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STABILITY OF HOMOMORPHISMS ON JB^*–TRIPLES ASSOCIATED TO A CAUCHY–JENSEN TYPE FUNCTIONAL EQUATION

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Abstract. In this paper, we investigate homomorphisms between JB^*-triples, and derivations on JB^*-triples associated to the following Cauchy–Jensen type additive functional equation

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+z}{2}+y\right)+f\left(\frac{y+z}{2}+x\right)=2[f(x)+f(y)+f(z)].$$

The concept of Hyers-Ulam-Rassias stability originated from Th. M. Rassias' stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms : *Let* $(G_1, *)$ *be a group and let* $(G_2, \diamond, d)$ *be a metric group with the metric* $d(\cdot,\cdot)$. *Given* $\epsilon > 0$, *does there exist a* $\delta(\epsilon) > 0$ *such that if a mapping* $h : G_1 \to G_2$ *satisfies the inequality*

$$d(h(x*y), h(x)\diamond h(y)) < \delta$$

for all $x, y \in G_1$, *then there is a homomorphism* $H : G_1 \to G_2$ *with*

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. In 1941, Hyers [6] considered the case of approximately additive mappings in Banach spaces and satisfying the well-known weak Hyers inequality controlled by a positive constant. In 1978, Th. M. Rassias [17] provided a generalization of Hyers'theorem which allows the *Cauchy difference to be unbounded.*

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THEOREM 1.1. (Th. M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leqslant \frac{2\epsilon}{2-2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. Also, if the mapping $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is $\mathbb{R}$-linear.

In 1990, Th. M. Rassias [18] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geqslant 1$. In 1991, Z. Gajda [4] following the same approach as in Th. M. Rassias [17], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [4], as well as by Th. M. Rassias and P. Šemrl [22] that one cannot prove a Th. M. Rassias' type theorem when $p = 1$. The counterexamples of Z. Gajda [4], as well as of Th. M. Rassias and P. Šemrl [22] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. P. Găvruta [5], S. Jung [10], who among others studied the Hyers–Ulam–Rassias stability of functional equations. The inequality (1.1) that was introduced for the first time by Th. M. Rassias [17] provided a lot of influence in the development of a generalization of the Hyers–Ulam stability concept. This new concept is known as *Hyers–Ulam–Rassias stability* of functional equations (cf. the books of P. Czerwik [3], D. H. Hyers, G. Isac and Th. M. Rassias [7]).

J. M. Rassias [16] following the spirit of the innovative approach of Th. M. Rassias [17] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

P. Găvruta [5] provided a further generalization of Th. M. Rassias' theorem. In 1996, G. Isac and Th. M. Rassias [9] applied the Hyers–Ulam–Rassias stability theory to prove fixed point theorems and study some new applications in Nonlinear Analysis. In [8], D. H. Hyers, G. Isac and Th. M. Rassias studied the asymptoticity aspect of Hyers–Ulam stability of mappings. During the past few years several mathematicians have published on various generalizations and applications of Hyers–Ulam stability and Hyers–Ulam–Rassias stability to a number of functional equations and mappings, for example: quadratic functional equation, invariant means, multiplicative mappings – superstability, bounded nth differences, convex functions, generalized orthogonality functional equation, Euler–Lagrange functional equation and Navier–Stokes equations. Several mathematicians have contributed works on these subjects; we mention a few: C. Park [11]–[15], Th. M. Rassias [19]–[21], F. Skof [25].

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 5, 11, 14, 20]).

We recall that a (complex) JB^*-triple is a complex Banach space $\mathcal{J}$ with a continuous triple product $\mathcal{J}\times\mathcal{J}\times\mathcal{J}\ni(x,y,z)\mapsto\{x,y,z\}\in\mathcal{J}$ which is bilinear and symmetric in the outer variables and conjugate linear in the middle variable, and satisfies:

(i) (Jordan Identity) $L(a,b)\{x,y,z\}=\{L(a,b)x,y,z\}-\{x,L(b,a)y,z\}+\{x,y,L(a,b)z\}$ for all $a,b,x,y,z\in\mathcal{J}$, where $L(a,b)x:=\{a,b,x\}$;

(ii) The operator $L(a,a):\mathcal{J}\to\mathcal{J}$ is an hermitian operator with non-negative spectrum;

(iii) $\|L(a,a)a\|_{\mathcal{J}}=\|a\|_{\mathcal{J}}^3$ for all $a\in\mathcal{J}$.

Every C^*-algebra is a (complex) JB^*-triple with respect to $\{a,b,c\}=\frac{1}{2}(ab^*c+cb^*a)$. Also, every JB^*-algebra is a JB^*-triple with respect to $\{a,b,c\}=(a\circ b^*)\circ c+(c\circ b^*)\circ a-(a\circ c)\circ b^*$. Conversely, every JB^*-triple with a unitary element e (that is, $\{e,e,a\}=\{a,e,e\}=a$ for every a) is a unital JB^*-algebra with product $a\circ b=\{a,e,b\}$, involution $a^*=\{e,a,e\}$, and unit e. We refer to [2], [23] and [24] for recent surveys on the theory of JB^*-triples.

A $\mathbb{C}$-linear mapping $H:\mathcal{J}_1\to\mathcal{J}_2$ between JB^*-triples is called a JB^*-triple homomorphism if

$$H(\{x,y,z\})=\{H(x),H(y),H(z)\}$$

for all $x,y,z\in\mathcal{J}_1$. A $\mathbb{C}$-linear mapping $d:\mathcal{J}\to\mathcal{J}$ is called a JB^*-triple derivation if

$$d(\{x,y,z\})=\{d(x),y,z\}+\{x,d(y),z\}+\{x,y,d(z)\}$$

for all $x,y,z\in\mathcal{J}$ (see [12]).

2. Stability of homomorphisms between JB^*-triples

Throughout this section, assume that $\mathcal{J}$ is a JB^*-triple with norm $\|.\|_{\mathcal{J}}$ and that $\mathcal{K}$ is a JB^*-triple with norm $\|.\|_{\mathcal{K}}$.

We will use the following Lemma in this paper:

LEMMA 2.1. *[13] Let X and Y be linear spaces and let $f:X\to Y$ be an additive mapping such that $f(\mu x)=\mu f(x)$ for all $x\in X$ and all $\mu\in\mathbb{T}^1$. Then the mapping f is $\mathbb{C}$-linear.*

LEMMA 2.2. *Let X be a uniquely 2-divisible abelian group and Y a linear space. A mapping $f:X\to Y$ satisfies*

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+z}{2}+y\right)+f\left(\frac{y+z}{2}+x\right)=2[f(x)+f(y)+f(z)] \quad (2.1)$$

for all $x,y,z\in X$ if and only if $f:X\to Y$ is additive.

Proof. Suppose that f satisfies (2.1). Letting $y = z = x$ in (2.1), we get $f(2x) = 2f(x)$ for all $x \in X$. So $f(0) = 0$ and $2f(x/2) = f(x)$ for all $x \in X$. Therefore by letting $y = -x$ and $z = 0$ in (2.1), we get $f(-x) = -f(x)$ for all $x \in X$. Letting $z = -y$ in (2.1), we get

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = f(x) \tag{2.2}$$

for all $x, y \in X$. Replacing x and y by $x+y$ and $x-y$ in (2.2), respectively, we infer that $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. So the mapping $f : X \to Y$ is additive.

It is clear that each additive mapping satisfies (2.1). $\square$

Let X and Y be linear spaces. For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x,y,z) := f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x+z}{2} + y\right) + f\left(\frac{y+z}{2} + x\right) \\ - 2f(x) - 2f(y) - 2f(z), \end{aligned}$$

$$\begin{aligned} D_\mu f(x,y,z) := f\left(\frac{\mu x+\mu y}{2} + \mu z\right) + f\left(\frac{\mu x+\mu z}{2} + \mu y\right) + f\left(\frac{\mu y+\mu z}{2} + \mu x\right) \\ - 2\mu f(x) - 2\mu f(y) - 2\mu f(z) \end{aligned}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x, y, z \in X$.

NOTATION. Let X be a linear space. $x \in X^\star$ means $x \in X$ or $x \in X \setminus \{0\}$.

PROPOSITION 2.3. *Let X and Y be linear spaces and let $f : X \to Y$ be a mapping such that*

$$D_\mu f(x,y,z) = 0 \tag{2.3}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in X$. Then the mapping $f : X \to Y$ is $\mathbb{C}$-linear.

