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SPECTRAL PROPERTIES OF A CLASS OF SINGULAR DIFFERENTIAL OPERATORS

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Abstract. We consider the operator

$$A_0 f = (-1)^n \frac{1}{v(t)} \left(D_\rho^n\right)^* \left[u^2(t) D_\rho^n \left(\frac{f(t)}{v(t)}\right)\right],$$

where

$$D_\rho^n f(t) = \frac{d^k}{dt^k}\left[\rho(t)\frac{d^m f(t)}{dt^m}\right], \quad \left(D_\rho^n\right)^* f(t) = \frac{d^m}{dt^m}\left[\rho(t)\frac{d^k f(t)}{dt^k}\right], \quad k+m=n.$$

Our main aim is to prove some spectral properties of a natural extension of this operator. In order to prove this we need to prove some properties of a function space, connected to the operator D_ρ^n, and some embedding theorems of independent interest.

1. Introduction

Let $I=(0,\infty)$, $k,m,l \in N$, $n=k+m$, $u(\cdot)$, $v(\cdot)$, and $\rho(\cdot)$ be infinite differentiable and positive weight functions on I. Moreover, $\|\cdot\|_{p,u}$ denotes the usual norm of the Lebesgue space $L_{p,u}(I) \equiv L_{p,u}$:

$$\|f\|_{p,u} = \|uf\|_p = \left(\int_I |u(t)f(t)|^p dt\right)^{\frac{1}{p}}, \quad 1<p<\infty.$$

For a function $f: I \to R$ introduce the following two differential operators:

$$D_\rho^i f(t) = \begin{cases} \frac{d^i f(t)}{dt^i}, & i=0,\dots,m-1, \\ \frac{d^{i-m}}{dt^{i-m}}\left[\rho(t)\frac{d^m f(t)}{dt^m}\right], & i=m,\dots,n, \end{cases}$$

and

$$\left(D_\rho^i\right)^* f(t) = \begin{cases} \frac{d^i f(t)}{dt^i}, & i=0,\dots,k-1, \\ \frac{d^{i-k}}{dt^{i-k}}\left[\rho(t)\frac{d^k f(t)}{dt^k}\right], & i=k,\dots,n, \end{cases}$$

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where all derivatives are understood in the generalized sense.

In $L_2(I)$ we consider the operator

$$A_0 f = (-1)^n \frac{1}{v(t)} \left(D_\rho^n\right)^* \left[u^2(t) D_\rho^n \left(\frac{f(t)}{v(t)}\right)\right],$$

defined on the set $C_0^\infty(I)$ of all infinitely differentiable and finitely supported functions on I.

The operator A_0 is non–negative and symmetric in L_2. Therefore, it has the Friedrichs self–adjoint extension in L_2 (see e.g. [14]). Denote this extension by A.

The main aim of this paper is to establish necessary and sufficient conditions for positive definiteness, discreteness of the spectrum and resolvent nuclearity of the operator A in dependence on the behaviour of the weight functions u, v, and ρ in neighbourhoods of the boundary points of I.

The theory of spectral analysis of differential operators, in particular, the spectral characteristics pointed out above, is very important in mathematics, which can be seen from a great number of works on this theme. When $\rho(\cdot) \equiv 1$ the operator A has attracted a lot of interest for various reasons and been studied in a number of works, see e.g. [2], [3], and the references given their. A special method to study spectral properties presented in this paper can be found, for example, in [1], [11], and [12]. In the monographs [8], [13] a complete presentation of this method is given and applied. The method is based on embedding theorems of some function spaces. The results concerning properties of these function spaces have independent interest and, in turn, are based on Hardy type inequalities (see [7]). In particular, in [1] the spectral properties of the operator A with $\rho(\cdot) \equiv 1$ were obtained due to the well–known Muchenhoupt's result on the Hardy inequality with weights [9]. When the conditions on the weight u is that there are traces in one of the endpoints of the interval $(0,\infty)$, the spectral properties of A with $\rho(\cdot) \equiv 1$ were studied in [15], and these results were obtained due to a Hardy type inequality for the Riemann–Liouville integral operator [16]. Moreover, the results from [16] gave a new impulse to study Hardy type inequalities for different kinds of integral operators because of the usefulness of such estimates in spectral theory. The scheme of connection between estimates for integral operators and spectral properties of differential operators such as A is given in [10].

The paper is organized as follows: In Section 2 the new results concerning a special function space, connected to the operator D_ρ^n, are described and discussed. In order not to disturb our presentation all proofs of the results in Section 2 are collected in Section 3. Finally, our main results concerning the spectral properties of the operator A can be found in Section 4.

2. Properties of function spaces connected to the operator D_ρ^n

Let $w_p^n(u,\rho;I) \equiv w_p^n$ be a set of functions $f: I \to R$, for which $D_\rho^n f$ has sense and the following seminorm is finite:

$$\|f\|_{w_p^n} \equiv \|u D_\rho^n f\|_p, \; 1 < p < \infty.$$

Now the first aim is to define the behaviour of $f \in w_p^n$ at the endpoints of I in dependence on the behaviour of u and ρ at the endpoints of I, and obtain for $f \in w_p^n$ an integral representation provided the following conditions

$$\lim_{t\to 0+} D_\rho^i f(t) \equiv D_\rho^i f(0) = 0,\ i = 0,\dots,l-1,$$

$$\lim_{t\to\infty} D_\rho^j f(t) \equiv D_\rho^j f(\infty) = 0,\ j = l,\dots,n-1,$$

hold, where $1 \leqslant l \leqslant n-1$.

In order to be able to reach this aim we weaken the conditions on the weight functions u and ρ, namely, we suppose that $\rho \in L_1^{loc}$, $\rho^{-1} \equiv \frac{1}{\rho} \in L_1^{loc}$, and $u \in L_p^{loc}$, $u^{-1} \equiv \frac{1}{u} \in L_{p'}^{loc}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

REMARK 1. The notation $A \ll B$ means $A \leqslant cB$, where the constant $c > 0$ may depend on unessential parameters. We write $A \approx B$ instead of $A \ll B \ll A$.

The studying of this problem divides into three cases depending on the relation between m and l: (1) $m < l$, (2) $m > l$, and (3) $m = l$.

THEOREM 1. *(1) Let* $1 \leqslant m < l \leqslant n-1$, $n \geqslant 3$.
Suppose that $\forall t > 0$ *and* $\forall z > 0$ *the following conditions hold:*

$$u^{-1}(s)s^{n-l-1}\int_0^s \rho^{-1}(x)x^{l-m}dx \in L_{p'}(0,t), \tag{1}$$

$$u^{-1}(s)s^{n-l}\int_s^z \rho^{-1}(x)x^{l-m-1}dx \in L_{p'}(0,t), \tag{2}$$

$$u^{-1}(s)s^{n-l-1} \in L_{p'}(t,\infty). \tag{3}$$

Then $\forall f \in w_p^n$ *there exist traces* $D_\rho^i f(0)$, $i = m,\dots,l-1$, *and numbers* c_i, $i = l,\dots,n-1$, *and* a_i, $i = 0,...,m-1$, *which are uniquely defined from the relations:*

$$\lim_{t\to\infty}\left(D_\rho^l f(t) - \sum_{i=l}^{n-1} c_i \frac{t^{i-l}}{(i-l)!}\right)^{(j)} = 0,\ j = 0,\dots,n-l-1, \tag{4}$$

$$\lim_{t\to 0}\left(f(t) - \sum_{i=m}^{l-1} D_\rho^i f(0)\int_1^t \frac{(t-s)^{m-1}}{(m-1)!}\rho^{-1}(s)\frac{s^{i-m}}{(i-m)!}ds\right)^{(j)} = a_j,\ j = 0,\dots,m-1. \tag{5}$$

