Journal of Mathematical Inequalities
Volume 1, Number 4 (2007), 449–471

# MEANINGFUL INEQUALITIES

GRAHAME BENNETT

(*communicated by P. Mercer*)

*Abstract.* The fundamental Theorem on Means serves to “explain” many recently published inequalities, and even to suggest new versions. Our aim here is to explore the possibilities.

## 1. Introduction

The following elementary inequalities have attracted a great deal of attention (see references).

THEOREM 1. (Minc/Sathre [22], Theorem 1).

$$1 < \frac{(n+1)!^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} < \frac{n+1}{n} \qquad (n = 1, 2, \dots). \tag{1}$$

THEOREM 2. (Martins [20], Theorem 1). *If $r > 0$, then the sequence*

$$\frac{1^r + 2^r + \cdots + n^r}{n(n!)^{\frac{r}{n}}} \qquad (n = 1, 2, \dots) \tag{2}$$

*increases with $n$.*

THEOREM 3. ([3], Lemma 8). *If $0 \leqslant \alpha \leqslant 1$, then the sequence*

$$\frac{n(n+1)^{\alpha}}{1^{\alpha} + 2^{\alpha} + \cdots + n^{\alpha}} \qquad (n = 1, 2, \dots) \tag{3}$$

*increases with $n$. If $\alpha \leqslant 0$ or $\alpha \geqslant 1$, the sequence decreases.*

THEOREM 4. (Alzer [1]). *If $r > 0$, then*

$$\frac{n}{n+1} < \left( \frac{(n+1)\sum_{i=1}^{n} i^r}{n \sum_{i=1}^{n+1} i^r} \right)^{\frac{1}{r}} \qquad (n = 1, 2, \dots). \tag{4}$$

*Mathematics subject classification* (2000): 26D15.
*Key words and phrases*: Alzer’s inequality, golden mean, Martins’ inequality.


Paper JMI-01-39

We have listed Theorems 1–4 in the order of their discovery, and in the precise form in which they were published. Even then, it is obvious that they can all be rephrased as *monotonicity results*, but this point has been missed by several authors. A vexing consequence is that the simple assertion,

$$x_n \leqslant x_{n+1} \qquad (n = 1, 2, \dots) \tag{5}$$

is replaced by the more combersome version,

$$x_n \leqslant x_{n+m} \qquad (m, n = 1, 2, \dots),$$

in many of the references.

Our main observation is that Theorems 1–4 are predictable results. There are no surprises at all here, and, indeed, much more general assertions are to be expected. Inequalities (2) and (4), in particular, ought to be valid even when $r < 0$. These insights come courtesy of the fundamental Theorem on Means ([17], Theorem 16), according to which the $p^{th}$ *power means,*

$$L_n^p(\boldsymbol{a}) = \left( \frac{a_1^p + a_2^p + \cdots a_n^p}{n} \right)^{\frac{1}{p}}, \tag{6}$$

increase with $p$. Here $\boldsymbol{a}$ is a sequence of positive terms, $n$ is a positive integer and $p$ is an extended real number. When $p = 0$, (6) is to be interpreted, via continuity, as the *geometric mean,* $(a_1 a_2 \dots a_n)^{\frac{1}{n}}$. Similarly, when $p = -\infty$, (6) becomes $\min\{a_1, a_2, \dots, a_n\}$, and, when $p = \infty$, $\max\{a_1, a_2, \dots, a_n\}$.

Theorems 1–4, in effect, turn the fundamental theorem on its head by asserting monotonicity in $n$ rather than in $p$. To see this, let us suppose that $p > q$: then, certainly,

$$x_n := \frac{L_n^p(\boldsymbol{a})}{L_n^q(\boldsymbol{a})} \geqslant 1 \qquad (n = 1, 2, \dots). \tag{7}$$

But the first term, $x_1$, *is* 1, so that the sequence $\boldsymbol{x}$, if monotonic, must increase with $n$. This intuition serves to "explain" Theorems 1–4, and even to suggest new results.