Proof. Letting $y = z = 0$ in (2.3) and using Lemma 2.2, we get $f(\mu x) = \mu f(x)$ for all $x \in X$. Now by using Lemmas 2.1 and 2.2, we infer that the mapping $f : X \to Y$ is $\mathbb{C}$-linear. $\square$

LEMMA 2.4. *Let X and Y be linear spaces. A mapping $f : X \to Y$ satisfies (2.1) for all $x, y, z \in X \setminus \{0\}$ if and only if $f : X \to Y$ is additive.*

Proof. Suppose that f satisfies (2.1). Letting $y = z = x$ in (2.1), we get

$$f(2x) = 2f(x) \tag{2.4}$$

for all $x \in X \setminus \{0\}$. Letting $y = z = -x$ in (2.1), we get

$$2f(-x) + 2f(x) = f(0) \tag{2.5}$$

for all $x \in X \setminus \{0\}$. Letting $y = 3x, z = -x$ in (2.1) and using (2.4), we get

$$f(3x) = f(x) - 2f(-x) \tag{2.6}$$

for all $x \in X \setminus \{0\}$. It follows from (2.4) that $2f(x/2) = f(x)$ for all $x \in X \setminus \{0\}$. So by letting $y = x$ and $z = 2x$ in (2.1) and using (2.4), we get

$$f(5x) + f(3x) = 8f(x) \tag{2.7}$$

for all $x \in X \setminus \{0\}$. Putting $y = 5x$ and $z = -x$ in (2.1) and using (2.5), we get

$$f(5x) - f(3x) = 2f(x) - f(0) \tag{2.8}$$

for all $x \in X \setminus \{0\}$. It follows from (2.7) and (2.8) that

$$2f(3x) = 6f(x) + f(0) \tag{2.9}$$

for all $x \in X \setminus \{0\}$. It follows from (2.6) and (2.9) that

$$4[f(x) + f(-x)] + f(0) = 0 \tag{2.10}$$

for all $x \in X \setminus \{0\}$. It follows from (2.5) and (2.10) that $f(0) = 0$. Hence it follows from (2.5) that f is odd. Therefore by letting $z = -x$ in (2.1), we get

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{y-x}{2}\right) = f(y) \tag{2.11}$$

for all $x, y \in X \setminus \{0\}$. Since f is odd, then (2.11) holds for all $x, y \in X$. Replacing x and y by $x - y$ and $x + y$ in (2.11), respectively, we get $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. So the mapping $f : X \to Y$ is additive.

It is clear that each additive mapping satisfies (2.1). □

PROPOSITION 2.5. *Let X and Y be linear spaces and let $f : X \to Y$ be a mapping satisfying (2.3) for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in X \setminus \{0\}$. Then the mapping $f : X \to Y$ is $\mathbb{C}$-linear.*

Proof. Letting $x = y = z$ in (2.3) and using Lemma 2.4, we get $f(\mu x) = \mu f(x)$ for all $x \in X$. Now by using Lemmas 2.1 and 2.4, we infer that the mapping $f : X \to Y$ is $\mathbb{C}$-linear. □

THEOREM 2.6. *Let $\varphi : \mathcal{J} \times \mathcal{J} \times \mathcal{J} \to [0, \infty)$ and $\psi : \mathcal{J} \times \mathcal{J} \times \mathcal{J} \to [0, \infty)$ be functions such that*

$$\widetilde{\varphi}(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} \varphi(2^n x, 2^n x, 2^n x) < \infty, \quad \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n z) = 0, \tag{2.12}$$

$$\lim_{n \to \infty} \frac{1}{8^n} \psi(2^n x, 2^n y, 2^n z) = 0 \tag{2.13}$$

for all $x, y, z \in \mathcal{J}$. Suppose that $f : \mathcal{J} \to \mathcal{K}$ is a mapping satisfying

$$\|D_\mu f(x, y, z)\|_{\mathcal{K}} \leqslant \varphi(x, y, z), \tag{2.14}$$

$$\left\| f(\{x, y, z\}) - \left\{ f(x), f(y), f(z) \right\} \right\|_{\mathcal{K}} \leqslant \psi(x, y, z) \tag{2.15}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{J}$. Then there exists a unique JB^*-triple homomorphism $H : \mathcal{J} \to \mathcal{K}$ such that

$$\left\|f(x) - H(x)\right\|_{\mathcal{K}} \leqslant \frac{1}{6}\widetilde{\varphi}(x) \tag{2.16}$$

for all $x \in \mathcal{J}$.

Proof. Letting $\mu = 1$ and $x = y = z$ in (2.14), we get

$$\|3f(2x) - 6f(x)\|_{\mathcal{K}} \leqslant \varphi(x, x, x) \tag{2.17}$$

for all $x \in \mathcal{J}$. If we replace x by $2^n x$ in (2.17) and divide both sides of (2.17) by $3 \times 2^{n+1}$, we get

$$\left\|\frac{1}{2^{n+1}}f(2^{n+1}x) - \frac{1}{2^n}f(2^n x)\right\|_{\mathcal{K}} \leqslant \frac{1}{3 \times 2^{n+1}}\varphi(2^n x, 2^n x, 2^n x)$$

for all $x \in \mathcal{J}$ and all non-negative integers n. Hence

$$\begin{aligned}\left\|\frac{1}{2^{n+1}}f(2^{n+1}x) - \frac{1}{2^m}f(2^m x)\right\|_{\mathcal{K}} &= \left\|\sum_{k=m}^{n}\left[\frac{1}{2^{k+1}}f(2^{k+1}x) - \frac{1}{2^k}f(2^k x)\right]\right\|_{\mathcal{K}} \\ &\leqslant \sum_{k=m}^{n}\left\|\frac{1}{2^{k+1}}f(2^{k+1}x) - \frac{1}{2^k}f(2^k x)\right\|_{\mathcal{K}} \\ &\leqslant \frac{1}{6}\sum_{k=m}^{n}\frac{1}{2^k}\varphi(2^k x, 2^k x, 2^k x)\end{aligned} \tag{2.18}$$

for all $x \in \mathcal{J}$ and all non-negative integers $n \geqslant m \geqslant 0$. It follows from (2.12) and (2.18) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence in $\mathcal{K}$ for all $x \in \mathcal{J}$. Since $\mathcal{K}$ is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges for all $x \in \mathcal{J}$. Thus one can define the mapping $H : \mathcal{J} \to \mathcal{K}$ by

$$H(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in \mathcal{J}$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.18) we get (2.16). It follows from (2.12) that

$$\begin{aligned}\left\|D_\mu H(x, y, z)\right\|_{\mathcal{K}} &= \lim_{n\to\infty}\frac{1}{2^n}\left\|D_\mu f(2^n x, 2^n y, 2^n z)\right\|_{\mathcal{K}} \\ &\leqslant \lim_{n\to\infty}\frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z) = 0\end{aligned}$$

for all $x, y, z \in \mathcal{J}$. So $D_\mu H(x, y, z) = 0$ for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{J}$. By Proposition 2.3 the mapping $H : \mathcal{J} \to \mathcal{K}$ is $\mathbb{C}$-linear.

It follows from (2.13) and (2.15) that

$$\begin{aligned}
&\left\|H(\{x,y,z\}) - \left\{H(x),H(y),H(z)\right\}\right\|_{\mathcal{K}} \\
&= \lim_{n\to\infty} \frac{1}{8^n}\left\|f(\{2^n x, 2^n y, 2^n z\}) - \left\{f(2^n x), f(2^n y), f(2^n z)\right\}\right\|_{\mathcal{K}} \\
&\leqslant \lim_{n\to\infty} \frac{1}{8^n}\psi(2^n x, 2^n y, 2^n z) = 0
\end{aligned}$$

for all $x,y,z \in \mathcal{J}$. Therefore

$$H(\{x,y,z\}) = \left\{H(x),H(y),H(z)\right\}$$

for all $x,y,z \in \mathcal{J}$. Therefore the mapping $H : \mathcal{J} \to \mathcal{K}$ is a JB^* -triple homomorphism.

Now, let $T : \mathcal{J} \to \mathcal{K}$ be another JB^* -triple homomorphism satisfying (2.16). Then we have from (2.12) that

$$\begin{aligned}
\|H(x) - T(x)\|_{\mathcal{K}} &= \lim_{n\to\infty} \frac{1}{2^n}\|f(2^n x) - T(2^n x)\|_{\mathcal{K}} \\
&\leqslant \frac{1}{6}\lim_{n\to\infty}\frac{1}{2^n}\widetilde{\varphi}(2^n x) \\
&= \frac{1}{6}\lim_{n\to\infty}\sum_{k=n}^{\infty}\frac{1}{2^k}\varphi(2^k x, 2^k x, 2^k x) = 0
\end{aligned}$$

for all $x \in \mathcal{J}$. So $H(x) = T(x)$ for all $x \in \mathcal{J}$. This proves the uniqueness of H. Thus the mapping $H : \mathcal{J} \to \mathcal{K}$ is a unique JB^* -triple homomorphism satisfying (2.16). □

COROLLARY 2.7. *Let ϵ, θ be non-negative real numbers and let $p_1, p_2, p_3, q_1, q_2, q_3$ be real numbers such that $p_1, p_2, p_3 < 1$ and $q_1, q_2, q_3 < 3$. Suppose that $f : \mathcal{J} \to \mathcal{K}$ is a mapping satisfying*

$$\|D_\mu f(x,y,z)\|_{\mathcal{K}} \leqslant \theta(\|x\|_{\mathcal{J}}^{p_1} + \|y\|_{\mathcal{J}}^{p_2} + \|z\|_{\mathcal{J}}^{p_3}), \tag{2.19}$$

$$\left\|f(\{x,y,z\}) - \left\{f(x),f(y),f(z)\right\}\right\|_{\mathcal{K}} \leqslant \epsilon(\|x\|_{\mathcal{J}}^{q_1} + \|y\|_{\mathcal{J}}^{q_2} + \|z\|_{\mathcal{J}}^{q_3}) \tag{2.20}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in \mathcal{J}^\star$. Then there exists a unique JB^* -triple homomorphism $H : \mathcal{J} \to \mathcal{K}$ such that

$$\left\|f(x) - H(x)\right\|_{\mathcal{K}} \leqslant \frac{\theta}{3}\sum_{i=1}^{3}\frac{1}{2-2^{p_i}}\|x\|_{\mathcal{J}}^{p_i}$$

for all $x \in \mathcal{J}^\star$.