In addition, $\forall f \in w_p^n$ *the following representation holds:*

$$f(t) = \sum_{\nu=0}^{m-1} a_\nu \frac{t^\nu}{\nu!} + \sum_{\nu=m}^{l-1} D_\rho^\nu f(0)\int_1^t \frac{(t-s)^{m-1}}{(m-1)!}\rho^{-1}(s)\frac{s^{\nu-m}}{(\nu-m)!}ds$$

$$+\sum_{\nu=l}^{n-1} c_\nu \int_0^t \frac{(t-s)^{m-1}}{(m-1)!}\rho^{-1}(s)\frac{s^{\nu-m}}{(\nu-m)!}ds+(-1)^{n-l}$$

$$\times\int_0^\infty\left(\int_0^{\min\{t,s\}}\frac{(s-z)^{n-l-1}}{(n-l-1)!}\int_z^t\frac{(t-x)^{m-1}}{(m-1)!}\rho^{-1}(x)\frac{(x-z)^{l-m-1}}{(l-m-1)!}dxdz\right)D_\rho^n f(s)ds. \quad (6)$$

(2) Let $1\leqslant l<m\leqslant n-1$*,* $n\geqslant 3$*.*

Suppose that $\forall t>0$ *the following conditions hold:*

$$u^{-1}(s)\int_0^s \rho^{-1}(x)x^{m-l}(s-x)^{k-1}dx\in L_{p'}(0,t), \quad (7)$$

$$u^{-1}(s)\int_t^s \rho^{-1}(x)x^{m-l-1}(s-x)^{k-1}dx\in L_{p'}(t,\infty). \quad (8)$$

Then $\forall f\in w_p^n$ *there exist numbers* c_i*,* $i=m,\dots,n-1$*, and* a_i*,* $i=0,...,m-1$*, which are uniquely defined from the relations:*

$$\lim_{t\to\infty}\left(D_\rho^m f(t)-\sum_{i=m}^{n-1}c_i\frac{t^{i-m}}{(i-m)!}\right)^{(j)}=0,\ j=0,\dots,n-m-1, \quad (9)$$

$$\lim_{t\to\infty}\left(f(t)-\sum_{i=m}^{n-1}c_i\int_1^t\frac{(t-s)^{m-1}}{(m-1)!}\rho^{-1}(s)\frac{s^{i-m}}{(i-m)!}ds-\sum_{i=l}^{m-1}a_i\frac{t^i}{i!}\right)^{(j)}=0, \quad (10)$$

$$j=l,\dots,m-1,$$

$$\lim_{t\to 0}\left(f(t)-\sum_{i=m}^{n-1}c_i\int_1^t\frac{(t-s)^{m-1}}{(m-1)!}\rho^{-1}(s)\frac{s^{i-m}}{(i-m)!}ds\right)^{(j)}=a_j,\ j=0,\dots,l-1. \quad (11)$$

In addition, $\forall f\in w_p^n$ *the following representation holds:*

$$f(t)=\sum_{\nu=0}^{m-1}a_\nu\frac{t^\nu}{\nu!}+\sum_{\nu=m}^{n-1}c_\nu\int_1^t\frac{(t-s)^{m-1}}{(m-1)!}\rho^{-1}(s)\frac{s^{i-m}}{(i-m)!}ds+(-1)^{n-l}$$

$$\times\int_0^\infty\left(\int_0^{\min\{t,s\}}\frac{(t-z)^{l-1}}{(l-1)!}\int_z^s\frac{(s-x)^{k-1}}{(k-1)!}\rho^{-1}(x)\frac{(x-z)^{m-l-1}}{(m-l-1)!}dxdz\right)D_\rho^n f(s)ds. \quad (12)$$

(3) Let $1 \leqslant m = l \leqslant n-1$, $n \geqslant 2$.

Suppose that $\forall t > 0$ *the following conditions hold:*

$$u^{-1}(s) \int_0^s \rho^{-1}(x)(s-x)^{k-1} dx \in L_{p'}(0,t), \tag{13}$$

$$u^{-1}(s) s^{k-1} \in L_{p'}(t,\infty). \tag{14}$$

Then $\forall f \in w_p^n$ *there exist traces* $D_\rho^i f(0)$, $i = 0,\dots,m-1$, *and numbers* c_i, $i = m,\dots,n-1$, *which are uniquely defined from the relation (9).*

In addition, $\forall f \in w_p^n$ *the following representation holds:*

$$f(t) = \sum_{\nu=0}^{m-1} D_\rho^\nu f(0) \frac{t^\nu}{\nu!} + \sum_{\nu=m}^{n-1} c_\nu \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} \rho^{-1}(s) \frac{s^{i-m}}{(i-m)!} ds$$

$$+ (-1)^{n-l} \int_0^\infty \left(\int_0^{\min\{t,s\}} \frac{(t-x)^{m-1}}{(m-1)!} \rho^{-1}(x) \frac{(s-x)^{k-1}}{(k-1)!} dx \right) D_\rho^n f(s) ds. \tag{15}$$

From (6), (12), and (15) it easily follows

COROLLARY 1. *Let* $f \in w_p^n$. *Suppose that the conditions of Theorem 1 for the case (1)* $m < l$ *hold. Then* $D_\rho^j f(\infty) = 0$, $j = l,\dots,n-1$, *if and only if* $c_j = 0$, $j = l,\dots,n-1$*; and if* $D_\rho^i f(0) = 0$, $i = m,\dots,l-1$, *then* $D_\rho^j f(0) = 0$, $j = 0,\dots,m-1$, *if and only if* $a_j = 0$, $j = 0,\dots,m-1$.

Suppose that the conditions of Theorem 1 for the case (2) $m > l$ *hold. Then* $D_\rho^j f(\infty) = 0$, $j = l,\dots,n-1$, *and* $D_\rho^j f(0) = 0$, $j = 0,\dots,l-1$, *if and only if* $a_i = 0$, $i = 0,\dots,m-1$, *and* $c_i = 0$, $i = m,\dots,n-1$.

Suppose that the conditions of Theorem 1 for the case (3) $m = l$ *hold. Then* $D_\rho^j f(\infty) = 0$, $j = m,\dots,n-1$, *if and only if* $c_j = 0$, $j = m,\dots,n-1$.

In each of the representations (6), (12), and (15) there are n constants: in the case (1) $m < l$ they are a_j, $j = 0,\dots,m-1$, $D_\rho^j f(0)$, $j = m,\dots,l-1$, and c_j, $j = l,\dots,n-1$; in the case (2) $m > l$ they are a_j, $j = 0,\dots,m-1$, and c_j, $j = m,\dots,n-1$; in the case (3) $m = l$ they are $D_\rho^j f(0)$, $j = 0,\dots,m-1$, and c_j, $j = m,\dots,n-1$. Denote them by $\alpha_i = \alpha_i(f)$, $i = 0,\dots,n-1$. Then on w_p^n the following functional can be defined:

$$\|f\|_{W_p^n} = \|u D_\rho^n f\|_p + \sum_{\nu=0}^{n-1} |\alpha_\nu(f)|, \tag{16}$$

which turns w_p^n into a normalized space $W_p^n(u,\rho;I) \equiv W_p^n$. By standard methods we can prove that W_p^n is a Banach space.

Let $\dot{W}_p^n = \{f : f \in W_p^n, \alpha_\nu(f) = 0, \nu = 0,\dots,n-1\}$. Notice that $\alpha_\nu(f) = 0$, $\nu = 0,\dots,n-1$, is equivalent to that $D_\rho^i f(0) = 0$, $i = 0,\dots,l-1$, and $D_\rho^i f(\infty) = 0$, $i = l,\dots,n-1$.

From the definition of $\alpha_\nu(f)$ it follows that $\alpha_\nu(f)=0$, $\nu=0,1,\dots,n-1$, for $f\in C_0^\infty(I)$.

Denote by $\overset{\circ}{W}{}_2^n(u,\rho,I)\equiv\overset{\circ}{W}{}_2^n$ the closure of the set $C_0^\infty(I)$ with respect to the norm (16).

