Journal of Mathematical Inequalities
Volume 1, Number 3 (2007), 425–436

SOME IMPROVEMENTS OF GRÜSS TYPE INEQUALITY

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(*communicated by A. Čižmešija*)

Abstract. In this paper some inequalities related to Chebyshev's functional are proved. The underlying function spaces are L^p spaces with weight function and exponents which need not be conjugate. Various bounds obtained in this and previous papers are compared.

1. Introduction

Several papers on various improvements of Grüss type inequalities appeared in recent time.

For two measurable functions $f, g : [a,b] \to \mathbf{R}$ denote by $T(f,g)$ Chebyshev's functional:

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx. \tag{1}$$

S. S. Dragomir ([1]) proved the following result:

THEOREM A. *Let $f, g : [a,b] \to \mathbf{R}$ be two mappings, differentiable on (a,b). If $f' \in L_\alpha(a,b)$ and $g' \in L_\beta(a,b)$, with $\alpha > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then we have the inequality*

$$\begin{aligned} |T(f,g)| &\leqslant \frac{1}{2}\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b |x-y|\left|\int_x^y |f'(t)|^\alpha dt\right| dx\,dy\right)^{\frac{1}{\alpha}} \times \\ &\quad \times \left(\frac{1}{(b-a)^2}\int_a^b\int_a^b |x-y|\left|\int_x^y |g'(t)|^\beta dt\right| dx\,dy\right)^{\frac{1}{\beta}} \\ &\leqslant \frac{b-a}{6}\|f'\|_\alpha \|g'\|_\beta. \end{aligned} \tag{2}$$

J. Pečarić and B. Tepeš ([4]) improved this result and showed that

$$|T(f,g)| \leqslant \frac{b-a}{8}\|f'\|_\alpha \|g'\|_\beta. \tag{3}$$

Mathematics subject classification (2000): 26D15.
Key words and phrases: Grüss inequality, Chebyshev's functional.

Paper JMI-01-36

We shall further generalize this improvement by considering weighted version of this Grüss type inequality.

Let $p(x)$ be nonnegative integrable function defined on $[a,b]$. It is well known that Chebyshev's functional

$$T(f,g;p) := \int_a^b p(x)dx \int_a^b f(x)g(x)p(x)dx - \int_a^b f(x)p(x)dx \int_a^b g(x)p(x)dx \quad (4)$$

can be written in the Korkine's form

$$T(f,g;p) = \frac{1}{2}\int_a^b \int_a^b p(x)p(y)[f(x)-f(y)][g(x)-g(y)]dx\,dy \quad (5)$$

The following theorem is proved in [1].

THEOREM B. *Let $f,g:[a,b]\to\mathbf{R}$ be two mappings, differentiable on (a,b) and $p:[a,b]\to[0,\infty)$ is integrable on $[a,b]$. If $f'\in L_\alpha(a,b)$, $g'\in L_\beta(a,b)$ with $\alpha>1$ and $\frac{1}{\alpha}+\frac{1}{\beta}=1$, then we have the inequality*

$$\begin{aligned} |T(f,g;p)| &\leqslant \frac{1}{2}\left(\int_a^b\int_a^b p(x)p(y)|x-y|\left|\int_x^y |f'(t)|^\alpha dt\right| dx\,dy\right)^{\frac{1}{\alpha}} \times \\ &\quad \times \left(\int_a^b\int_a^b p(x)p(y)|x-y|\left|\int_x^y |g'(t)|^\beta dt\right| dx\,dy\right)^{\frac{1}{\beta}} \\ &\leqslant \frac{1}{2}\left(\int_a^b\int_a^b |x-y|p(x)p(y)dx\,dy\right)\|f'\|_\alpha\|g'\|_\beta. \end{aligned} \quad (6)$$

Let us denote

$$J_\alpha(f) := \int_a^b\int_a^b p(x)p(y)|x-y|\left|\int_x^y |f'(t)|^\alpha dt\right| dx\,dy \quad (7)$$