Proof. The result follows by Proposition 2.5 and Theorem 2.6. □

THEOREM 2.8. *Let $\Phi : \mathcal{J}\times\mathcal{J}\times\mathcal{J}\to[0,\infty)$ and $\Psi : \mathcal{J}\times\mathcal{J}\times\mathcal{J}\to[0,\infty)$ be functions such that*

$$\widetilde{\Phi}(x) := \sum_{n=1}^{\infty} 2^n\Phi\Big(\frac{x}{2^n},\frac{x}{2^n},\frac{x}{2^n}\Big) < \infty, \quad \lim_{n\to\infty} 2^n\Phi\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big) = 0, \tag{2.21}$$

$$\lim_{n\to\infty} 8^n\Psi\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big) = 0 \tag{2.22}$$

for all $x,y,z\in\mathcal{J}$. Suppose that $f:\mathcal{J}\to\mathcal{K}$ is a mapping satisfying

$$\|D_\mu f(x,y,z)\|_{\mathcal{K}} \leqslant \Phi(x,y,z), \tag{2.23}$$

$$\Big\|f(\{x,y,z\}) - \Big\{f(x),f(y),f(z)\Big\}\Big\|_{\mathcal{K}} \leqslant \Psi(x,y,z) \tag{2.24}$$

for all $\mu\in\mathbb{T}^1$ and all $x,y,z\in\mathcal{J}$. Then there exists a unique JB^*-triple homomorphism $H:\mathcal{J}\to\mathcal{K}$ such that

$$\big\|f(x) - H(x)\big\|_{\mathcal{K}} \leqslant \frac{1}{6}\widetilde{\Phi}(x) \tag{2.25}$$

for all $x\in\mathcal{J}$.

Proof. Letting $\mu = 1$ and $x = y = z$ in (2.23), we get

$$\|f(2x) - 2f(x)\|_{\mathcal{K}} \leqslant \frac{1}{3}\Phi(x,x,x) \tag{2.26}$$

for all $x\in\mathcal{J}$. If we replace x by $\frac{x}{2^{n+1}}$ in (2.26) and multiply both sides of (2.26) to 2^n, we get

$$\Big\|2^{n+1}f\Big(\frac{x}{2^{n+1}}\Big) - 2^n f\Big(\frac{x}{2^n}\Big)\Big\|_{\mathcal{K}} \leqslant \frac{2^n}{3}\Phi\Big(\frac{x}{2^{n+1}},\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\Big)$$

for all $x\in\mathcal{J}$ and all non-negative integers n. Hence

$$\begin{aligned}\Big\|2^{n+1}f\Big(\frac{x}{2^{n+1}}\Big) - 2^m f\Big(\frac{x}{2^m}\Big)\Big\|_{\mathcal{K}} &= \Big\|\sum_{k=m}^{n}\Big[2^{k+1}f\Big(\frac{x}{2^{k+1}}\Big) - 2^k f\Big(\frac{x}{2^k}\Big)\Big]\Big\|_{\mathcal{K}}\\ &\leqslant \sum_{k=m}^{n}\Big\|2^{k+1}f\Big(\frac{x}{2^{k+1}}\Big) - 2^k f\Big(\frac{x}{2^k}\Big)\Big\|_{\mathcal{K}}\\ &\leqslant \frac{1}{6}\sum_{k=m}^{n}2^{k+1}\Phi\Big(\frac{x}{2^{k+1}},\frac{x}{2^{k+1}},\frac{x}{2^{k+1}}\Big)\end{aligned} \tag{2.27}$$

for all $x\in\mathcal{J}$ and all non-negative integers $n\geqslant m\geqslant 0$. It follows from (2.21) and (2.27) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence in $\mathcal{K}$ for all $x\in\mathcal{J}$. Since $\mathcal{K}$ is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges for all $x\in\mathcal{J}$. Thus one can define the mapping $H:\mathcal{J}\to\mathcal{K}$ by

$$H(x) := \lim_{n\to\infty} 2^n f\Big(\frac{x}{2^n}\Big)$$

for all $x\in\mathcal{J}$. Moreover, letting $m = 0$ and passing the limit $n\to\infty$ in (2.27) we get (2.25). The rest of the proof is similar to the proof of Theorem 2.6. □

COROLLARY 2.9. *Let $\epsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3$ be non-negative real numbers such that $p_1, p_2, p_3 > 1$ and $q_1, q_2, q_3 > 3$. Suppose that $f : \mathcal{J} \to \mathcal{K}$ is a mapping satisfying (2.19) and (2.20) for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{J}$. Then there exists a unique JB^*-triple homomorphism $H : \mathcal{J} \to \mathcal{K}$ such that*

$$\left\|f(x) - H(x)\right\|_{\mathcal{K}} \leqslant \frac{\theta}{3}\sum_{i=1}^{3}\frac{1}{2^{p_i}-2}\|x\|_{\mathcal{J}}^{p_i}$$

for all $x \in \mathcal{J}$.

3. Homomorphisms between JB^*-triples

THEOREM 3.1. *Let $\epsilon, \theta, p_1, p_2, p_3$ be non-negative real numbers and let q_1, q_2, q_3 be real numbers such that $p_i > 0$ and $q_j \neq 1$ for some $1 \leqslant i, j \leqslant 3$. Suppose that $f : \mathcal{J} \to \mathcal{K}$ is a mapping satisfying*

$$\left\|D_\mu f(x,y,z)\right\|_{\mathcal{K}} \leqslant \theta \|x\|_{\mathcal{J}}^{p_1}\|y\|_{\mathcal{J}}^{p_2}\|z\|_{\mathcal{J}}^{p_3}, \tag{3.1}$$

$$\left\|f(\{x,y,z\}) - \left\{f(x), f(y), f(z)\right\}\right\|_{\mathcal{K}} \leqslant \epsilon \|x\|_{\mathcal{J}}^{q_1}\|y\|_{\mathcal{J}}^{q_2}\|z\|_{\mathcal{J}}^{q_3} \tag{3.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{J}^\star$. Then the mapping $f : \mathcal{J} \to \mathcal{K}$ is a JB^*-triple homomorphism.

Proof. Without any loss of generality, we suppose $p_1 > 0$. By letting $x = y = z = 0$ in (3.1), we get $f(0) = 0$. Letting $x = y = 0$ and replacing z by $2z$ in (3.1), we get

$$f(2\mu z) + 2f(\mu z) = 2\mu f(2z) \tag{3.3}$$

for all $\mu \in \mathbb{T}^1$ and all $z \in \mathcal{J}$. Letting $\mu = 1$ in (3.3), we get

$$f(2z) = 2f(z) \tag{3.4}$$

for all $z \in \mathcal{J}$. We get from (3.3) and (3.4) that $f(\mu z) = \mu f(z)$ for all $\mu \in \mathbb{T}^1$ and all $z \in \mathcal{J}$. Therefore f is an odd function.

Letting $x = 0$ and replacing y and z by $2y$ and $2z$ in (3.1), respectively, we get

$$f(y+2z) + f(z+2y) + f(y+z) = 4f(y) + 4f(z) \tag{3.5}$$

for all $y, z \in \mathcal{J}$. Replacing y by $y+z$ and z by $-z$ in (3.5) and using the oddness of f, we get

$$f(y-z) + f(2y+z) + f(y) = 4f(y+z) - 4f(z) \tag{3.6}$$

for all $y, z \in \mathcal{J}$. Replacing y by z and z by y in (3.6) and using the oddness of f, we get

$$-f(y-z) + f(2z+y) + f(z) = 4f(y+z) - 4f(y) \tag{3.7}$$

for all $y, z \in \mathcal{J}$. Adding (3.6) to (3.7) we have

$$f(y+2z) + f(z+2y) = 8f(y+z) - 5f(y) - 5f(z) \tag{3.8}$$

for all $y, z \in \mathcal{J}$. Now, by (3.5) and (3.8), we have $f(y+z) = f(y) + f(z)$ for all $y, z \in \mathcal{J}$. Hence by Lemma 2.1 the mapping $f : \mathcal{J} \to \mathcal{K}$ is $\mathbb{C}$-linear.