THEOREM 2. *Suppose that (1), (2), and (3) hold in the case (1) $m<l$, (7) and (8) hold in the case (2) $m>l$, and (13) and (14) hold in the case (3) $m=l$. Moreover, assume that for $t>0$ the following conditions hold:*

for the case (1) $m<l$:

$$\rho^{-1}(s)\notin L_1(t,\infty), \tag{17}$$

$$u^{-1}(s)s^{n-l-1}\notin L_{p'}(0,t), \tag{18}$$

$$u^{-1}(s)s^{n-l}\notin L_{p'}(t,\infty), \tag{19}$$

for the case (2) $m>l$:

$$\rho^{-1}(s)s^{m-l-1}\notin L_1(0,t), \tag{20}$$

$$u^{-1}(s)s^{k-1}\notin L_{p'}(0,t), \tag{21}$$

$$u^{-1}(s)\int_0^s\rho^{-1}(x)x^{m-l}(s-x)^{k-1}dx\notin L_{p'}(t,\infty), \tag{22}$$

for the case (3) $m=l$: (17), (21), and

$$u^{-1}(s)s^{k-1}\int_0^s\rho^{-1}(x)dx\notin L_{p'}(t,\infty). \tag{23}$$

Then $\overset{\circ}{W}{}_2^n=\dot{W}_p^n$.

Thus, if the conditions of Theorem 2 hold, then we have

$$\overset{\circ}{W}{}_2^n=\{f:f\in W_p^n,D_\rho^if(0)=0,i=0,\dots,l-1,D_\rho^if(\infty)=0,i=l,\dots,n-1\}.$$

Now we shall consider the problem of continuousness and compactness of the embedding $\overset{\circ}{W}{}_2^n\hookrightarrow L_{q,v}$, when $1<p\leqslant q<\infty$.

Let $\varphi_1(s,x)=\begin{cases}x^{l-m}s^{n-l-1}, & x\leqslant s,\\ x^{l-m-1}s^{n-l}, & x>s,\end{cases}$ and $\varphi_2(s,x)=\begin{cases}x^{m-l}s^{l-1}, & x\leqslant s,\\ x^{m-l-1}s^{l}, & x>s.\end{cases}$

We define $T^1_{p,q}(t)$ and $T^2_{p,q}(t)$ by

$$\begin{cases} \left(\int\limits_t^\infty v^q(y)\left(\int\limits_0^t u^{-p'}(s)\left(\int\limits_0^y \rho^{-1}(x)(y-x)^{m-1}\varphi_1(s,x)dx\right)^{p'} ds\right)^{\frac{q}{p'}} dy\right)^{\frac{1}{q}}, m<l, \\ \left(\int\limits_t^\infty u^{-p'}(y)\left(\int\limits_0^t v^q(s)\left(\int\limits_0^y \rho^{-1}(x)(y-x)^{k-1}\varphi_2(s,x)dx\right)^{q} ds\right)^{\frac{p'}{q}} dy\right)^{\frac{1}{p'}}, m>l, \\ \left(\int\limits_t^\infty v^q(y)\left(\int\limits_0^t u^{-p'}(s)\left(\int\limits_0^s \rho^{-1}(\tau)(y-\tau)^{m-1}(s-\tau)^{k-1}d\tau\right)^{p'} ds\right)^{\frac{q}{p'}} dy\right)^{\frac{1}{q}}, m=l, \end{cases}$$

and

$$\begin{cases} \left(\int\limits_0^t v^q(s)\left(\int\limits_0^s \rho^{-1}(x)x^{l-m}(s-x)^{m-1}dx\right)^{q} ds\right)^{\frac{1}{q}}\left(\int\limits_t^\infty u^{-p'}(y)y^{p'(n-l-1)}dy\right)^{\frac{1}{p'}}, m<l, \\ \left(\int\limits_t^\infty v^q(y)y^{q(l-1)}dy\right)^{\frac{1}{q}}\left(\int\limits_0^t u^{-p'}(s)\left(\int\limits_0^s \rho^{-1}(x)x^{m-l}(s-x)^{k-1}dx\right)^{p'} ds\right)^{\frac{1}{p'}}, m>l, \\ \left(\int\limits_t^\infty u^{-p'}(y)\left(\int\limits_0^t v^q(s)\left(\int\limits_0^s \rho^{-1}(\tau)(y-\tau)^{k-1}(s-\tau)^{m-1}d\tau\right)^{q} ds\right)^{\frac{p'}{q}} dy\right)^{\frac{1}{p'}}, m=l, \end{cases}$$

respectively. Moreover, let

$$T^i_{p,q} = \sup_{t>0} T^i_{p,q}(t),\ i=1,2;\ T_{p,q} = \max\{T^1_{p,q}, T^2_{p,q}\}.$$

THEOREM 3. *Let $1<p\leqslant q<\infty$, $n\geqslant 3$ for $m\neq l$ and $n\geqslant 2$ for $m=l$. Suppose that u and ρ satisfy the conditions of Theorem 2 concerning the relations between l and m. Then the embedding $\mathring{W}^n_p \hookrightarrow L_{q,v}$*
(a) is continuous if and only if $T_{p,q}<\infty$, and, in addition, $\|E\|\approx T_{p,q}$, where $\|E\|$ is a norm of the embedding operator $E:\mathring{W}^n_2 \to L_{q,v}$;
(b) is compact if and only if $T_{p,q}<\infty$ and

$$\lim_{t\to 0} T^i_{p,q}(t) = \lim_{t\to\infty} T^i_{p,q}(t) = 0,\ i=1,2.$$

Since $D^i_\rho f(0)=0$, $i=0,\dots,l-1$, and $D^i_\rho f(\infty)=0$, $i=l,\dots,n-1$, for $f\in\mathring{W}^n_2$, in the integral representations of $f\in W^n_p$ (see Theorem 1) it follows that $f\in\mathring{W}^n_2$ can be expressed by

$$f(t) = (-1)^{n-l}\int\limits_0^\infty K(t,s)D^n_\rho f(s)ds,\ \forall f\in\mathring{W}^n_2, \tag{24}$$

where

$$K(t,s)=\begin{cases}\int\limits_0^{\min\{t,s\}}\frac{(s-z)^{n-l-1}}{(n-l-1)!}\int\limits_z^t\frac{(t-x)^{m-1}}{(m-1)!}\rho^{-1}(x)\frac{(x-z)^{l-m-1}}{(l-m-1)!}dxdz, & m<l,\\ \int\limits_0^{\min\{t,s\}}\frac{(t-z)^{l-1}}{(l-1)!}\int\limits_z^s\frac{(s-x)^{k-1}}{(k-1)!}\rho^{-1}(x)\frac{(x-z)^{m-l-1}}{(m-l-1)!}dxdz, & m>l,\\ \int\limits_0^{\min\{t,s\}}\frac{(t-x)^{m-1}}{(m-1)!}\rho^{-1}(x)\frac{(s-x)^{k-1}}{(k-1)!}dx, & m=l.\end{cases} \tag{25}$$

Further we need the following Lemma:

LEMMA 1. *Let $1<p<\infty$. Suppose that u and ρ satisfy the conditions of Theorem 1 concerning the relations between l and m. Then $\forall t\in I$ the relation*

$$J(t)=\sup_{f\in\mathring{W}_2^n}\frac{|f(t)|}{\|uD_\rho^n f\|_p}=\left(\int\limits_0^\infty K^{p'}(t,s)u^{-p'}(s)ds\right)^{\frac{1}{p'}} \tag{26}$$

holds, where $K(t,s)$ is defined by (25).