Further improvements can be obtained by analyzing this integral. We have

$$\begin{aligned} J_\alpha(f) &= 2\int_a^b dx \int_a^x (x-y)\int_y^x |f'(t)|^\alpha dt\, p(x)p(y)dy \\ &= 2\int_a^b |f'(t)|^\alpha dt \int_t^b dx \int_a^t (x-y)p(x)p(y)dy \end{aligned} \quad (8)$$

Let us denote

$$F(t) := 2\int_t^b dx \int_a^t (x-y)p(x)p(y)dy \quad (9)$$

This function can be written in the following form:

$$F(t) = 2\int_t^b xp(x)dx \int_a^t p(y)dy - 2\int_a^t xp(x)dx \int_t^b p(y)dy. \quad (10)$$

From (8) and (9) it follows immediately:

$$|J_\alpha(f)| \leqslant \max_{a \leqslant t \leqslant b} F(t) \cdot \|f'\|_\alpha^\alpha. \tag{11}$$

Hence, we may state the following theorem.

THEOREM 1. *Let $f, g : [a,b] \to \mathbf{R}$ be two mappings, absolutely continuous and differentiable on (a,b) and $p : [a,b] \to [0,\infty)$ is integrable on $[a,b]$. If $f' \in L_\alpha(a,b)$, $g' \in L_\beta(a,b)$ with $\alpha > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then we have the inequality*

$$\begin{aligned} |T(f,g;p)| &\leqslant \frac{1}{2}|J_\alpha(f)|^{1/\alpha}|J_\beta(g)|^{1/\beta} \\ &\leqslant \frac{1}{2}\left(\max_{a \leqslant t \leqslant b} F(t)\right) \cdot \|f'\|_\alpha \|g'\|_\beta. \end{aligned} \tag{12}$$

Therefore, we shall search for the maximum of F. From

$$\begin{aligned} F'(t) &= -2\int_a^t (t-x)p(t)p(x)dx + 2\int_t^b (x-t)p(t)p(x)dx \\ &= 2p(t)\left[\int_a^b xp(x)dx - t\int_a^b p(x)dx\right] \end{aligned}$$

it follows that the extremum of this function is obtained at the first moment of weight function:

$$t_0 = \frac{\int_a^b xp(x)dx}{\int_a^b p(x)dx}. \tag{13}$$

From the sign of the first derivative it is obvious that this extremum is the local maximum.

2. Symmetric case

Let us suppose that the weight function is symmetric, i.e.

$$p(x) = p(a+b-x), \qquad \forall x \in [a,b]. \tag{14}$$

Then we have

$$\begin{aligned} \int_a^b xp(x)dx &= \int_a^b (a+b-u)p(a+b-u)du \\ &= (a+b)\int_a^b p(x)dx - \int_a^b xp(x)dx. \end{aligned}$$

Hence, it follows that

$$\int_a^b xp(x)dx = \frac{a+b}{2}\int_a^b p(x)dx$$

and therefore, the abscissa (13) of the point of maximum of function $F(t)$ is

$$t_0 = \frac{a+b}{2}. \tag{15}$$

In this case, using (10), it is easy to obtain:

$$F(t_0) = \left(\int_a^b p(x)dx\right)\left(\int_{\frac{a+b}{2}}^b xp(x)dx - \int_a^{\frac{a+b}{2}} xp(x)dx\right) \tag{16}$$

In the special case $p(x) = 1/(b-a)$, it follows from (16):

$$F(t_0) = \frac{1}{b-a}\left(\left[\frac{b^2}{2} - \frac{(a+b)^2}{8}\right] - \left[\frac{(a+b)^2}{8} - \frac{a^2}{2}\right]\right) = \frac{b-a}{4},$$

and (3) follows.

3. General case

We shall improve Theorem 1 by removing the assumption that α and β are conjugate exponents.