Without any loss of generality, we may assume that $q_1 \neq 1$. Let $q_1 > 1$. It follows from (3.2) that

$$\begin{aligned}
&\left\| f(\{x,y,z\}) - \left\{ f(x), f(y), f(z) \right\} \right\|_{\mathcal{K}} \\
&= \lim_{n\to\infty} 2^n \left\| f\left(\left\{ \frac{x}{2^n}, y, z \right\} \right) - \left\{ f\left(\frac{x}{2^n}\right), f(y), f(z) \right\} \right\|_{\mathcal{K}} \\
&\leqslant \epsilon \lim_{n\to\infty} \frac{2^n}{2^{nq_1}} \|x\|_{\mathcal{J}}^{q_1} \|y\|_{\mathcal{J}}^{q_2} \|z\|_{\mathcal{J}}^{q_3} = 0
\end{aligned}$$

for all $x, y, z \in \mathcal{J}^\star$. Therefore

$$f(\{x,y,z\}) = \{f(x), f(y), f(z)\} \tag{3.9}$$

for all $x, y, z \in \mathcal{J}^\star$. Since $f(0) = 0$, then (3.9) holds for all $x, y, z \in \mathcal{J}$ when $q_i < 0$ for some $2 \leqslant i \leqslant 3$. Similarly, for $q_1 < 1$, we get (3.9). So the mapping $f : \mathcal{J} \to \mathcal{K}$ is a JB^*-triple homomorphism. □

We will use the following lemma in the proof of the next theorem.

LEMMA 3.2. *Let X and Y be linear spaces. An odd mapping $f : X \to Y$ satisfies*

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x}{2} + y\right) + f\left(\frac{y}{2} + x\right) = 2[f(x) + f(y)] \tag{3.10}$$

for all $x, y \in X \setminus \{0\}$ if and only if $f : X \to Y$ is additive.

Proof. Suppose that f satisfies (3.10). Since f is odd, then $f(0) = 0$. Letting $y = x$ in (3.10), we get

$$f\left(\frac{3x}{2}\right) = \frac{3}{2} f(x) \tag{3.11}$$

for all $x \in X \setminus \{0\}$. Letting $y = 2x$ in (3.10) and using (3.11), we get

$$f\left(\frac{5x}{2}\right) = f(2x) + \frac{1}{2} f(x) \tag{3.12}$$

for all $x \in X \setminus \{0\}$. Letting $y = -2x$ in (3.10) and using the oddness of f, we get

$$f\left(\frac{3x}{2}\right) + f\left(\frac{x}{2}\right) = 2f(2x) - 2f(x) \tag{3.13}$$

for all $x \in X \setminus \{0\}$. It follows from (3.13) that

$$f(3x) + f(x) = 2f(4x) - 2f(2x) \tag{3.14}$$

for all $x \in X \setminus \{0\}$. Letting $y = 4x$ in (3.10) and using (3.11) and (3.12), we get

$$5f(3x) = 4f(4x) - 2f(2x) + 3f(x) \tag{3.15}$$

for all $x \in X \setminus \{0\}$. It follows from (3.14) and (3.15) that

$$3f(4x) = 4f(2x) + 4f(x) \tag{3.16}$$

for all $x \in X \setminus \{0\}$. It follows from (3.11) and (3.13) that

$$7f(x) + 2f\left(\frac{x}{2}\right) = 4f(2x)$$

for all $x \in X \setminus \{0\}$. Replacing x by $2x$ in the last equation, we get

$$4f(4x) = 7f(2x) + 2f(x) \tag{3.17}$$

for all $x \in X \setminus \{0\}$. It follows from (3.16) and (3.17) that $f(2x) = 2f(x)$ for all $x \in X \setminus \{0\}$. Since $f(0) = 0$, then $f(2x) = 2f(x)$ for all $x \in X$. Therefore (3.10) holds for all $x, y \in X$. Hence the mapping f satisfies (3.5) for all $y, z \in X$. Using the proof of Theorem 3.1, we get that the mapping $f : X \to Y$ is additive.

It is clear that each additive mapping satisfies (3.10). □

THEOREM 3.3. *Let ϵ, θ be non-negative real numbers and let $p_1, p_2, p_3, q_1, q_2, q_3$ be real numbers such that $p_i p_j < 0$ for some $1 \leqslant i < j \leqslant 3$ and $q_j \neq 1$ for some $1 \leqslant j \leqslant 3$. Suppose that $f : \mathcal{J} \to \mathcal{K}$ is a mapping satisfying (3.1) and (3.2) for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{J}^\star$. Then the mapping $f : \mathcal{J} \to \mathcal{K}$ is a JB^*-triple homomorphism.*

Proof. Without any loss of generality, we may assume that $p_3 > 0$. Let $\mu = 1$. Letting $z = 0$ in (3.1), we get

$$f\left(\frac{x+y}{2}\right) + f\left(\frac{x}{2} + y\right) + f\left(\frac{y}{2} + x\right) = 2[f(x) + f(y) + f(0)] \tag{3.18}$$

for all $x, y \in \mathcal{J} \setminus \{0\}$. We show that f is additive.

Letting $y = -x$ in (3.18), we get

$$f\left(\frac{x}{2}\right) + f\left(\frac{-x}{2}\right) = 2[f(x) + f(-x)] + f(0) \tag{3.19}$$

for all $x \in \mathcal{J} \setminus \{0\}$. It follows from (3.19) that

$$f(x) + f(-x) = 2[f(2x) + f(-2x)] + f(0) \tag{3.20}$$

$$f\left(\frac{3x}{2}\right) + f\left(\frac{-3x}{2}\right) = 2[f(3x) + f(-3x)] + f(0) \tag{3.21}$$

for all $x \in \mathcal{J} \setminus \{0\}$. Letting $y = x$ in (3.18), we get

$$2f\left(\frac{3x}{2}\right) = 3f(x) + 2f(0) \tag{3.22}$$

for all $x \in \mathcal{J} \setminus \{0\}$. It follows from (3.22) that

$$2\left[f\left(\frac{3x}{2}\right) + f\left(\frac{-3x}{2}\right)\right] = 3[f(x) + f(-x)] + 4f(0) \tag{3.23}$$

$$2[f(3x) + f(-3x)] = 3[f(2x) + f(-2x)] + 4f(0) \tag{3.24}$$

for all $x \in \mathcal{J} \setminus \{0\}$. It follows from (3.21) and (3.23) that

$$3[f(x) + f(-x)] + 2f(0) = 4[f(3x) + f(-3x)] \tag{3.25}$$

for all $x \in \mathcal{J} \setminus \{0\}$. It follows from (3.24) and (3.25) that

$$f(x) + f(-x) = 2[f(2x) + f(-2x) + f(0)] \tag{3.26}$$

for all $x \in \mathcal{J} \setminus \{0\}$. Now, we get from (3.20) and (3.26) that $f(0) = 0$. Hence (3.26) implies that

$$f(x) + f(-x) = 2[f(2x) + f(-2x)] \tag{3.27}$$

for all $x \in \mathcal{J} \setminus \{0\}$. Letting $y = -2x$ in (3.18) and using (3.22) (with $f(0) = 0$), we get

$$f\left(\frac{-x}{2}\right) + \frac{3}{2}f(-x) = 2[f(x) + f(-2x)]$$

for all $x \in \mathcal{J} \setminus \{0\}$. It follows from the last equation that

$$\begin{aligned} &\left[f\left(\frac{x}{2}\right) + f\left(\frac{-x}{2}\right)\right] + \frac{3}{2}\left[f(x) + f(-x)\right] \\ &= 2[f(x) + f(-x)] + 2[f(2x) + f(-2x)] \end{aligned} \tag{3.28}$$

for all $x \in \mathcal{J} \setminus \{0\}$. Since $f(0) = 0$, then it follows from (3.19), (3.27) and (3.28) that $f(-x) = -f(x)$ for all $x \in \mathcal{J} \setminus \{0\}$. Since $f(0) = 0$, then f is odd. Therefore the odd mapping $f : \mathcal{J} \to \mathcal{K}$ satisfies (3.10) for all $x, y \in \mathcal{J} \setminus \{0\}$. So by Lemma 3.2, the mapping f is additive. Therefore by letting $z = 0$ and $y = x$ in (3.1), we get $f(\mu x) = \mu f(x)$ for all $x \in \mathcal{J} \setminus \{0\}$. Since $f(0) = 0$, then $f(\mu x) = \mu f(x)$ for all $x \in \mathcal{J}$. So by Lemma 2.1, the mapping f is $\mathbb{C}$-linear.