The right-hand side of Theorem 1, for example, may be rephrased succinctly as

$$\frac{n}{(n!)^{\frac{1}{n}}} \nearrow \tag{8}$$

and the explanation is

$$\frac{L^\infty}{L^0} \nearrow. \tag{9}$$

[We are dealing here with the sequence $\boldsymbol{a} = (1, 2, 3, \dots)$, the numerator in (8) is $L_n^\infty(\boldsymbol{a})$; the denominator, $L_n^0(\boldsymbol{a})$.] The left-hand side of Theorem 1 has a similar interpretation

$$\frac{(n!)^{\frac{1}{n}}}{1} \nearrow \quad \text{viz.} \quad \frac{L^0}{L^{-\infty}} \nearrow. \tag{10}$$

By the same token, Theorem 2 is of the form

$$\frac{L^r}{L^0} \nearrow \qquad (r > 0) \tag{11}$$

and it is reasonable to expect another result corresponding to

$$\frac{L^r}{L^0} \searrow \qquad (r < 0), \tag{12}$$

i.e. Martins' inequality, (2), ought to be valid as stated *for all* $r \in \mathbb{R}$.

Evaluating the $L^1$–means, we see that Theorem 3 admits the interpretations

$$\frac{L^1}{L^\alpha} \nearrow \quad (\alpha \leqslant 1) \quad \text{and} \quad \frac{L^1}{L^\alpha} \searrow \quad (\alpha \geqslant 1), \tag{13}$$

and, furthermore, we expect Theorem 4,

$$\frac{L^\infty}{L^r} \nearrow \qquad (r > 0), \tag{14}$$

to be valid when $r < 0$ as well.

We shall refer to Theorems 1–4, and their extensions described below, as *meaningful inequalities,* because they give exactly the monotonicities that we would expect from the Theorem on Means. It is important to emphasize that the fundamental theorem offers no help at all with the proofs. By dictating the directions of whatever monotonicities might be present, however, it provides a powerful tool for suggesting new inequalities. Moreover, since the Theorem on Means is valid for arbitrary sequences (of positive terms), there are many possibilities to explore.

Abstract versions of Theorems 1–4 are discussed, respectively, in sections 2.–5. Our analysis provides sharper results than those previously published, but it fails to be definitive. It is only in section 6., where we discuss integral analogues of Theorems 1–4, that some of our results can pretend to be complete.

## 2. Minc / Sathre

With $(1, 2, 3, \dots)$ replaced by any *increasing* sequence of positive terms, $\boldsymbol{a} = (a_1, a_2, \dots)$, the Theorem on Means suggests the following generalizations of Theorem 1.

$$(a_1 a_2 \dots a_n)^{\frac{1}{n}} \nearrow \tag{15}$$

and

$$\frac{a_n}{(a_1 a_2 \dots a_n)^{\frac{1}{n}}} \nearrow . \tag{16}$$

The first suggestion $\left(\frac{L^0}{L^{-\infty}} \nearrow\right)$ is a triviality since the geometric means of a (strictly) increasing sequence are themselves (strictly) increasing. (In fairness to Minc and Sathre, it must be pointed out that they, too, recognize Theorem 1 as a triviality. Their paper [22] contains other inequalities of far greater significance, but it is Theorem 1 that has come to be known as *the* Minc/Sathre inequality.)

The second suggestion, (16), is much more interesting because it fails to be true without some additional restriction on the sequence $\boldsymbol{a}$. Qi and Luo [30] gave the first extension of Theorem 1 by showing that (16) is valid for sequences of the form

$$\boldsymbol{a} = (n, n+1, n+2, \dots), \tag{17}$$

$n$ being a positive integer. Their proof is based upon Stirling's formula: a simpler approach to a more general result is given by

PROPOSITION 1. *An arithmetic progression of positive terms*

$$\boldsymbol{a} = (a, a+d, a+2d, \dots) \qquad (a > 0,\ d \geqslant 0) \tag{18}$$

*satisfies the Minc/Sathre inequality, (16), if* $d \leqslant a$.