In this section we assume only $\alpha > 1$, $\beta > 1$ and $\gamma > 1$. Let us denote by α', β' and γ' the corresponding conjugate exponents. From Hölder's inequality it follows

$$\begin{aligned} |f(x) - f(y)| &\leqslant |x-y|^{1/\alpha'} \left|\int_y^x |f'(t)|^\alpha dt\right|^{1/\alpha}, \\ |g(x) - g(y)| &\leqslant |x-y|^{1/\beta'} \left|\int_y^x |g'(t)|^\beta dt\right|^{1/\beta}. \end{aligned} \tag{17}$$

Therefore, we have the following estimation for Chebyshev's functional:

$$\begin{aligned} |T(f,g;p)| &\leqslant \frac{1}{2}\int_a^b \int_a^b p(x)p(y)|f(x) - f(y)||g(x) - g(y)|dx\,dy \\ &\leqslant \frac{1}{2}\int_a^b \int_a^b p(x)p(y)|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}} \left|\int_y^x |f'(t)|^\alpha dt\right|^{\frac{1}{\alpha}} \left|\int_y^x |g'(t)|^\beta dt\right|^{\frac{1}{\beta}} dx\,dy \\ &\leqslant \frac{1}{2}\left(\int_a^b \int_a^b p(x)p(y)|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}} \left|\int_y^x |f'(t)|^\alpha dt\right|^{\frac{\gamma}{\alpha}} dx\,dy\right)^{\frac{1}{\gamma}} \times \\ &\qquad \times \left(\int_a^b \int_a^b p(x)p(y)|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}} \left|\int_y^x |g'(t)|^\beta dt\right|^{\frac{\gamma'}{\beta}} dx\,dy\right)^{\frac{1}{\gamma'}} \\ &\leqslant \frac{1}{2}\left(\int_a^b \int_a^b p(x)p(y)|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}} dx\,dy\right) \|f'\|_\alpha \|g'\|_\beta. \end{aligned} \tag{18}$$

In the special case $p(x)=\frac{1}{b-a}$, we have:

$$|T(f,g)| \leqslant \frac{(b-a)^{\frac{1}{\alpha'}+\frac{1}{\beta'}}}{\left(\frac{1}{\alpha'}+\frac{1}{\beta'}+1\right)\left(\frac{1}{\alpha'}+\frac{1}{\beta'}+2\right)}\|f'\|_\alpha\|g'\|_\beta$$

$$= \frac{(b-a)^{2-\frac{1}{\alpha}-\frac{1}{\beta}}}{\left(3-\frac{1}{\alpha}-\frac{1}{\beta}\right)\left(4-\frac{1}{\alpha}-\frac{1}{\beta}\right)}\|f'\|_\alpha\|g'\|_\beta \tag{19}$$

From (18) we can obtain, by taking an appropriate limit, some estimations which are not covered earlier. Let us note only the following three:

COROLLARY 1. *In the case* $\alpha=\beta=1$ *we have:*

$$|T(f,g;p)| \leqslant \frac{1}{2}\|f'\|_1\|g'\|_1\left(\int_a^b p(x)dx\right)^2. \tag{20}$$

If $\alpha=\beta=\infty$, *then*

$$|T(f,g;p)| \leqslant \frac{1}{2}\|f'\|_\infty\|g'\|_\infty\left(\int_a^b\int_a^b p(x)p(y)(x-y)^2dx\,dy\right) \tag{21}$$

and, in the case $\alpha=1$, $\beta=\infty$ *we have:*

$$|T(f,g;p)| \leqslant \frac{1}{2}\|f'\|_1\|g'\|_\infty\int_a^b\int_a^b p(x)p(y)|x-y|\,dx\,dy. \tag{22}$$

COROLLARY 2. *Let* $p(x)=\frac{1}{b-a}$. *Then in the case of* $\alpha=\beta=1$ *we have:*

$$T(f,g) \leqslant \frac{1}{2}||f'||_1||g'||_1. \tag{23}$$

If $\alpha=\beta=\infty$, *then*

$$T(f,g) \leqslant \frac{1}{12}(b-a)^2||f'||_\infty||g'||_\infty, \tag{24}$$

and, in the case $\alpha=1$, $\beta=\infty$, *we have:*

$$T(f,g) \leqslant \frac{1}{6}(b-a)||f'||_1||g'||_\infty. \tag{25}$$

Let us note that the bound in (18) is not so accurate as in (12). We can improve it in the case when $\frac{1}{\alpha}+\frac{1}{\beta}<1$. Assume first that $p(x)=\frac{1}{b-a}$. We have:

THEOREM 2. *Let* $f, g : [a,b] \to \mathbf{R}$ *be two mappings, absolutely continuous and differentiable on* (a,b). *If* $f' \in L_\alpha(a,b)$ *and* $g' \in L_\beta(a,b)$, *with* $\alpha > 1$, $\beta > 1$ *and* $\frac{1}{\alpha} + \frac{1}{\beta} \leqslant 1$, *then the following inequality holds:*

$$|T(f,g)| \leqslant \frac{(b-a)^{2-\frac{1}{\alpha}-\frac{1}{\beta}}}{\left(3-\frac{1}{\alpha}-\frac{1}{\beta}\right)\left(4-\frac{1}{\alpha}-\frac{1}{\beta}\right)} \left[1-2^{\frac{1}{\alpha}+\frac{1}{\beta}-3}\right]^{\frac{1}{\alpha}+\frac{1}{\beta}} ||f'||_\alpha ||g'||_\beta. \tag{26}$$

Proof. Let us denote by α', β' and γ' the conjugate values of α, β and γ. From (18) we have:

$$|T(f,g)| \leqslant \frac{1}{2(b-a)^2} \left[\int_a^b \int_a^b |x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}} \left(\int_y^x |f'(t)|^\alpha \, dt\right)^{\frac{\gamma}{\alpha}} dxdy\right]^{\frac{1}{\gamma}} \times$$
$$\times \left[\int_a^b \int_a^b |x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}} \left(\int_y^x |g'(t)|^\beta \, dt\right)^{\frac{\gamma'}{\beta}} dxdy\right]^{\frac{1}{\gamma'}}. \tag{27}$$

If γ, γ' are conjugate values such that $1 \leqslant \gamma \leqslant \alpha$, $1 \leqslant \gamma' \leqslant \beta$ then $\frac{1}{\alpha} + \frac{1}{\beta} \leqslant \frac{1}{\gamma} + \frac{1}{\gamma'} = 1$. Thus, the condition $\frac{1}{\alpha} + \frac{1}{\beta} \leqslant 1$ can be realized by setting such values for γ, γ' which satisfy $1 - \frac{\gamma}{\alpha} \geqslant 0$, $1 - \frac{\gamma'}{\beta} \geqslant 0$. In this case, let us consider:

$$\begin{aligned}
I &= \int_a^b \int_a^b |x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}} \left(\int_y^x |f'(t)|^\alpha \, dt\right)^{\frac{\gamma}{\alpha}} dxdy \\
&= \int_a^b \int_a^b \left\{|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}}\right\}^{1-\frac{\gamma}{\alpha}} \left\{|x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}} \int_y^x |f'(t)|^\alpha dt\right\}^{\frac{\gamma}{\alpha}} dxdy \\
&\leqslant \left[\int_a^b \int_a^b |x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}} dxdy\right]^{1-\frac{\gamma}{\alpha}} \left[\int_a^b \int_a^b |x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}} \int_y^x |f'(t)|^\alpha dt \cdot dxdy\right]^{\frac{\gamma}{\alpha}} \\
&= J_1^{1-\frac{\gamma}{\alpha}} J_2^{\frac{\gamma}{\alpha}}
\end{aligned} \tag{28}$$

where we denote

$$\begin{aligned}
J_1 &= \int_a^b \int_a^b |x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}} dxdy, \\
J_2 &= \int_a^b \int_a^b |x-y|^{\frac{1}{\alpha'}+\frac{1}{\beta'}} \int_y^x |f'(t)|^\alpha dt \cdot dxdy.
\end{aligned}$$