The rest of the proof is similar to the proof of Theorem 3.1. □

THEOREM 3.4. *Let $\epsilon, \theta, p_1, p_2, p_3$ be non-negative real numbers and let q_1, q_2, q_3 be real numbers such that $q_1 + q_2 + q_3 \neq 3$ and $p_i > 0$ for some $1 \leqslant i \leqslant 3$. Suppose that $f : \mathcal{J} \to \mathcal{K}$ is a mapping satisfying (3.1) and (3.2) for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{J}^\star$. Then the mapping $f : \mathcal{J} \to \mathcal{K}$ is a JB^*-triple homomorphism.*

Proof. It follows from the proof of Theorem 3.1 that the mapping $f : \mathcal{J} \to \mathcal{K}$ is $\mathbb{C}$-linear. Let $q_1 + q_2 + q_3 > 3$. It follows from (3.2) that

$$\begin{aligned} &\left\| f(\{x,y,z\}) - \left\{f(x), f(y), f(z)\right\} \right\|_{\mathcal{K}} \\ &= \lim_{n\to\infty} 8^n \left\| f\left(\left\{\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right\}\right) - \left\{f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right)\right\} \right\|_{\mathcal{K}} \\ &\leqslant \epsilon \lim_{n\to\infty} \frac{8^n}{2^{n(q_1+q_2+q_3)}} \|x\|_{\mathcal{J}}^{q_1} \|y\|_{\mathcal{J}}^{q_2} \|z\|_{\mathcal{J}}^{q_3} = 0 \end{aligned}$$

for all $x, y, z \in \mathcal{J}^\star$. Therefore we get (3.9) for all $x, y, z \in \mathcal{J}^\star$. Since $f(0) = 0$, then (3.9) holds for all $x, y, z \in \mathcal{J}$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3$. Similarly, for $q_1 + q_2 + q_3 < 3$, we get (3.9). So the mapping $f : \mathcal{J} \to \mathcal{K}$ is a JB^*-triple homomorphism. □

THEOREM 3.5. *Let ϵ, θ be non-negative real numbers and let $p_1, p_2, p_3, q_1, q_2, q_3$ be real numbers such that $p_i p_j < 0$ for some $1 \leqslant i < j \leqslant 3$ and $q_1 + q_2 + q_3 \neq 3$. Suppose that $f : \mathcal{J} \to \mathcal{K}$ is a mapping satisfying (3.1) and (3.2) for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{J}^\star$. Then the mapping $f : \mathcal{J} \to \mathcal{K}$ is a JB^*-triple homomorphism.*

Proof. It follows from the proof of Theorem 3.3 that the mapping $f : \mathcal{J} \to \mathcal{K}$ is $\mathbb{C}$-linear. The rest of the proof is similar to the proof of Theorem 3.4. □

REMARK 3.6. If we replace the condition $q_1 + q_2 + q_3 \neq 3$ in Theorems 3.4 and 3.5 by $q_i + q_j \neq 2$ for some $1 \leqslant i < j \leqslant 3$, then by using the similar proof of Theorem 3.4, we get that the mapping $f : \mathcal{J} \to \mathcal{K}$ is a JB^*-triple homomorphism.

4. Homomorphisms between unital JB^*-triples

Throughout this section, assume that $\mathcal{J}$ is a unital JB^*-triple with norm $\|.\|_{\mathcal{J}}$, unit e and that $\mathcal{K}$ a JB^*-triple with norm $\|.\|_{\mathcal{K}}$ and unit e'.

We investigate homomorphisms between unital JB^*-triples, associated to the functional equation $D_\mu f(x,y,z) = 0$.

THEOREM 4.1. *Let ϵ, θ be non-negative real numbers and let $p_1, p_2, p_3, q_1, q_2, q_3$ real numbers such that $p_1, p_2, p_3 < 1$, $q_1, q_2 < 2$ and $q_3 < 3$. Suppose that $f : \mathcal{J} \to \mathcal{K}$ is a mapping with $f(0) = 0$ satisfying (2.19) and (2.20) for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{J}^\star$. If there exists a real number $\lambda > 1$ $(0 < \lambda < 1)$ and an element $x_0 \in \mathcal{J}$ such that $\lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x_0) = e'$ $(\lim_{n\to\infty} \lambda^n f(\frac{x_0}{\lambda^n}) = e')$, then the mapping $f : \mathcal{J} \to \mathcal{K}$ is a JB^*-triple homomorphism.*

Proof. By Corollary 2.7 there exists a unique JB^*-triple homomorphism $H : \mathcal{J} \to \mathcal{K}$ such that

$$\left\|f(x) - H(x)\right\|_{\mathcal{K}} \leqslant \frac{\theta}{3} \sum_{i=1}^{3} \frac{1}{2 - 2^{p_i}} \|x\|_{\mathcal{J}}^{p_i} \tag{4.1}$$

for all $x \in \mathcal{J}^\star$. It follows from (4.1) that

$$H(x) = \lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x), \qquad \Big(H(x) = \lim_{n\to\infty} \lambda^n f(\frac{x}{\lambda^n})\Big) \tag{4.2}$$

for all $x \in \mathcal{J}$ and all real number $\lambda > 1$ $(0 < \lambda < 1)$. Therefore by the assumption, we get that $H(x_0) = e'$. Let $\lambda > 1$ and $\lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x_0) = e'$. It follows from (2.20) that

$$\begin{aligned}
&\Big\|\{H(x), H(y), H(z)\} - \{H(x), H(y), f(z)\}\Big\|_{\mathcal{K}} \\
&= \Big\|H(\{x,y,z\}) - \{H(x), H(y), f(z)\}\Big\|_{\mathcal{K}} \\
&= \lim_{n\to\infty} \frac{1}{\lambda^{2n}} \Big\|f(\{\lambda^n x, \lambda^n y, z\}) - \Big\{f(\lambda^n x), f(\lambda^n y), f(z)\Big\}\Big\|_{\mathcal{K}} \\
&\leqslant \epsilon \lim_{n\to\infty} \frac{1}{\lambda^{2n}} \Big[\lambda^{nq_1} \|x\|_{\mathcal{J}}^{q_1} + \lambda^{nq_2} \|y\|_{\mathcal{J}}^{q_2} + \|z\|_{\mathcal{J}}^{q_3})\Big] = 0
\end{aligned}$$

for all $x, y, z \in \mathcal{J} \setminus \{0\}$. So $\{H(x), H(y), H(z)\} = \{H(x), H(y), f(z)\}$ for all $x, y, z \in \mathcal{J} \setminus \{0\}$. Letting $x = y = x_0$ in the last equality, we get $f(z) = H(z)$ for all $z \in \mathcal{J} \setminus \{0\}$. Since $H(0) = f(0) = 0$, then $f = H$. Similarly, one can show that $H(z) = f(z)$ for all $z \in \mathcal{J}$ when $0 < \lambda < 1$ and $\lim_{n\to\infty} \lambda^n f(\frac{x_0}{\lambda^n}) = e'$. Therefore the mapping $f : \mathcal{J} \to \mathcal{K}$ is a JB^*-triple homomorphism. □

REMARK 4.2. Theorem 4.1 will be valid if we replace the conditions $q_1, q_2 < 2$ and $q_3 < 3$ by $q_2, q_3 < 2$ and $q_1 < 3$.

THEOREM 4.3. *Let $\epsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3$ be non-negative real numbers such that $p_1, p_2, p_3 > 1$ and $q_1, q_2, q_3 > 2$. Suppose that $f : \mathcal{J} \to \mathcal{K}$ is a mapping satisfying (2.19) and*

$$\begin{aligned}&\Big\|f(\{x,y,z\}) - \Big\{f(x), f(y), f(z)\Big\}\Big\|_{\mathcal{K}} \\ &\leqslant \epsilon\Big(\|x\|_{\mathcal{J}}^{q_1}\|y\|_{\mathcal{J}}^{q_2} + \|y\|_{\mathcal{J}}^{q_2}\|z\|_{\mathcal{J}}^{q_3} + \|x\|_{\mathcal{J}}^{q_1}\|z\|_{\mathcal{J}}^{q_3}\Big)\end{aligned} \tag{4.3}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{J}$. If there exists a real number $\lambda > 1$ $(0 < \lambda < 1)$ and an element $x_0 \in \mathcal{J}$ such that $\lim_{n\to\infty} \lambda^n f(\frac{x_0}{\lambda^n}) = e'$ $(\lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x_0) = e')$, then the mapping $f : \mathcal{J} \to \mathcal{K}$ is a JB^*-triple homomorphism.