3. Proofs

Proof of Theorem 1. Let us begin with the case (1) $m<l$.
Put

$$D_\rho^l f(t):=\varphi(t).$$

Then $D_\rho^n f(t)=\varphi^{(n-l)}(t)$ and $\varphi\in L_{p,u}^{n-l}$, where $L_{p,u}^{n-l}$ is Kudryavcev's class [6] with the following seminorm:

$$\|\varphi\|_{L_{p,u}^{n-l}}=\left\|u\varphi^{(n-l)}\right\|_p.$$

From the condition (3) and Kudryavcev's theorem [6] there exist numbers c_i, $i=l,\dots,n-1$, which are defined by φ from the relation:

$$\lim_{t\to\infty}\left(\varphi(t)-\sum_{i=l}^{n-1}c_i\frac{t^{i-l}}{(i-l)!}\right)^{(j)}=0,\ j=0,\dots,n-l-1,$$

which is equivalent to (4), and the following representation holds:

$$\varphi(t)=\sum_{i=l}^{n-1}c_i\frac{t^{i-l}}{(i-l)!}+(-1)^{n-l}\int\limits_t^\infty\frac{(s-t)^{n-l-1}}{(n-l-1)!}\varphi^{(n-l)}(s)ds,$$

i.e.,

$$D_\rho^l f(t)=\sum_{i=l}^{n-1}c_i\frac{t^{i-l}}{(i-l)!}+(-1)^{n-l}\int\limits_t^\infty\frac{(s-t)^{n-l-1}}{(n-l-1)!}D_\rho^n f(s)ds. \tag{27}$$

Let us show that from (1) and (2) we get the finiteness of the expression:

$$\psi_i(t) = \int_0^t (t-x)^i \int_x^\infty (s-x)^{n-l-1} |D_\rho^n f(s)| ds dx$$

for $0 \leqslant i \leqslant l-m-1$ and $\forall t > 0$.

Indeed,

$$\psi_i(t) = \int_0^t (t-x)^i \int_x^t (s-x)^{n-l-1} |D_\rho^n f(s)| ds dx$$

$$+ \int_0^t (t-x)^i \int_t^\infty (s-x)^{n-l-1} |D_\rho^n f(s)| ds dx$$

$$= \int_0^t |D_\rho^n f(s)| \int_0^s (t-x)^i (s-x)^{n-l-1} dx ds + \int_t^\infty |D_\rho^n f(s)| \int_0^t (t-x)^i (s-x)^{n-l-1} dx ds$$

$$\approx t^i \int_0^t s^{n-l} |D_\rho^n f(s)| ds + t^{i+1} \int_t^\infty s^{n-l-1} |D_\rho^n f(s)| ds. \tag{28}$$

For $t \geqslant 1$, in view of (28), (2), and (3) we have

$$\psi_i(t) \ll t^i \|u D_\rho^n f\|_p \left(\left(\int_0^t |u^{-1}(s) s^{n-l}|^{p'} ds \right)^{\frac{1}{p'}} \right.$$

$$\left. + t \left(\int_t^\infty |u^{-1}(s) s^{n-l-1}|^{p'} ds \right)^{\frac{1}{p'}} \right) < \infty.$$

For $0 \leqslant t < 1$, according to (28), (2), and (3) we have

$$\psi_i(t) \ll t^i \int_0^t s^{n-l} |D_\rho^n f(s)| ds + t^i \int_t^1 s^{n-l} |D_\rho^n f(s)| ds + t^{i+1} \int_1^\infty s^{n-l-1} |D_\rho^n f(s)| ds$$

$$\leqslant t^i \|u D_\rho^n f\|_p \left(\left(\int_0^1 |u^{-1}(s) s^{n-l}|^{p'} ds \right)^{\frac{1}{p'}} + t \left(\int_1^\infty |u^{-1}(s) s^{n-l-1}|^{p'} ds \right)^{\frac{1}{p'}} \right) < \infty.$$

Therefore, the equality (27) can be integrated $l-m$ times on $(0,t)$:

$$D_\rho^m f(t) = \sum_{i=l}^{n-1} c_i \frac{t^{i-m}}{(i-m)!} + \sum_{i=m}^{l-1} D_\rho^i f(0) \frac{t^{i-m}}{(i-m)!}$$

$$+(-1)^{n-l}\int_0^t \frac{(t-x)^{l-m-1}}{(l-m-1)!}\int_x^\infty \frac{(s-x)^{n-l-1}}{(n-l-1)!}D_\rho^n f(s)dsdx.$$

Let us rewrite the obtained equality in the form:

$$\frac{d^m}{dt^m}\left(f(t)-\sum_{i=m}^{l-1}D_\rho^i f(0)\int_1^t \frac{(t-s)^{m-1}}{(m-1)!}\rho^{-1}(s)\frac{s^{i-m}}{(i-m)!}ds\right)$$

$$=\rho^{-1}(t)\sum_{i=l}^{n-1}c_i\frac{t^{i-m}}{(i-m)!}$$

$$+(-1)^{n-l}\rho^{-1}(t)\int_0^t \frac{(t-x)^{l-m-1}}{(l-m-1)!}\int_x^\infty \frac{(s-x)^{n-l-1}}{(n-l-1)!}D_\rho^n f(s)dsdx. \tag{29}$$

Now we shall prove that the right–hand side of the last equality can be integrated m times on $(0,t)$. From (1) $\forall t>0$ it follows that $\rho^{-1}(s)s^{l-m}dx \in L_1(0,t)$. This yields that the first summand of the right–hand side of (29) can be integrated m times. To show that the second summand can be integrated m times we need to prove that $\forall t>0$:

$$I(t)=\int_0^t (t-x)^{m-1}\rho^{-1}(x)\psi_{l-m-1}(x)dx<\infty. \tag{30}$$

From (28) we get

$$I(t)\approx\int_0^t (t-x)^{m-1}\rho^{-1}(x)x^{l-m-1}\int_0^x s^{n-l}|D_\rho^n f(s)|dsdx$$

$$+\int_0^t (t-x)^{m-1}\rho^{-1}(x)x^{l-m}\int_x^\infty s^{n-l-1}|D_\rho^n f(s)|dsdx$$

$$\leqslant t^{m-1}\left(\int_0^t |D_\rho^n f(s)|s^{n-l}\int_s^t \rho^{-1}(x)x^{l-m-1}dxds\right.$$

$$+\int_0^t |D_\rho^n f(s)|s^{n-l-1}\int_0^s \rho^{-1}(x)x^{l-m}dxds$$

$$\left.+\int_t^\infty |D_\rho^n f(s)|s^{n-l-1}\int_0^t \rho^{-1}(x)x^{l-m}dxds\right)$$

$$\leqslant t^{m-1}\|uD_\rho^n f\|_p\left(\left(\left(\int_0^t |u^{-1}(s)s^{n-l}\int_s^t \rho^{-1}(x)x^{l-m-1}dx|^{p'}ds\right)^{\frac{1}{p'}}+\right.\right.$$

$$+\left(\int_0^t |u^{-1}(s)s^{n-l-1}\int_0^s \rho^{-1}(x)x^{l-m}dx|^{p'}ds\right)^{\frac{1}{p'}}.$$

$$\left.+\int_0^t \rho^{-1}(x)x^{l-m}dx\left(\int_t^\infty |u^{-1}(s)s^{n-l-1}|^{p'}ds\right)^{\frac{1}{p'}}\right).$$

Now, by using (1), (2), and (3) we obtain that (30) holds. Hence, we can integrate (29) m times on $(0,t)$:

$$f(t)=\sum_{\nu=0}^{m-1}a_\nu\frac{t^\nu}{\nu!}+\sum_{\nu=m}^{l-1}D_\rho^\nu f(0)\int_1^t\frac{(t-s)^{m-1}}{(m-1)!}\rho^{-1}(s)\frac{s^{\nu-m}}{(\nu-m)!}ds$$

$$+\sum_{\nu=l}^{n-1}c_\nu\int_0^t\frac{(s-t)^{m-1}}{(m-1)!}\rho^{-1}(s)\frac{s^{\nu-m}}{(\nu-m)!}ds$$

$$+(-1)^{n-l}\int_0^t\rho^{-1}(\tau)\frac{(t-\tau)^{m-1}}{(m-1)!}\int_0^\tau\frac{(\tau-x)^{l-m-1}}{(l-m-1)!}\int_x^\infty\frac{(s-x)^{n-l-1}}{(n-l-1)!}D_\rho^n f(s)dsdxd\tau, \tag{31}$$

where a_i, $i=0,\dots,m-1$, are from (5).

By changing the order of integration in the last summand of (31), we get (6).