We have for J_1 and J_2:

$$J_1 = 2\int_a^b dx \int_a^x (x-y)^{\frac{1}{\alpha'}+\frac{1}{\beta'}} dy = \frac{2(b-a)^{\frac{1}{\alpha'}+\frac{1}{\beta'}+2}}{\left(\frac{1}{\alpha'}+\frac{1}{\beta'}+1\right)\left(\frac{1}{\alpha'}+\frac{1}{\beta'}+2\right)}, \tag{29}$$

$$\begin{aligned} J_2 &= 2\int_a^b dx \int_a^x (x-y)^{\frac{1}{\alpha'}+\frac{1}{\beta'}} dy \int_y^x |f'(t)|^\alpha dt \\ &= 2\int_a^b |f'(t)|^\alpha dt \int_t^b dx \int_a^t (x-y)^{\frac{1}{\alpha'}+\frac{1}{\beta'}} dy \\ &= \frac{2}{\left(\frac{1}{\alpha'}+\frac{1}{\beta'}+1\right)\left(\frac{1}{\alpha'}+\frac{1}{\beta'}+2\right)} \times \\ &\quad \times \int_a^b |f'(t)|^\alpha \left[(b-a)^{\frac{1}{\alpha'}+\frac{1}{\beta'}+2} - (t-a)^{\frac{1}{\alpha'}+\frac{1}{\beta'}+2} - (b-t)^{\frac{1}{\alpha'}+\frac{1}{\beta'}+2}\right] dt. \end{aligned}$$

By computing extremal values of the function $\varphi(t) = (b-a)^r - (t-a)^r - (b-t)^r$, $t \in [a,b]$, $r = \frac{1}{\alpha'} + \frac{1}{\beta'} + 2$ we obtain:

$$\max_{a \leqslant t \leqslant b} \varphi = \varphi\left(t_0 = \frac{a+b}{2}\right) = (b-a)^r - \frac{(b-a)^r}{2^{r-1}} = (b-a)^r[1-2^{1-r}]$$

and so we have:

$$J_2 \leqslant \frac{2(b-a)^{\frac{1}{\alpha'}+\frac{1}{\beta'}+2}}{\left(\frac{1}{\alpha'}+\frac{1}{\beta'}+1\right)\left(\frac{1}{\alpha'}+\frac{1}{\beta'}+2\right)} \left[1-2^{-\frac{1}{\alpha'}-\frac{1}{\beta'}-1}\right] ||f'||_\alpha^\alpha. \tag{30}$$

Thus we obtain the following estimate for I:

$$\begin{aligned} I &\leqslant J_1^{1-\frac{\gamma}{\alpha}} J_2^{\frac{\gamma}{\alpha}} \\ &\leqslant \frac{2(b-a)^{\frac{1}{\alpha'}+\frac{1}{\beta'}+2}}{\left(\frac{1}{\alpha'}+\frac{1}{\beta'}+1\right)\left(\frac{1}{\alpha'}+\frac{1}{\beta'}+2\right)} \left[1-2^{-\frac{1}{\alpha'}-\frac{1}{\beta'}-1}\right]^{\frac{\gamma}{\alpha}} ||f'||_\alpha^\gamma. \end{aligned}$$

Using (27) we finally obtain:

$$\begin{aligned} |T(f,g)| \leqslant \frac{1}{2(b-a)^2} &\left\{ \frac{2(b-a)^{\frac{1}{\alpha'}+\frac{1}{\beta'}+2}}{\left(\frac{1}{\alpha'}+\frac{1}{\beta'}+1\right)\left(\frac{1}{\alpha'}+\frac{1}{\beta'}+2\right)} \left[1-2^{-\frac{1}{\alpha'}-\frac{1}{\beta'}-1}\right]^{\frac{\gamma}{\alpha}} ||f'||_\alpha^\gamma \right\}^{\frac{1}{\gamma}} \times \\ &\times \left\{ \frac{2(b-a)^{\frac{1}{\alpha'}+\frac{1}{\beta'}+2}}{\left(\frac{1}{\alpha'}+\frac{1}{\beta'}+1\right)\left(\frac{1}{\alpha'}+\frac{1}{\beta'}+2\right)} \left[1-2^{-\frac{1}{\alpha'}-\frac{1}{\beta'}-1}\right]^{\frac{\gamma'}{\beta}} ||g'||_\beta^{\gamma'} \right\}^{\frac{1}{\gamma'}} \end{aligned}$$