Proof. By Theorem 2.8 there exists a unique JB^*-triple homomorphism $H : \mathcal{J} \to \mathcal{K}$ such that

$$\big\|f(x) - H(x)\big\|_{\mathcal{K}} \leqslant \frac{\theta}{3}\sum_{i=1}^{3}\frac{1}{2^{p_i} - 2}\|x\|_{\mathcal{J}}^{p_i} \tag{4.4}$$

for all $x \in \mathcal{J}$. It follows from (4.4) that

$$H(x) = \lim_{n\to\infty} \lambda^n f(\frac{x}{\lambda^n}), \qquad \Big(H(x) = \lim_{n\to\infty}\frac{1}{\lambda^n} f(\lambda^n x)\Big) \tag{4.5}$$

for all $x \in \mathcal{J}$ and all real number $\lambda > 1$ $(0 < \lambda < 1)$. Therefore by the assumption, we get that $H(x_0) = e'$. Let $\lambda > 1$ and $\lim_{n\to\infty} \lambda^n f(\frac{x_0}{\lambda^n}) = e'$. It follows from (4.3) that

$$\begin{aligned}&\Big\|\{H(x), H(y), H(z)\} - \{H(x), H(y), f(z)\}\Big\|_{\mathcal{K}} \\ &= \Big\|H(\{x,y,z\}) - \{H(x), H(y), f(z)\}\Big\|_{\mathcal{K}} \\ &= \lim_{n\to\infty} \lambda^{2n}\Big\|f\Big(\Big\{\frac{x}{\lambda^n}, \frac{y}{\lambda^n}, z\Big\}\Big) - \Big\{f\Big(\frac{x}{\lambda^n}\Big), f\Big(\frac{y}{\lambda^n}\Big), f(z)\Big\}\Big\|_{\mathcal{K}} \\ &\leqslant \epsilon \lim_{n\to\infty} \lambda^{2n}\Big[\frac{1}{\lambda^{n(q_1+q_2)}}\|x\|_{\mathcal{J}}^{q_1}\|y\|_{\mathcal{J}}^{q_2} + \frac{1}{\lambda^{nq_2}}\|y\|_{\mathcal{J}}^{q_2}\|z\|_{\mathcal{J}}^{q_3} + \frac{1}{\lambda^{nq_1}}\|x\|_{\mathcal{J}}^{q_1}\|z\|_{\mathcal{J}}^{q_3})\Big] = 0\end{aligned}$$

for all $x, y, z \in \mathcal{J}$. So $\{H(x), H(y), H(z)\} = \{H(x), H(y), f(z)\}$ for all $x, y, z \in \mathcal{J}$. Letting $x = y = x_0$ in the last equality, we get $f(z) = H(z)$ for all $z \in \mathcal{J}$. Similarly, one can show that $H(z) = f(z)$ for all $z \in \mathcal{J}$ when $0 < \lambda < 1$ and $\lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x_0) = e'$. Therefore the mapping $f : \mathcal{J} \to \mathcal{K}$ is a JB^*-triple homomorphism. □

5. Derivations and Stability of derivations on JB^* -triples

Throughout this section, assume that $\mathcal{J}$ is a JB^* -triple with norm $\|\cdot\|_{\mathcal{J}}$.

In this section we prove the Hyers–Ulam–Rassias stability of derivations on JB^* -triples for the functional equation $D_\mu f(x,y,z) = 0$.

THEOREM 5.1. *Let $\varphi : \mathcal{J}\times\mathcal{J}\times\mathcal{J} \to [0,\infty)$ and $\psi : \mathcal{J}\times\mathcal{J}\times\mathcal{J} \to [0,\infty)$ be functions such that*

$$\widetilde{\varphi}(x) := \sum_{n=0}^{\infty} \frac{1}{2^n}\varphi(2^n x, 2^n x, 2^n x) < \infty, \quad \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z) = 0, \tag{5.1}$$

$$\lim_{n\to\infty} \frac{1}{8^n}\psi(2^n x, 2^n y, 2^n z) = 0 \tag{5.2}$$

for all $x,y,z \in \mathcal{J}$. Suppose that $f : \mathcal{J} \to \mathcal{J}$ is a mapping satisfying

$$\|D_\mu f(x,y,z)\|_{\mathcal{J}} \leqslant \varphi(x,y,z), \tag{5.3}$$

$$\Big\| f(\{x,y,z\}) - \{f(x),y,z\} - \{x,f(y),z\} - \{x,y,f(z)\} \Big\|_{\mathcal{J}} \leqslant \psi(x,y,z) \tag{5.4}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in \mathcal{J}$. Then there exists a unique JB^* -triple derivation $D : \mathcal{J} \to \mathcal{J}$ such that

$$\big\| f(x) - D(x) \big\|_{\mathcal{J}} \leqslant \frac{1}{6}\widetilde{\varphi}(x) \tag{5.5}$$

for all $x \in \mathcal{J}$.

Proof. By the proof of Theorem 2.6, there exists a unique $\mathbb{C}$-linear mapping $D : \mathcal{J} \to \mathcal{J}$ satisfying (5.5) and

$$D(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in \mathcal{J}$. It follows from (5.2) and (5.4) that

$$\begin{aligned}
&\Big\| D(\{x,y,z\}) - \{D(x),y,z\} - \{x,D(y),z\} - \{x,y,D(z)\} \Big\|_{\mathcal{J}} \\
&= \lim_{n\to\infty} \frac{1}{8^n} \Big\| f(\{2^n x, 2^n y, 2^n z\}) - \{f(2^n x), 2^n y, 2^n z\} \\
&\quad - \{2^n x, f(2^n y), 2^n z\} - \{2^n x, 2^n y, f(2^n z)\} \Big\|_{\mathcal{J}} \\
&\leqslant \lim_{n\to\infty} \frac{1}{8^n}\psi(2^n x, 2^n y, 2^n z) = 0
\end{aligned}$$

for all $x,y,z \in \mathcal{J}$. So

$$D(\{x,y,z\}) = \{D(x),y,z\} + \{x,D(y),z\} + \{x,y,D(z)\}$$

for all $x,y,z \in \mathcal{J}$. Therefore the mapping $D : \mathcal{J} \to \mathcal{J}$ is a JB^* -triple derivation. □

THEOREM 5.2. *Let $\varphi : \mathcal{J}\times\mathcal{J}\times\mathcal{J} \to [0,\infty)$ be a function satisfying (5.1). Suppose that the function $\psi : \mathcal{J}\times\mathcal{J}\times\mathcal{J} \to [0,\infty)$ satisfies in one of the following conditions*

(i) $\lim_{n\to\infty} \frac{1}{4^n}\psi(2^n x, 2^n y, z) = 0$;
(ii) $\lim_{n\to\infty} \frac{1}{4^n}\psi(x, 2^n y, 2^n z) = 0$;
(iii) $\lim_{n\to\infty} \frac{1}{4^n}\psi(2^n x, y, 2^n z) = 0$
for all $x, y, z \in \mathcal{J}$. *Let* $f : \mathcal{J} \to \mathcal{J}$ *be a mapping satisfying (5.3) and (5.4). Then the mapping* $f : \mathcal{J} \to \mathcal{J}$ *is a* JB^**-triple derivation.*

Proof. By the proof of Theorem 2.6, there exists a $\mathbb{C}$-linear mapping $D : \mathcal{J} \to \mathcal{J}$ defined by

$$D(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in \mathcal{J}$. We show that if the mapping ψ satisfies in one of the conditions (i), (ii) or (iii), then $f = D$.

Let ψ satisfy in (i) (we have a similar proof if ψ satisfies in (ii) or (iii)). It follows from (5.4) that

$$\begin{aligned}
&\Big\| D(\{x,y,z\}) - \{D(x),y,z\} - \{x,D(y),z\} - \{x,y,f(z)\} \Big\|_{\mathcal{J}} \\
&= \lim_{n\to\infty} \frac{1}{4^n} \Big\| f(\{2^n x, 2^n y, z\}) - \{f(2^n x), 2^n y, z\} \\
&\quad - \{2^n x, f(2^n y), z\} - \{2^n x, 2^n y, f(z)\} \Big\|_{\mathcal{J}} \\
&\leqslant \lim_{n\to\infty} \frac{1}{4^n} \psi(2^n x, 2^n y, z) = 0
\end{aligned}$$

for all $x, y, z \in \mathcal{J}$. Therefore

$$D(\{x,y,z\}) = \{D(x),y,z\} + \{x,D(y),z\} + \{x,y,f(z)\} \tag{5.6}$$

for all $x, y, z \in \mathcal{J}$. Replacing z by $2z$ in (5.6), we get

$$2D(\{x,y,z\}) = 2\{D(x),y,z\} + 2\{x,D(y),z\} + \{x,y,f(2z)\} \tag{5.7}$$

for all $x, y, z \in \mathcal{J}$. It follows from (5.6) and (5.7) that

$$\{x,y,f(2z) - 2f(z)\} = 0$$

for all $x, y, z \in \mathcal{J}$. Letting $x = y = f(2z) - 2f(z)$ in the last equation, we get

$$\|f(2z) - 2f(z)\|_{\mathcal{J}}^3 = \Big\| \Big\{ f(2z) - 2f(z), f(2z) - 2f(z), f(2z) - 2f(z) \Big\} \Big\|_{\mathcal{J}} = 0$$

for all $z \in \mathcal{J}$. So $f(2z) = 2f(z)$ for all $z \in \mathcal{J}$. By using induction, we infer that $f(2^n z) = 2^n f(z)$ for all $z \in \mathcal{J}$ and all $n \in \mathbb{Z}$. Therefore $D(x) = f(x)$ for all $x \in \mathcal{J}$. Hence it follows from (5.6) that the mapping $f : \mathcal{J} \to \mathcal{J}$ is a JB^*-triple derivation. □