*Proof.* Since $d \leqslant a$, we have, for $n = 1, 2, \dots,$

$$\frac{n}{n+1}\left(1 + \frac{d}{a+(n-1)d}\right) + \frac{1}{n+1}(1) \leqslant 1 + \frac{d}{a+nd}. \tag{19}$$

Replacing the (weighted) arithmetic mean on the left-hand side of (19) by the corresponding geometric mean, we deduce that

$$\left(1 + \frac{d}{a+(n-1)d}\right)^{\frac{n}{n+1}} \leqslant 1 + \frac{d}{a+nd}, \tag{20}$$

in other words,

$$\left(\frac{a_{n+1}}{a_n}\right)^{n} \leqslant \left(\frac{a_{n+2}}{a_{n+1}}\right)^{n+1}. \tag{21}$$

The Minc/Sathre inequality, (16), now follows from the representation

$$\frac{a_n}{(a_1 a_2 \dots a_n)^{\frac{1}{n}}} = \left(1\left(\frac{a_2}{a_1}\right)\left(\frac{a_3}{a_2}\right)^2 \cdots \left(\frac{a_n}{a_{n-1}}\right)^{n-1}\right)^{\frac{1}{n}}, \tag{22}$$

which expresses the terms on the left as the geometric means of the increasing sequence on the right. □

Proposition 1 suggests the following

PROBLEM 1. Which arithmetic progressions satisfy the Minc/Sathre inequality (16)?

It can be shown that there is a number, $\delta_{MS}$, which we call the *Minc/Sathre index,* such that (16) is satisfied whenever $d \leqslant \delta_{MS} a$, and fails whenever $d > \delta_{MS} a$. Machine calculations suggest that

$$2 \leqslant \delta_{MS} < 3, \tag{23}$$

but the exact value is unknown. (The lower bound is confirmed in section 5..)

Theorems 2, 3 and 4 are all stronger than the Minc/Sathre inequality. Taking $r^{th}$ roots in (2), then making $r \to \infty$, or making $r \to 0^+$ in (4), we recover the right-hand side of (1), at any rate, with $<$ replaced by $\leqslant$. Theorems 2 and 3 even provide an improvement,

$$\frac{(n+1)!^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} \leqslant \frac{n+2}{n+1} \qquad \left(< \frac{n+1}{n}\right), \tag{24}$$

as is seen by setting $r = 1$ in (2), or by taking $\alpha^{th}$ roots in (3), then making $\alpha \to 0$.

Martins and Alzer both use inequality (24) in their work. Martins' proof ([20], Lemma 1) is short and simple; Alzer's ([1], Lemma 1) is short as well, but not so simple, being based upon properties of the gamma and digamma functions. The simplest proof of all is given in

PROPOSITION 2. *The sequence*

$$\frac{n+1}{(n!)^{\frac{1}{n}}} \qquad (n=1,2,\dots) \tag{25}$$

*increases strictly with* $n$.

*Proof.*

$$\frac{n+1}{(n!)^{\frac{1}{n}}} = \left(\left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\cdots\left(\frac{n+1}{n}\right)^n\right)^{\frac{1}{n}}.$$