$$\leqslant \frac{1}{2(b-a)^2}\cdot\frac{2(b-a)^{\frac{1}{\alpha'}+\frac{1}{\beta'}+2}}{\left(\frac{1}{\alpha'}+\frac{1}{\beta'}+1\right)\left(\frac{1}{\alpha'}+\frac{1}{\beta'}+2\right)}\left[1-2^{-\frac{1}{\alpha'}-\frac{1}{\beta'}-1}\right]^{\frac{1}{\alpha}+\frac{1}{\beta}}||f'||_\alpha||g'||_\beta$$

wherefrom it follows (26).

REMARK 1. For comparison, the estimate (26) improves (19), since $\left[1-2^{\frac{1}{\alpha}+\frac{1}{\beta}-3}\right]^{\frac{1}{\alpha}+\frac{1}{\beta}} < 1$ for $0 < \frac{1}{\alpha}+\frac{1}{\beta} \leqslant 1$. In the case of $\frac{1}{\alpha}+\frac{1}{\beta} = 0$, i.e. $\alpha=\beta=\infty$, (19) and (26) coincide.

A better estimation is described in the following theorem.

THEOREM 3. *Let $f,g : [a,b] \to \mathbf{R}$ be two mapings, absolutely continuous and differentiable on (a,b) and $p : [a,b] \to [0,\infty)$ is integrable on $[a,b]$. If $f' \in L_\alpha(a,b)$, $g' \in L_\beta(a,b)$ with $\alpha > 1$, $\beta > 1$ and $\frac{1}{\alpha}+\frac{1}{\beta} \leqslant 1$, then we have the inequality:*

$$|T(f,g;p)| \leqslant \frac{1}{2}M\cdot\max F(t)^{\frac{1}{\alpha}+\frac{1}{\beta}}||f'||_\alpha||g'||_\beta \tag{31}$$

where

$$M = \left[\int_a^b\int_a^b p(x)p(y)(x-y)^2dxdy\right]^{1-\frac{1}{\alpha}-\frac{1}{\beta}},$$

$$F(t) = 2\int_t^b p(x)\,dx\int_a^t(x-y)p(y)\,dy. \tag{32}$$

Proof. Denote

$$\frac{1}{\delta} := 1-\frac{1}{\alpha}-\frac{1}{\beta}. \tag{33}$$

Then $\frac{1}{\alpha'}+\frac{1}{\beta'} = 1-\frac{1}{\alpha}+1-\frac{1}{\beta} = 1+\frac{1}{\delta}$. Using (17), it is easy to obtain the following estimate:

$$\begin{aligned}&|f(x)-f(y)|\cdot|g(x)-g(y)|\\&\quad\leqslant |x-y|^{\frac{1}{\alpha'}}\left(\int_y^x|f'(t)|^\alpha dt\right)^{\frac{1}{\alpha}}|x-y|^{\frac{1}{\beta'}}\left(\int_y^x|g'(t)|^\beta dt\right)^{\frac{1}{\beta}}\\&\quad= |x-y|\left(\int_y^x dt\right)^{\frac{1}{\delta}}\left(\int_y^x|f'(t)|^\alpha dt\right)^{\frac{1}{\alpha}}\left(\int_y^x|g'(t)|^\beta dt\right)^{\frac{1}{\beta}}.\end{aligned} \tag{34}$$