COROLLARY 5.3. *Let* ϵ, θ *be non-negative real numbers and let* $p_1, p_2, p_3, q_1, q_2, q_3$ *be real numbers such that* $p_1, p_2, p_3 < 1$ *and* $q_i, q_j < 2$ *for some* $1 \leqslant i < j \leqslant 3$. *Suppose that* $f : \mathcal{J} \to \mathcal{J}$ *is a mapping with* $f(0) = 0$ *satisfying*

$$\|D_\mu f(x,y,z)\|_{\mathcal{J}} \leqslant \theta(\|x\|_{\mathcal{J}}^{p_1} + \|y\|_{\mathcal{J}}^{p_2} + \|x\|_{\mathcal{J}}^{p_3}), \tag{5.8}$$

$$\Big\|f(\{x,y,z\}) - \{f(x),y,z\} - \{x,f(y),z\} - \{x,y,f(z)\}\Big\|_{\mathcal{J}} \leqslant \epsilon(\|x\|_{\mathcal{J}}^{q_1} + \|y\|_{\mathcal{J}}^{q_2} + \|z\|_{\mathcal{J}}^{q_3}) \tag{5.9}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x,y,z \in \mathcal{J}^\star$. *Then the mapping* $f : \mathcal{J} \to \mathcal{J}$ *is a* JB^* *-triple derivation.*

THEOREM 5.4. *Let* $\varphi : \mathcal{J}\times\mathcal{J}\times\mathcal{J} \to [0,\infty)$ *be a function satisfying (5.1). Suppose that the function* $\psi : \mathcal{J}\times\mathcal{J}\times\mathcal{J} \to [0,\infty)$ *satisfies in one of the following conditions*

(i) $\lim_{n\to\infty} \frac{1}{2^n}\psi(2^n x,y,z) = 0$;
(ii) $\lim_{n\to\infty} \frac{1}{2^n}\psi(x,2^n y,z) = 0$;
(iii) $\lim_{n\to\infty} \frac{1}{2^n}\psi(x,y,2^n z) = 0$

for all $x,y,z \in \mathcal{J}$. *Let* $f : \mathcal{J} \to \mathcal{J}$ *be a mapping satisfying (5.3) and (5.4). Then the mapping* $f : \mathcal{J} \to \mathcal{J}$ *is a* JB^* *-triple derivation.*

Proof. By the proof of Theorem 2.6, there exists a $\mathbb{C}$-linear mapping $D : \mathcal{J} \to \mathcal{J}$ defined by

$$D(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in \mathcal{J}$. We show that if the mapping ψ satisfies in one of the conditions (i), (ii) or (iii), then $f = D$.

Let ψ satisfy in (i) (we have a similar proof if ψ satisfies in (ii) or (iii)). It follows from (5.4) that

$$\begin{aligned}&\Big\|D(\{x,y,z\}) - \{D(x),y,z\} - \{x,f(y),z\} - \{x,y,f(z)\}\Big\|_{\mathcal{J}}\\ &= \lim_{n\to\infty} \frac{1}{2^n}\Big\|f(\{2^n x,y,z\}) - \{f(2^n x),y,z\} - \{2^n x,f(y),z\} - \{2^n x,y,f(z)\}\Big\|_{\mathcal{J}}\\ &\leqslant \lim_{n\to\infty} \frac{1}{2^n}\psi(2^n x,y,z) = 0\end{aligned}$$

for all $x,y,z \in \mathcal{J}$. Therefore

$$D(\{x,y,z\}) = \{D(x),y,z\} + \{x,f(y),z\} + \{x,y,f(z)\} \tag{5.10}$$

for all $x,y,z \in \mathcal{J}$.

The rest of the proof is similar to the proof Theorem 5.2. □

COROLLARY 5.5. *Let* ϵ, θ *be non-negative real numbers and let* p_1,p_2,p_3,q_1,q_2,q_3 *be real numbers such that* $p_1,p_2,p_3 < 1$ *and* $q_i < 1$ *for some* $1 \leqslant i \leqslant 3$. *Suppose that* $f : \mathcal{J} \to \mathcal{J}$ *is a mapping with* $f(0) = 0$ *satisfying (5.8) and (5.9) for all* $\mu \in \mathbb{T}^1$ *and all* $x,y,z \in \mathcal{J}^\star$. *Then the mapping* $f : \mathcal{J} \to \mathcal{J}$ *is a* JB^* *-triple derivation.*

THEOREM 5.6. *Let* $\Phi : \mathcal{J}\times\mathcal{J}\times\mathcal{J} \to [0,\infty)$ *and* $\Psi : \mathcal{J}\times\mathcal{J}\times\mathcal{J} \to [0,\infty)$ *be functions such that*

$$\widetilde{\Phi}(x) := \sum_{n=1}^{\infty} 2^n \Phi\Big(\frac{x}{2^n},\frac{x}{2^n},\frac{x}{2^n}\Big) < \infty, \quad \lim_{n\to\infty} 2^n \Phi\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big) = 0, \tag{5.11}$$

$$\lim_{n\to\infty} 8^n \Psi\Big(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\Big) = 0 \tag{5.12}$$

for all $x,y,z \in \mathcal{J}$. *Suppose that* $f : \mathcal{J} \to \mathcal{J}$ *is a mapping satisfying*

$$\|D_\mu f(x,y,z)\|_{\mathcal{J}} \leqslant \Phi(x,y,z), \tag{5.13}$$

$$\Big\|f(\{x,y,z\}) - \{f(x),y,z\} - \{x,f(y),z\} - \{x,y,f(z)\}\Big\|_{\mathcal{J}} \leqslant \Psi(x,y,z) \tag{5.14}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x,y,z \in \mathcal{J}$. *Then there exists a unique* JB^**-triple derivation* $D : \mathcal{J} \to \mathcal{J}$ *such that*

$$\big\|f(x) - D(x)\big\|_{\mathcal{J}} \leqslant \frac{1}{6}\widetilde{\Phi}(x) \tag{5.15}$$

for all $x \in \mathcal{J}$.

Proof. By the proof of Theorem 2.8, there exists a unique $\mathbb{C}$-linear mapping $D : \mathcal{J} \to \mathcal{J}$ satisfying (5.15) and

$$D(x) := \lim_{n\to\infty} 2^n f\Big(\frac{x}{2^n}\Big)$$

for all $x \in \mathcal{J}$.

The rest of the proof is similar to the proof of Theorem 5.1. □

THEOREM 5.7. *Let* $\Phi : \mathcal{J}\times\mathcal{J}\times\mathcal{J} \to [0,\infty)$ *be a function satisfying (5.11). Suppose that the function* $\Psi : \mathcal{J}\times\mathcal{J}\times\mathcal{J} \to [0,\infty)$ *satisfies in one of the following conditions*

(i) $\lim_{n\to\infty} 4^n\Psi(\frac{x}{2^n}, \frac{y}{2^n}, z) = 0$;
(ii) $\lim_{n\to\infty} 4^n\Psi(x, \frac{y}{2^n}, \frac{z}{2^n}) = 0$;
(iii) $\lim_{n\to\infty} 4^n\Psi(\frac{x}{2^n}, y, \frac{z}{2^n}) = 0$

for all $x,y,z \in \mathcal{J}$. *Let* $f : \mathcal{J} \to \mathcal{J}$ *be a mapping satisfying (5.13) and (5.14). Then the mapping* $f : \mathcal{J} \to \mathcal{J}$ *is a* JB^**-triple derivation.*

Proof. By the proof of Theorem 2.8, there exists a $\mathbb{C}$-linear mapping $D : \mathcal{J} \to \mathcal{J}$ defined by

$$D(x) := \lim_{n\to\infty} 2^n f\Big(\frac{x}{2^n}\Big)$$

for all $x \in \mathcal{J}$.