Let us turn to the case (2) $m>l$. As in the first case supposing that $D_\rho^m f(t)=\varphi(t)$ we have $\varphi\in L_{p,u}^k$. From the condition (8) $\forall t>0$ we get $u^{-1}(s)s^{k-1}\in L_{p'}(t,\infty)$ and, hence, by Kudryavcev's theorem [6] there exist numbers c_i, $i=0,\dots,k-1$, which are defined by φ from the relation (9), and the following relation holds:

$$D_\rho^m f(t)=\sum_{i=m}^{n-1}c_i\frac{t^{i-m}}{(i-m)!}+(-1)^k\int_t^\infty\frac{(s-t)^{k-1}}{(k-1)!}D_\rho^n f(s)ds,$$

or

$$f^{(m)}(t)=\sum_{i=m}^{n-1}c_i\rho^{-1}(t)\frac{t^{i-m}}{(i-m)!}+(-1)^k\rho^{-1}(t)\int_t^\infty\frac{(s-t)^{k-1}}{(k-1)!}D_\rho^n f(s)ds. \tag{32}$$

The last equality can be rewritten in the form:

$$\left(f(t)-\sum_{i=m}^{n-1}c_i\int_1^t\frac{(t-s)^{m-1}}{(m-1)!}\rho^{-1}(s)\frac{s^{i-m}}{(i-m)!}ds\right)^{(m)}$$

$$= (-1)^k \rho^{-1}(t) \int_t^\infty \frac{(s-t)^{k-1}}{(k-1)!} D_\rho^n f(s) ds.$$

In the same way as in the first case it is easy to show that from (8) it follows that the right–hand side of the last equality can be integrated $m-l$ times on (t,∞).

Put

$$\psi(t) = f(t) - \sum_{i=m}^{n-1} c_i \int_1^t \frac{(t-s)^{m-1}}{(m-1)!} \rho^{-1}(s) \frac{s^{i-m}}{(i-m)!} ds.$$

After integration of the both sides of the last equality on (t,∞) we can conclude that there exists $\psi^{(m-1)}(\infty) \equiv \lim_{t\to\infty} \psi^{(m-1)}(t) = a_{m-1}$, and the following equality is true:

$$\psi^{(m-1)}(t) - a_{m-1} = (-1)^{k+1} \int_t^\infty \rho^{-1}(x) \int_x^\infty \frac{(s-x)^{k-1}}{(k-1)!} D_\rho^n f(s) ds\, dx$$

or

$$\left(\psi(t) - a_{m-1}\frac{t^{m-1}}{(m-1)!}\right)^{(m-1)} = (-1)^{k+1} \int_t^\infty \rho^{-1}(x) \int_x^\infty \frac{(s-x)^{k-1}}{(k-1)!} D_\rho^n f(s) ds\, dx,$$

where

$$\lim_{t\to\infty} \left(\psi(t) - a_{m-1}\frac{t^{m-1}}{(m-1)!}\right)^{(m-1)} = 0.$$

If we continue to integrate till $m-l$ times we find numbers a_i, $i = l,\dots,m-1$, which are defined from the relations:

$$\lim_{t\to\infty} \left(\psi(t) - \sum_{i=l}^{m-1} a_i \frac{t^i}{i!}\right)^{(j)} = 0,\ j = l,\dots,m-1,$$

i.e., from (10). Moreover, the following equality is true:

$$\psi^{(l)}(t) = \sum_{i=l}^{m-1} a_i \frac{t^{i-l}}{(i-l)!} + (-1)^{n-l} \int_t^\infty \frac{(x-t)^{m-l-1}}{(m-l-1)!} \rho^{-1}(x) \int_x^\infty \frac{(s-x)^{k-1}}{(k-1)!} D_\rho^n f(s) ds\, dx.$$

Again from (7) and (8) it follows that the right–hand side of the last equality can be integrated l times on $(0,t)$. Hence, l times integration of the both sides of the last equality yields:

$$\psi(t) = \sum_{i=0}^{m-1} a_i \frac{t^i}{i!} + (-1)^{n-l}$$

$$\times \int_0^t \frac{(t-z)^{l-1}}{(l-1)!} \int_z^\infty \frac{(x-z)^{m-l-1}}{(m-l-1)!} \rho^{-1}(x) \int_x^\infty \frac{(s-x)^{k-1}}{(k-1)!} D_\rho^n f(s) ds\, dx\, dz,$$

or (12) after changing of the order of integration in the last summand, where

$$a_i = \lim_{t\to 0} \psi^{(i)}(t),\ 0 = 1,\dots,l-1,$$

which is the same as (11).

The correctness of the case (3) $m = l$ is obvious from (32) because, due to (13) and (14), we can integrate the right–hand side of (32) on $(0,t)$ and get (15) and (9).

□

Proof of Theorem 2. Since $C_0^\infty(I) \subset \dot{W}_p^n$, it yields that $\mathring{W}_2^n \subset \dot{W}_p^n$.

Let us prove the reversed inclusion $\mathring{W}_2^n \supset \dot{W}_p^n$.

Due to (16) the space W_p^n is isomorphic to the space $L_{p,u} \times R^n$. Hence, for $1 < p < \infty$ we have that the space W_p^n is reflexive and $(L_{p,u} \times R^n)^* = L_{p',u^{-1}} \times R^n = (W_p^n)^*$ up to isometry. Therefore, $\forall F \in (W_p^n)^*$ there exist a unique function $g \in L_{p',u^{-1}}$ and a set $\beta = (\beta_0, \beta_1, \dots, \beta_{n-1}) \in R^n$, such that $\forall f \in W_p^n$:

$$F(f) = \int_0^\infty g(s) D_\rho^n f(s) ds + \sum_{i=0}^{n-1} \alpha_i \beta_i,\ \alpha_i = \alpha_i(f),\ \beta_i = \beta_i(g). \tag{33}$$

Let $B = \{F : F \in (W_p^n)^*, F(f) = 0, f \in C_0^\infty(I)\}$. From the reflexivity of W_p^n and the denseness of $C_0^\infty(I)$ in $\mathring{W}_2^n$ we get $\mathring{W}_2^n = \{f : f \in W_p^n, F(f) = 0, \forall F \in B\}$.

Hence, it is obvious that $\mathring{W}_2^n \supset \dot{W}_p^n$ is equivalent to the condition: $F(f) = 0$, $\forall f \in \dot{W}_p^n$, holds $\forall F \in B$.

From (33) $\forall \varphi \in C_0^\infty(I)$ and $\forall F \in B$ we have

$$F(\varphi) = \int_0^\infty g(t) D_\rho^n \varphi(t) dt = 0.$$

Therefore, g is a solution of the equation

$$(D_\rho^n)^* g(t) = 0,$$

i.e.,

$$\frac{d^m}{dt^m} \rho(t) \frac{d^k}{dt^k} g(t) = 0. \tag{34}$$

Hence, there exist constants $\gamma_i = \gamma_i(g)$, $i = 0, \dots, n-1$, such that

$$g(t) = \sum_{i=0}^{n-1} \gamma_i \omega_i(t), \tag{35}$$

where $\omega_i(\cdot)$, $i = 0, \dots, n-1$, is a fundamental system of solutions of (34).

Now the aim is to prove that $g(\cdot) \equiv 0$. We now point out a scheme for the proof of this crucial fact. Firstly, we prove that (35) can be divided into two groups of sums. The first group consists of functions $\omega_i(\cdot)$ such that $u^{-1}(\cdot)\omega_i(\cdot)$ belong to $L_{p'}(0,t)$,

$t > 0$, but do not belong to $L_{p'}(t,\infty)$, $t > 0$, and the second one consists of functions $\omega_i(\cdot)$ such that $u^{-1}(\cdot)\omega_i(\cdot)$ belong to $L_{p'}(t,\infty)$, $t > 0$, but do not belong to $L_{p'}(0,t)$, $t > 0$. Secondly, we prove that the functions $\omega_i(\cdot)$ inside of each of the groups do not have a same order in neighbourhoods of the appropriate endpoints. This yields that we can present $g(\cdot)$ in the following form:

$$g = \varphi + \psi,$$

where $\forall t > 0$:

$$u^{-1}\varphi \in L_{p'}(0,t),\ u^{-1}\varphi \notin L_{p'}(t,\infty),$$

$$u^{-1}\psi \in L_{p'}(t,\infty),\ u^{-1}\psi \notin L_{p'}(0,t).$$

But by the assumption it yields that $u^{-1}g \in L_{p'}(0,\infty)$. Hence, since $u^{-1}\varphi \in L_{p'}(0,t)$ we should have $u^{-1}\psi \in L_{p'}(0,t)$, but we have $u^{-1}\psi \notin L_{p'}(0,t)$. This contradiction gives that $\psi(t) = 0$, $\forall t > 0$. The same arguments for the interval (t,∞) imples that $\varphi(t) = 0$, $\forall t > 0$. Therefore, $g(t) \equiv 0$, $\forall t \in I$.