Applying again Hölder's inequality, we obtain as before:

$$
\begin{aligned}
|T(f,g;p)| &\leqslant \frac{1}{2}\int_a^b\int_a^b p(x)p(y)|x-y|\left(\int_y^x dt\right)^{\frac{1}{\delta}}\left(\int_y^x |f'(t)|^{\alpha}dt\right)^{\frac{1}{\alpha}}\times\\
&\quad\times\left(\int_y^x |g'(t)|^{\beta}dt\right)^{\frac{1}{\beta}} dxdy\\
&= \frac{1}{2}\int_a^b\int_a^b\left(\int_y^x p(x)p(y)|x-y|\,dt\right)^{\frac{1}{\delta}}\left(p(x)p(y)|x-y|\int_y^x |f'(t)|^{\alpha}dt\right)^{\frac{1}{\alpha}}\times\\
&\quad\times\left(p(x)p(y)|x-y|\int_y^x |g'(t)|^{\beta}dt\right)^{\frac{1}{\beta}} dxdy\\
&\leqslant \frac{1}{2}\left(\int_a^b\int_a^b p(x)p(y)(x-y)^2dxdy\right)^{\frac{1}{\delta}}(J_{\alpha}(f))^{\frac{1}{\alpha}}\left(J_{\beta}(g)\right)^{\frac{1}{\beta}}.
\end{aligned}
$$

Using (11) we finally obtain:

$$
\begin{aligned}
|T(f,g;p)| &\leqslant \frac{1}{2}M\left[\max_{a\leqslant t\leqslant b}F(t)\right]^{\frac{1}{\alpha}}||f'||_{\alpha}\left[\max_{a\leqslant t\leqslant b}F(t)\right]^{\frac{1}{\beta}}||g'||_{\beta}\\
&= \frac{1}{2}M\left[\max_{a\leqslant t\leqslant b}F(t)^{\frac{1}{\alpha}+\frac{1}{\beta}}\right]||f'||_{\alpha}||g'||_{\beta}.
\end{aligned}
$$

In the case $p(x)=\frac{1}{b-a}$ we have:

$$
\max_{a\leqslant t\leqslant b}F(t)=\frac{1}{4}(b-a),
$$
$$
M=\left[\frac{1}{6}(b-a)^2\right]^{1-\frac{1}{\alpha}-\frac{1}{\beta}},
$$

$$
\begin{aligned}
|T(f,g)| &\leqslant \frac{1}{2}\cdot 6^{-1+\frac{1}{\alpha}+\frac{1}{\beta}}(b-a)^{2-\frac{2}{\alpha}-\frac{2}{\beta}}\cdot 4^{-\frac{1}{\alpha}-\frac{1}{\beta}}(b-a)^{\frac{1}{\alpha}+\frac{1}{\beta}}||f'||_{\alpha}||g'||_{\beta}\\
&= \frac{1}{12}\left(\frac{3}{2}\right)^{\frac{1}{\alpha}+\frac{1}{\beta}}(b-a)^{2-\frac{1}{\alpha}-\frac{1}{\beta}}||f'||_{\alpha}||g'||_{\beta}.
\end{aligned}
$$

Thus we showed:

COROLLARY 3. *For* $\alpha>1$, $\beta>1$, $\frac{1}{\alpha}+\frac{1}{\beta}\leqslant 1$ *we have:*

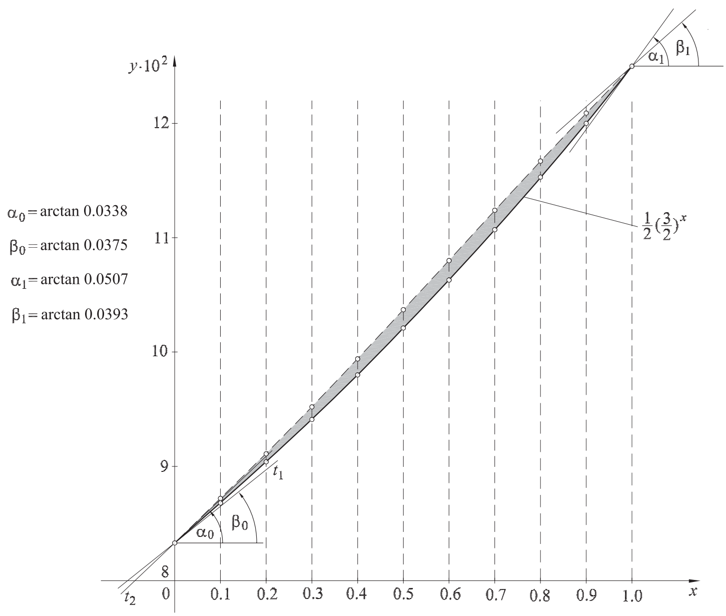
$$
|T(f,g)|\leqslant\frac{1}{12}\left(\frac{3}{2}\right)^{\frac{1}{\alpha}+\frac{1}{\beta}}(b-a)^{2-\frac{1}{\alpha}-\frac{1}{\beta}}||f'||_{\alpha}||g'||_{\beta}. \tag{35}
$$