The rest of the proof is similar to the proof of Theorem 5.2. □

COROLLARY 5.8. *Let* $\epsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3$ *be non-negative real numbers such that* $p_1, p_2, p_3 > 1$ *and* $q_1, q_2, q_3 > 2$. *Suppose that* $f : \mathcal{J} \to \mathcal{J}$ *is a mapping satisfying (5.8) and*

$$\begin{aligned}&\Big\|f(\{x,y,z\}) - \{f(x),y,z\} - \{x,f(y),z\} - \{x,y,f(z)\}\Big\|_{\mathcal{J}} \\ &\leqslant \epsilon\Big(\|x\|_{\mathcal{J}}^{q_1}\|y\|_{\mathcal{J}}^{q_2} + \|y\|_{\mathcal{J}}^{q_2}\|z\|_{\mathcal{J}}^{q_3} + \|x\|_{\mathcal{J}}^{q_1}\|z\|_{\mathcal{J}}^{q_3}\Big)\end{aligned} \tag{5.16}$$

for all $x,y,z \in \mathcal{J}$. *Then the mapping* $f : \mathcal{J} \to \mathcal{J}$ *is a* JB^**-triple derivation.*

THEOREM 5.9. *Let* $\Phi : \mathcal{J}\times\mathcal{J}\times\mathcal{J}\to[0,\infty)$ *be a function satisfying (5.11). Suppose that the function* $\Psi : \mathcal{J}\times\mathcal{J}\times\mathcal{J}\to[0,\infty)$ *satisfies in one of the following conditions*

(i) $\lim_{n\to\infty} 2^n\Psi(\frac{x}{2^n}, y, z) = 0;$
(ii) $\lim_{n\to\infty} 2^n\Psi(x, \frac{y}{2^n}, z) = 0;$
(iii) $\lim_{n\to\infty} 2^n\Psi(x, y, \frac{z}{2^n}) = 0$

for all $x, y, z \in \mathcal{J}$. *Let* $f : \mathcal{J}\to\mathcal{J}$ *be a mapping satisfying (5.13) and (5.14). Then the mapping* $f : \mathcal{J}\to\mathcal{J}$ *is a* JB^* *-triple derivation.*

Proof. By the proof of Theorem 2.8, there exists a $\mathbb{C}$-linear mapping $D : \mathcal{J}\to\mathcal{J}$ defined by

$$D(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in \mathcal{J}$.

The rest of the proof is similar to the proof of Theorem 5.4. □

THEOREM 5.10. *Let* $\epsilon, \theta, p_1, p_2, p_3$ *be non-negative real numbers and let* q_1, q_2, q_3 *be real numbers such that* $p_i > 0$ *and* $q_j \neq 1$ *for some* $1 \leqslant i, j \leqslant 3$. *Suppose that* $f : \mathcal{J}\to\mathcal{J}$ *is a mapping satisfying*

$$\left\|D_\mu f(x, y, z)\right\|_{\mathcal{J}} \leqslant \theta\|x\|_{\mathcal{J}}^{p_1}\|y\|_{\mathcal{J}}^{p_2}\|z\|_{\mathcal{J}}^{p_3}, \tag{5.17}$$

$$\begin{aligned}&\left\|f(\{x, y, z\}) - \{f(x), y, z\} - \{x, f(y), z\} - \{x, y, f(z)\}\right\|_{\mathcal{J}}\\&\leqslant \epsilon\|x\|_{\mathcal{J}}^{q_1}\|y\|_{\mathcal{J}}^{q_2}\|z\|_{\mathcal{J}}^{q_3}\end{aligned} \tag{5.18}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y, z \in \mathcal{J}^\star$. *Then the mapping* $f : \mathcal{J}\to\mathcal{J}$ *is a* JB^* *-triple derivation.*

Proof. Without any loss of generality, we may assume that $q_1 \neq 1$ and $p_1 > 0$. Therefore it follows from the proof of Theorem 3.1 that the mapping $f : \mathcal{J}\to\mathcal{J}$ is $\mathbb{C}$-linear. Let $q_1 < 1$. It follows from (5.18) that

$$\begin{aligned}&\left\|f(\{x, y, z\}) - \{f(x), y, z\} - \{x, f(y), z\} - \{x, y, f(z)\}\right\|_{\mathcal{J}}\\&= \lim_{n\to\infty}\frac{1}{2^n}\left\|f(\{2^n x, y, z\} - \{f(2^n x), y, z\} - \{2^n x, f(y), z\} - \{2^n x, y, f(z)\}\right\|_{\mathcal{J}}\\&\leqslant \epsilon\lim_{n\to\infty}\frac{2^{nq_1}}{2^n}\|x\|_{\mathcal{J}}^{q_1}\|y\|_{\mathcal{J}}^{q_2}\|z\|_{\mathcal{J}}^{q_3} = 0\end{aligned}$$

for all $x, y, z \in \mathcal{J}^\star$. Therefore

$$f(\{x, y, z\}) = \{f(x), y, z\} + \{x, f(y), z\} + \{x, y, f(z)\} \tag{5.19}$$

for all $x, y, z \in \mathcal{J}^\star$. Since $f(0) = 0$, then (5.19) holds for all $x, y, z \in \mathcal{J}$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3$. Similarly, we get (5.19) when $q_1 > 1$. So the mapping $f : \mathcal{J}\to\mathcal{J}$ is a JB^* -triple derivation. □

THEOREM 5.11. *Let* ϵ, θ *be non-negative real numbers and let* $p_1, p_2, p_3, q_1, q_2, q_3$ *be real numbers such that* $p_i p_j < 0$ *for some* $1 \leqslant i < j \leqslant 3$ *and* $q_j \neq 1$ *for some* $1 \leqslant j \leqslant 3$. *Suppose that* $f : \mathcal{J} \to \mathcal{J}$ *is a mapping satisfying (5.17) and (5.18) for all* $\mu \in \mathbb{T}^1$ *and all* $x, y, z \in \mathcal{J}^\star$. *Then the mapping* $f : \mathcal{J} \to \mathcal{J}$ *is a* JB^* *-triple derivation.*

Proof. It follows from the proof of Theorem 3.3 that the mapping $f : \mathcal{J} \to \mathcal{J}$ is $\mathbb{C}$-linear. The rest of the proof is similar to the proof of Theorem 5.10. □

THEOREM 5.12. *Let* $\epsilon, \theta, p_1, p_2, p_3$ *be non-negative real numbers and let* q_1, q_2, q_3 *be real numbers such that* $p_i > 0$ *and* $q_1 + q_2 + q_3 \neq 3$ *for some* $1 \leqslant i \leqslant 3$. *Suppose that* $f : \mathcal{J} \to \mathcal{J}$ *is a mapping satisfying (5.17) and (5.18) for all* $\mu \in \mathbb{T}^1$ *and all* $x, y, z \in \mathcal{J}^\star$. *Then the mapping* $f : \mathcal{J} \to \mathcal{J}$ *is a* JB^* *-triple derivation.*

Proof. It follows from the proof of Theorem 3.1 that the mapping $f : \mathcal{J} \to \mathcal{J}$ is $\mathbb{C}$-linear. Let $q_1 + q_2 + q_3 < 3$. It follows from (5.18) that

$$\begin{aligned}
&\Big\| f(\{x, y, z\}) - \{f(x), y, z\} - \{x, f(y), z\} - \{x, y, f(z)\} \Big\|_{\mathcal{J}} \\
&= \lim_{n\to\infty} \frac{1}{8^n} \Big\| f(\{2^n x, 2^n y, 2^n z\}) - \{f(2^n x), 2^n y, 2^n z\} \\
&\quad - \{2^n x, f(2^n y), 2^n z\} - \{2^n x, 2^n y, f(2^n z)\} \Big\|_{\mathcal{J}} \\
&\leqslant \epsilon \lim_{n\to\infty} \frac{2^{n(q_1+q_2+q_3)}}{8^n} \|x\|_{\mathcal{J}}^{q_1} \|y\|_{\mathcal{J}}^{q_2} \|z\|_{\mathcal{J}}^{q_3} = 0
\end{aligned}$$

for all $x, y, z \in \mathcal{J}^\star$. Therefore

$$f(\{x, y, z\}) = \{f(x), y, z\} + \{x, f(y), z\} + \{x, y, f(z)\} \tag{5.20}$$

for all $x, y, z \in \mathcal{J}^\star$. Since $f(0) = 0$, then (5.20) holds for all $x, y, z \in \mathcal{J}$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3$. Similarly, we get (5.20) when $q_1 + q_2 + q_3 > 3$. So the mapping $f : \mathcal{J} \to \mathcal{J}$ is a JB^* -triple derivation. □

THEOREM 5.13. *Let* ϵ, θ *be non-negative real numbers and let* $p_1, p_2, p_3, q_1, q_2, q_3$ *be real numbers such that* $p_i p_j < 0$ *for some* $1 \leqslant i < j \leqslant 3$ *and* $q_1 + q_2 + q_3 \neq 3$. *Suppose that* $f : \mathcal{J} \to \mathcal{J}$ *is a mapping satisfying (5.17) and (5.18) for all* $x, y, z \in \mathcal{J}^\star$. *Then the mapping* $f : \mathcal{J} \to \mathcal{J}$ *is a* JB^* *-triple derivation.*

Proof. It follows from the proof of Theorem 3.3 that the mapping $f : \mathcal{J} \to \mathcal{J}$ is $\mathbb{C}$-linear. The rest of the proof is similar to the proof of Theorem 5.12. □

REMARK 5.14. If we replace the condition $q_1 + q_2 + q_3 \neq 3$ in Theorems 5.12 and 5.13 by $q_i + q_j \neq 2$ for some $1 \leqslant i < j \leqslant 3$, then by using the similar proof of Theorem 5.12, we get that the mapping $f : \mathcal{J} \to \mathcal{J}$ is a JB^* -triple derivation.

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