To use this scheme we need to construct different fundamental systems of solutions of (34) suitable for each of the cases (1) $m < l$, (2) $m > l$, and (3) $m = l$.

Let us begin from the case (1) $m < l$. The following functions

$$\omega_i(t) = t^i,\ i = 0,\ldots,k-1, \tag{36}$$

$$\omega_i(t) = \int_0^t (t-x)^{(m-l)+i-1} \int_1^x \rho^{-1}(s)(x-s)^{(k+l-m)-i-1} s^{i-k} ds dx,$$

$$i = k,\ldots,k+l-m-1,$$

$$\omega_i(t) = \int_0^t \rho^{-1}(s)(t-s)^{k-1} s^{i-k} ds,\ i = k+l-m,\ldots,n-1,$$

form a fundamental system of solutions of (34).

Estimate ω_i, $i = k,\ldots,n-1$, from above in a right neighbourhood of zero. For $i = k,\ldots,k+l-m-1$, we have

$$|\omega_i(t)| = \int_0^t \rho^{-1}(s)s^{i-k} \int_0^s (t-x)^{(m-l)+i-1}(s-x)^{(k+l-m)-i-1} dx ds$$

$$+ \int_t^1 \rho^{-1}(s)s^{i-k} \int_0^t (t-x)^{(m-l)+i-1}(s-x)^{(k+l-m)-i-1} dx ds$$

$$\approx t^{(m-l)+i-1} \int_0^t \rho^{-1}(s)s^{l-m} ds + t^{(m-l)+i} \int_t^1 \rho^{-1}(s)s^{l-m-1} ds, \tag{37}$$

and for $i = k+l-m, \dots, n-1$ we have

$$\omega_i(t) \leqslant t^{k-1} \int_0^t \rho^{-1}(s) s^{i-k} ds. \tag{38}$$

Estimate ω_i, $i = k, \dots, n-1$, from below in a neighbourhood of infinity. For $i = k, \dots, k+l-m-1$, and for some $t > 1$, changing order of integration and integrating by parts, we have

$$\omega_i(t) = (-1)^{k+l-m-i} \int_0^1 \rho^{-1}(s) s^{i-k} \int_0^s (t-x)^{(m-l)+i-1} (s-x)^{(k+l-m)-i-1} dx ds$$

$$+ c(m,l,k,i) \int_1^t \rho^{-1}(s) s^{i-k} (t-s)^{k-1} ds. \tag{39}$$

If $k+l-m-i$ is an even number, then from (39) for some $t > 2$ we get

$$\omega_i(t) \geqslant c(m,l,k,i) \int_1^2 \rho^{-1}(s) s^{i-k} (t-s)^{k-1} ds \geqslant (t-2)^{k-1} \int_1^2 \rho^{-1}(s) s^{i-k} ds. \tag{40}$$

If $k+l-m-i$ is an odd number, then from (39) for some $t > 2$ we obtain

$$\omega_i(t) \geqslant c(m,l,k,i) \int_1^2 \rho^{-1}(s) s^{i-k} (t-s)^{k-1} ds - t^{(m-l)+i-1} \int_0^1 \rho^{-1}(s) s^{l-m} ds$$

$$= t^{k-1}(1 + \bar{\bar{o}}(1)), \ t \to \infty. \tag{41}$$

For $i = k+l-m, \dots, n-1$, and for some $t > 1$, we have:

$$\omega_i(t) \geqslant \int_0^1 \rho^{-1}(s)(t-s)^{k-1} s^{i-k} ds \geqslant (t-1)^{k-1} \int_0^1 \rho^{-1}(s) s^{i-k} ds. \tag{42}$$

When $i = n-l, \dots, n-1$, from (37), (38), (1), and (2) for $t > 0$ we have $u^{-1}\omega_i \in L_{p'}(0,t)$, but from (40), (41), (42), and (19) we have $u^{-1}\omega_i \notin L_{p'}(t,\infty)$. Using the L'Hospital rule and taking (17) into account it is easy to see that for $i = n-l+1, \dots, n-1$:

$$\lim_{t\to\infty} \frac{\omega_{i-1}(t)}{\omega_i(t)} = 0,$$

i.e., $\omega_{i-1}(t) = \bar{\bar{o}}(\omega_i(t))$, $i = n-l+1, \dots, n-1$, when $t \to \infty$.

When $i = 0, \dots, n-l-1$, from (3) it obviously follows that $u^{-1}\omega_i \in L_{p'}(t,\infty)$, and from (18) it follows that $u^{-1}\omega_i \notin L_{p'}(0,t)$. Moreover, $\omega_i(t) = \bar{\bar{o}}(\omega_{i-1}(t))$, $i = 1, \dots, n-l-1$, when $t \to 0$. Therefore, $g(t) = 0$, $\forall t \in I$.

Let us turn to the case (2) $m > l$. This case we divide into two cases: $m-l > k-1$ and $m-l \leqslant k-1$.

In its turn, the case $m-l > k-1$ we divide into two cases: $k=1$ and $k>1$.

Suppose $k=1$. Then the system (36),

$$\omega_i(t) = \int_1^t \rho^{-1}(s)(t-s)^{k-1}s^{i-k}ds, \ i=k,\dots,m-l, \tag{43}$$

and

$$\omega_i(t) = \int_0^t \rho^{-1}(s)(t-s)^{k-1}s^{i-k}ds, \ i=n-l,\dots,n-1, \tag{44}$$

is a fundamental system of solutions of (34). Notice that the system consists of n functions because here $m-l+1 = n-l$.

When $i=n-l,\dots,n-1$, according to (7) for $t>0$ we have $u^{-1}\omega_i \in L_{p'}(0,t)$, but in view of (22) we have $u^{-1}\omega_i \notin L_{p'}(t,\infty)$. Moreover, $\omega_{i-1}(t) = \bar{\bar{o}}(\omega_i(t))$, $j=n-l+1,\dots,n-1$, when $t\to\infty$.

When $i=0,\dots,m-l=n-l-1$, by (8) for $t>0$ we have $u^{-1}\omega_i \in L_{p'}(t,\infty)$, but by (21) we have $u^{-1}\omega_i \notin L_{p'}(0,t)$. Using the L'Hospital rule and taking (20) into account, we get $\lim\limits_{t\to 0}|\omega_i(t)| = \infty$, $i=1,\dots,m-l=n-l-1$, and $\omega_i(t) = \bar{\bar{o}}(\omega_{i-1}(t))$, $i=2,\dots,m-l=n-l-1$, when $t\to 0$. Therefore, $g(t)=0$, $\forall t\in I$.

Suppose $k>1$. Then to get a fundamental system of solutions of (34) we add to (36), (43), and (44) the following functions:

$$\omega_i(t) = \int_0^t (t-x)^{i-(m-l)-1}\int_1^x \rho^{-1}(s)(x-s)^{(n-l)-i-1}s^{i-k}dsdx, \tag{45}$$
$$i=m-l+1,\dots,n-l-1.$$

Let us notice that for $i=n-l,\dots,n-1$, we have the same situation as in the case $k=1$. Hence we need to consider the case when $i=0,\dots,n-l-1$.