REMARK 2. By elementary calculus we can show that $\frac{1}{12}\left(\frac{3}{2}\right)^x < \frac{1}{(3-x)(4-x)}$ for $x \in (0,1]$, thus the estimate (35) improves (19). In the case $x = \frac{1}{\alpha} + \frac{1}{\beta} = 0$, i.e. $\alpha = \beta = \infty$, (19) and (35) coincide.

REMARK 3. The estimate (35) is better than (26), since it holds that:

$$\frac{1}{12}\left(\frac{3}{2}\right)^x < \frac{(1-2^{x-3})^x}{(3-x)(4-x)}, \qquad x \in (0,1).$$

On the following picture graphs of functions appearing on the left-hand and the right-hand side of this inequality are plotted.



In cases $\alpha = \beta = \infty$ and $\alpha = 1$, $\beta = \infty$ (26) and (35) coincide.

4. Comparison with previous results

We shall compare estimate (3) from [4] with the following one which was proved much earlier in [3]:

$$|T(f,g)| \leqslant \frac{b-a}{4}\left[\frac{2^\alpha - 1}{\alpha(\alpha+1)}\right]^{1/\alpha}\left[\frac{2^\beta - 1}{\beta(\beta+1)}\right]^{1/\beta}\|f'\|_\alpha\|g'\|_\beta. \tag{36}$$

THEOREM 4. *It holds*

$$\frac{1}{2} \leqslant \left[\frac{2^\alpha - 1}{\alpha(\alpha+1)}\right]^{1/\alpha} \left[\frac{2^\beta - 1}{\beta(\beta+1)}\right]^{1/\beta}.$$

Therefore, inequality (3) *is stronger than* (36).

Proof. Let us calculate first the following integral

$$I_\alpha := \int_1^2 dx \int_0^1 (x-y)^{\alpha-1} dy = \frac{1}{\alpha}\int_1^2 [x^\alpha - (x-1)^\alpha] dx = 2 \cdot \frac{2^\alpha - 1}{\alpha(\alpha+1)}.$$

In particular, for $\alpha = 2$ it follows that

$$I_2 = \int_1^2 dx \int_0^1 (x-y) dy = 1.$$

By using Hölder's inequality, we obtain

$$\begin{aligned} 1 = I_2 &= \int_1^2 dx \int_0^1 (x-y) dy \\ &= \int_1^2 \int_0^1 (x-y)(x-y) \frac{1}{x-y} dy\, dx \\ &\leqslant \left(\int_1^2 \int_0^1 (x-y)^\alpha \frac{1}{x-y} dy\, dx\right)^{1/\alpha} \left(\int_1^2 \int_0^1 (x-y)^\beta \frac{1}{x-y} dy\, dx\right)^{1/\beta} \\ &= (I_\alpha)^{1/\alpha} (I_\beta)^{1/\beta} \\ &= 2 \cdot \left[\frac{2^\alpha - 1}{\alpha(\alpha+1)}\right]^{1/\alpha} \left[\frac{2^\beta - 1}{\beta(\beta+1)}\right]^{1/\beta}, \end{aligned}$$

which proves the theorem.

Acknowledgement. We wish to thanks to the referee for his observations and suggestions.

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