARY 21. *The identric mean $\mathcal{I}_r$ is upper universal for $r > 0$ and lower universal for $r < 0$.*

Proof. As $\mathcal{I}_r(a,b) = [\mathcal{I}(a^r,b^r)]^{1/r}$, the inequality $\mathcal{P}_{1/3} < \mathcal{I} < \mathcal{P}_1$ becomes $\mathcal{P}_{r/3} < \mathcal{I}_r < \mathcal{P}_r$ if $r > 0$ and $\mathcal{P}_{r/3} > \mathcal{I}_r > \mathcal{P}_r$ if $r < 0$. □

EXAMPLE 22. In [3] is proved that

$$\mathcal{A} < \mathcal{T},$$

giving that $\mathcal{T}$ is also upper universal.

REFERENCES

[1] P. S. BULLEN, *Handbook of Means and Their Inequalities*, Kluwer Acad. Publ., Dordrecht, 2003.
[2] J. SÁNDOR, *On certain inequalities for means*, J. Math. Anal. Appl. **189** (1995), no. 2, 602–606.
[3] G. TOADER, *An exponential mean*, "Babes-Bolyai" Univ. Preprint **7** (1988), 51–54.
[4] G. TOADER, S. TOADER, *Greek means and the Arithmetic-Geometric Mean*, RGMIA Monographs, Victoria University, 2005. (ON-LINE: http://rgmia.vu.edu.au/monographs).

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