Estimate (45) from above for $t\geqslant 1$:

$$|\omega_i(t)| \leqslant \int_0^1 (t-x)^{i-(m-l)-1}\int_x^1 \rho^{-1}(s)(s-x)^{(n-l)-i-1}s^{i-k}dsdx$$

$$+\int_1^t (t-x)^{i-(m-l)-1}\int_1^x \rho^{-1}(s)(x-s)^{(n-l)-i-1}s^{i-k}dsdx$$

$$\leqslant t^{i-(m-l)-1}\int_0^1 \rho^{-1}(s)s^{m-l}ds + \int_1^t \rho^{-1}(s)(x-s)^{k-1}s^{i-k}ds. \tag{46}$$

In the same way as in the case (1) $m < l$ we estimate (45) for $0 \leqslant t < 1$:

$$|\omega_i(t)| \approx t^{i-(m-l)-1} \int_0^t \rho^{-1}(s)s^{m-l}ds + t^{i-(m-l)} \int_t^1 \rho^{-1}(s)s^{m-l-1}ds. \qquad (47)$$

When $i = m-l+1, \dots, n-l-1$, from (46) and (8) for $t > 0$ it follows that $u^{-1}\omega_i \in L_{p'}(t,\infty)$, but from (47) and (21) it follows that $u^{-1}\omega_i \notin L_{p'}(0,t)$. Thus, when $i = 0, \dots, n-l-1$, we have $u^{-1}\omega_i \in L_{p'}(t,\infty)$ and $u^{-1}\omega_i \notin L_{p'}(0,t)$.

Arguing as in the case $k = 1$, we have $\lim_{t\to 0} |\omega_i(t)| = \infty$, $i = k, \dots, m-l$, and $\omega_i(t) = \bar{\bar{o}}(\omega_{i-1}(t))$, $i = k+1, \dots, m-l$, when $t \to 0$. At the same time $\lim_{t\to 0} |\omega_i(t)| = 0$ for $i = 1, \dots, k-1$, and $i = m-l+1, \dots, n-l-1$, and it is obvious that $\omega_i(t) = \bar{\bar{o}}(\omega_{i-1}(t))$, $i = 1, \dots, k-1$, when $t \to 0$, and due to (20) $\omega_i(t) = \bar{\bar{o}}(\omega_{i-1}(t))$, $i = m-l+2, \dots, n-l-1$, when $t \to 0$. However, again due to (20) for $t \to 0$ there are no functions ω_i, when $i = 0, \dots, k-1$, which have a same order as the functions ω_i, when $i = m-l+1, \dots, n-l-1$. Therefore, $g(t) = 0$, $\forall t \in I$.

Let us turn to the case $m-l \leqslant k-1$. Since still (2) $m > l$, then $k > 1$. In this case the functions from (36), (45) with $i = k, \dots, n-l-1$, and (44) form a fundamental system of solutions of (34). By arguing in the same way as above we find that $g(t) = 0$, $\forall t \in I$.

Now we shall turn to the case (3) $m = l$. The system of functions (36) and

$$\omega_i(t) = \int_0^t (t-x)^{k-1}\rho^{-1}(x)x^{i-k}dx, \; i = k, \dots, n-1,$$

is a fundamental system of solution of (34).

When $i = 0, \dots, k-1$, from (14) for $t > 0$ it follows that $u^{-1}\omega_i \in L_{p'}(t,\infty)$, and from (21) it follows that $u^{-1}\omega_i \notin L_{p'}(0,t)$. But $\omega_i(t) = \bar{\bar{o}}(\omega_{i-1}(t))$, $i = 1, \dots, k-1$, when $t \to 0$.

When $i = k, \dots, n-1$, from (13) for $t > 0$ it follows that $u^{-1}\omega_i \in L_{p'}(0,t)$, and from (23) it follows that $u^{-1}\omega_i \notin L_{p'}(t,\infty)$. But due to (17) we have $\omega_{i-1}(t) = \bar{\bar{o}}(\omega_i(t))$, $i = k+1, \dots, n-1$, when $t \to \infty$. Thus, again $g(t) = 0$, $\forall t \in I$.

For all three cases (1) $m < l$, (2) $m > l$ and (3) $m = l$ we obtain $g(t) = 0$ and

$$F(f) = \sum_{i=0}^{n-1} \alpha_i(f)\beta_i, \; \forall F \in B.$$

By the definition $\alpha_i(f) = 0$, $i = 0, 1, \dots, n-1$, for any $f \in \dot{W}_p^n$. Hence $\forall F \in B$ and $\forall f \in \dot{W}_p^n$ we get $F(f) = 0$, i.e., $\mathring{W}_2^n \supset \dot{W}_p^n$. This gives $\mathring{W}_2^n = \dot{W}_p^n$, and the proof is complete. □

Proof of Theorem 3. From (24) and (25) the problems of continuousness and compactness of the embedding $\mathring{W}_2^n \hookrightarrow L_{q,v}$ are equivalent to the problems of continuousness and compactness of the operator K from $L_{p,u}$ to $L_{q,v}$, respectively.

A criterion for the boundedness of the operator K from $L_{p,u}$ to $L_{q,v}$ in the case $1 < p \leqslant q < \infty$ was established in [4] and in the case $1 < q < p < \infty$ in [5].

The validity of (a) follows directly from the results of [4].

From the proofs of the results in [4] we can conclude that $\|K\| \approx T_{p,q}$, since K can be presented in the form

$$K \approx K_1 + K_2 + K_3 + K_4,$$

where K_i are integral operators with non–negative kernels. This implies that $\|K\| \approx \max\{\|K_i\|, i = 1,2,3,4\}$. Therefore, it is obvious that the operator K is compact from $L_{p,u}$ to $L_{q,v}$ if and only if all K_i, $i = 1,2,3,4$, are compact from $L_{p,u}$ to $L_{q,v}$. Thus, the validity of (b) again follows from the results of [4]. □

Proof of Lemma 1. Using Hölder's inequality in the right–hand side of (24), we get

$$|f(t)| \leqslant \left(\int_0^\infty K^{p'}(t,s) u^{-p'}(s) ds \right)^{\frac{1}{p'}} \|u D_\rho^n f\|_p.$$

Hence,

$$J(t) \leqslant \left(\int_0^\infty K^{p'}(t,s) u^{-p'}(s) ds \right)^{\frac{1}{p'}}. \tag{48}$$

Let $0 < \varepsilon < t < N < \infty$ and f_t be the solution of the problem

$$\begin{cases} D_\rho^n f(s) = F_t(s), \\ D_\rho^i f(0) = 0, \ i = 0, \ldots, l-1, \\ D_\rho^i f(\infty) = 0, \ i = l, \ldots, n-1, \end{cases}$$

where F_t has the form:

$$F_t(s) = \chi_{(\varepsilon,N)}(s) u^{-p'}(s) K^{p'-1}(t,s).$$

Then

$$f_t(x) = (-1)^{n-l} \int_\varepsilon^N K(x,s) K^{p'-1}(t,s) u^{-p'}(s) ds, \ \forall x \in I,$$

and

$$\|u D_\rho^n f_t\|_p = \left(\int_\varepsilon^N K^{p'}(t,s) u^{-p'}(s) ds \right)^{\frac{1}{p}}.$$

Therefore,

$$J(t) \geqslant \frac{|f_t(t)|}{\|u D_\rho^n f_t\|_p} = \left(\int_\varepsilon^N K^{p'}(t,s) u^{-p'}(s) ds \right)^{\frac{1}{p'}}.$$

By letting $\varepsilon \to 0$ and $N \to \infty$, we obtain the inequality

$$J(t) \geqslant \left(\int_0^\infty K^{p'}(t,s) u^{-p'}(s) ds \right)^{\frac{1}{p'}},$$

which together with (48) yields (26). □

4. Some spectral characteristics of the operator A

THEOREM 4. *Suppose that u and ρ satisfy the conditions of Theorem 2 with respect to relations between l and m for $p = 2$. Then:*
(a) the operator A is positive defined if and only if $T_{2,2} < \infty$, and, in addition, if $m(A)$ is the greatest lower bound of A, then $m(A) \approx (T_{2,2})^{-2}$;
(b) For the case $T_{2,2} < \infty$ the spectrum of A is discrete if and only if $\lim\limits_{t\to 0} T^i_{2,2}(t) = \lim\limits_{t\to\infty} T^i_{2,2}(t) = 0$.

Proof. The Friedrichs extension saves the greatest lower bound of A_0. Then A is positive defined if and only if A_0 is positive defined and $m(A) = m(A_0)$.

By definition

$$(A_0 f, f) \geqslant m(A_0)\|f\|_2^2, \ \forall f \in D(A_0).$$

Since

$$(A_0 f, f) = \|u D^n_\rho f / v\|_2^2, \ \forall f \in D(A_0),$$

then the positive definedness of A_0 is equivalent to the condition

$$\|u D^n_\rho f\|_2^2 \geqslant m(A_0)\|v f\|_2^2, \ \forall f \in C_0^\infty(I).$$

Due to the denseness of $C_0^\infty(I)$ in $\mathring{W}_2^n$ the last inequality is equivalent to the embedding $\mathring{W}_2^n \hookrightarrow L_{2,v}$, and, in addition, $m(A_0) = \|E\|^{-2}$, where E is the embedding operator $\mathring{W}_2^n \hookrightarrow L_{2,v}$.

Therefore, the validity of (a) of Theorem 4 follows from the validity of (a) of Theorem 3.

By the condition of (b) we have $T_{2,2} < \infty$. Then, according to (a), A is positive defined. Hence, by Rellih's lemma [8] the spectrum of A is discrete if and only if the set $\{f \in C_0^\infty, \ (A_0 f, f) \leqslant 1\}$ is compact in L_2, which is equivalent to the compactness of the embedding $\mathring{W}_2^n \hookrightarrow L_{2,v}$. Therefore, the validity of (b) of Theorem 4 follows from the validity of (b) of Theorem 3. □

THEOREM 5. *Suppose that u and ρ satisfy the conditions of Theorem 2 with respect to the relations between l and m for $p = 2$. Let A be positive defined and assume that the spectrum be discrete. Then the following assertions hold:*

(a) If $\{\lambda_m\}$ is a system of eigenvalues of A, enumerated in non–decreasing order with multiplicity, and $\{\varphi_m\}$ is a complete orthonormal system of the corresponding eigenfunctions in $L_2(I)$, then the equality

$$\sum_{m=1}^{\infty}\lambda_m^{-1}|\varphi_m(t)|^2 = v^2(t)\int_0^{\infty}K^2(t,s)u^{-2}(s)ds, \ \forall t\in I, \tag{49}$$

holds, where $K(t,s)$ is defined by (25).
(b) The operator A^{-1} is nuclear if and only if

$$M=\int_0^{\infty}\int_0^{\infty}K^2(t,s)v^2(t)u^{-2}(s)dsdt<\infty,$$

and, in addition,

$$M=\|A^{-1}\|_{\sigma_1}\equiv\sum_{m=1}^{\infty}\lambda_m^{-1}.$$

Proof. From the conditions on u and ρ of Theorem 1 it easily follows that

$$\int_0^{\infty}K^2(t,s)u^{-2}(s)ds<\infty, \ \forall t\in I,$$

and, according to (24) $f\in \overset{\circ}{W}{}_2^n$ is continuous on I.

Moreover, due to Lemma 1, we have

$$J(t)=\sup_{\overset{\circ}{W}{}_2^n}\frac{|f(t)|}{\|uD_\rho^n f\|_2}=\left(\int_0^{\infty}K^2(t,s)u^{-2}(s)ds\right)^{\frac{1}{2}}. \tag{50}$$

Denote by $\overset{\circ}{\widetilde{W}}{}_2^n$ the Hilbert space, which is obtained as the completion of $C_0^\infty(I)$ with respect to the norm $(Af,f)^{\frac{1}{2}}=\|uD_\rho^n f/v\|_2$. In this space $\{\lambda_m^{-\frac{1}{2}}\varphi_m\}_{m=1}^{\infty}$ is a complete orthonormal system.

Let $f\in\overset{\circ}{\widetilde{W}}{}_2^n$. Then $f(t)=\sum_{m=1}^{\infty}c_m\lambda_m^{-\frac{1}{2}}\varphi_m(t)$ in the sense of $\overset{\circ}{\widetilde{W}}{}_2^n$. Moreover, by the Parseval equality we get

$$\|f\|^2_{\overset{\circ}{\widetilde{W}}{}_2^n}=\sum_{m=1}^{\infty}|c_m|^2. \tag{51}$$

Since $\frac{\varphi_m}{v}$ and $\frac{f}{v}$ belong to $\overset{\circ}{W}{}_2^n$, we conclude that $\varphi_m(t)$ and $f(t)$ are continuous on I. Hence, $f(t)=\lim_{k\to\infty}\sum_{m=1}^{k}c_m\lambda_m^{-\frac{1}{2}}\varphi_m(t)$. Therefore, by also using the Cauchy – Bunyakovskii inequality and (51), we obtain

$$|f(t)|\leqslant\|f\|_{\overset{\circ}{\widetilde{W}}{}_2^n}\left(\sum_{m=1}^{\infty}\lambda_m^{-1}|\varphi_m(t)|^2\right)^{\frac{1}{2}}, \ t\in I. \tag{52}$$

Now, take any arbitrary integer $N > 0$ and $t \in I$ and consider the function $f_{N,t}(x) = \sum_{m=1}^{N} \lambda_m^{-1} \varphi_m(x) \overline{\varphi_m(t)}$. For this function we have

$$\|f_{N,t}\|^2_{\overset{\circ}{\widetilde{W}}{}_2^n} = (Af_{N,t}, f_{N,t}) = \sum_{m=1}^{N} \lambda_m^{-1} |\varphi_m(t)|^2,$$

$$f_{N,t}(t) = \sum_{m=1}^{N} \lambda_m^{-1} |\varphi_m(t)|^2.$$

Hence,

$$\sup_{f \in \overset{\circ}{\widetilde{W}}{}_2^n} \frac{|f(t)|}{\|f\|_{\overset{\circ}{\widetilde{W}}{}_2^n}} \geqslant \sup_{N>0} \frac{|f_{N,t}(t)|}{\|f_{N,t}\|_{\overset{\circ}{\widetilde{W}}{}_2^n}} = \lim_{N \to \infty} \left(\sum_{m=1}^{N} \lambda_m^{-1} |\varphi_m(t)|^2 \right)^{\frac{1}{2}}$$

$$= \left(\sum_{m=1}^{\infty} \lambda_m^{-1} |\varphi_m(t)|^2 \right)^{\frac{1}{2}},$$

which together with (52) yields

$$\sup_{f \in \overset{\circ}{\widetilde{W}}{}_2^n} \frac{|f(t)|}{\|f\|_{\overset{\circ}{\widetilde{W}}{}_2^n}} = \left(\sum_{m=1}^{\infty} \lambda_m^{-1} |\varphi_m(t)|^2 \right)^{\frac{1}{2}}. \tag{53}$$

But, on the other hand

$$\sup_{f \in \overset{\circ}{\widetilde{W}}{}_2^n} \frac{|f(t)|}{\|f\|_{\overset{\circ}{\widetilde{W}}{}_2^n}} = \sup_{f \in \overset{\circ}{\widetilde{W}}{}_2^n} \frac{|f(t)|}{\|uD_\rho^n f / v\|_2} = v(t) \sup_{f \in \overset{\circ}{W}{}_2^n} \frac{|f(t)|}{\|uD_\rho^n f\|_2}.$$

Therefore, by combining (50) and (53) we have (49).

Further the integration of (49) on I and the fact that $\{\varphi_m\}$ is orthonormal in $L_2(I)$ give

$$\sum_{m=1}^{\infty} \lambda_m^{-1} = \int_0^\infty \int_0^\infty K^2(t,s) v^2(t) u^{-2}(s) ds dt = M.$$

The validity of (b) follows and the proof is complete. □

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