□

An interesting improvement to the left side of Theorem 1, namely

$$\sqrt{\frac{n+1}{n}} \leqslant \frac{(n+1)!^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}}, \tag{26}$$

has been discovered by Qi. He studies arithmetic progressions, (18), for which

$$\frac{\sqrt{a_n}}{(a_1a_2\dots a_n)^{\frac{1}{n}}} \searrow, \tag{27}$$

(26) being the special case, $\boldsymbol{a}=(1,2,3\dots)$. Qi [25] shows that (27) holds whenever

$$a \geqslant 1 \quad \text{and} \quad d=1, \tag{28}$$

and he does the same thing again in [26]. The restriction to unit differences is removed in a joint paper with Guo [15]. Their extension, however, is illusory since the inequality underlying (27),

$$\sqrt{\frac{a_{n+1}}{a_n}} \leqslant \frac{(a_1a_2\dots a_{n+1})^{\frac{1}{n+1}}}{(a_1a_2\dots a_n)^{\frac{1}{n}}}, \tag{29}$$

is homogeneous in the $a_k$ s. [To deduce the case $a \geqslant d$, covered in [15], from (28), simply divide all terms by $d$.]

Inequality (27), in fact, is valid for every arithmetic progression (of positive terms). This observation is a consequence of the following result since any arithmetic progression is log-concave.

PROPOSITION 3. *If the positive sequence* $\boldsymbol{a}$ *is log-concave, i.e.,*

$$a_na_{n+2} \leqslant a_{n+1}^2 \qquad (n=1,2,\dots), \tag{30}$$

*then*

$$\frac{\sqrt{a_n}}{(a_1a_2\dots a_n)^{\frac{1}{n}}} \searrow. \tag{31}$$

*Proof.* The identity

$$\frac{a_n}{(a_1a_2\dots a_n)^{\frac{2}{n}}} = \left(\frac{a_1^{-1}}{1}\ \frac{a_2^0}{a_1^1}\ \frac{a_3^1}{a_2^2}\ \cdots\ \frac{a_n^{n-2}}{a_{n-1}^{n-1}}\right)^{\frac{1}{n}}$$

expresses the terms on the left as the geometric means of the sequence

$$\left(\frac{a_1^{-1}}{1}, \frac{a_2^0}{a_1^1}, \frac{a_3^1}{a_2^2}, \frac{a_4^2}{a_3^3}, \dots\right),$$

which is decreasing by virtue of (30). □

The most interesting aspect of inequality (26), and of Proposition 3, is that the square roots appearing therein are best possible. Replacing $\sqrt{a_n}$ by $a_n^{\varepsilon}$, with any $\varepsilon > \frac{1}{2}$, would certainly produce better inequalities, but such replacement is impossible. To see this, we observe that the first inequality in (31),

$$\frac{a_1^{\varepsilon}}{a_1} \geqslant \frac{a_2^{\varepsilon}}{\sqrt{a_1a_2}},$$

fails to hold when $\boldsymbol{a} = (1,2,3,\dots)$, unless $\varepsilon \leqslant \frac{1}{2}$.

Qi and Guo [29] go on to study sequences for which

$$\frac{(a_1a_2\dots a_n)^{\frac{1}{n}}}{\sqrt{n}} \nearrow . \tag{32}$$

No applications are offered, save for the obvious one, (26), with $a_n = n$. They show that (32) holds under the bizarre hypothesis

$$\frac{a_{n+2}}{a_{n+1}} \geqslant \frac{n+2}{n+1}\left(\frac{n(n+2)}{(n+1)^2}\right)^{\frac{n}{2}} \qquad (n = 0,1,2,\dots). \tag{33}$$

Their result follows at once from the identity

$$\frac{(a_1a_2\dots a_n)^{\frac{1}{n}}}{\sqrt{n}} = \left(\frac{a_1}{\sqrt{1}}\left(\frac{0}{1}\right)^{\frac{0}{2}} \frac{a_2}{\sqrt{2}}\left(\frac{1}{2}\right)^{\frac{1}{2}} \cdots \frac{a_n}{\sqrt{n}}\left(\frac{n-1}{n}\right)^{\frac{n-1}{2}}\right)^{\frac{1}{n}},$$

condition (33) being equivalent to the assertion that

$$\frac{a_{n+1}}{n+1}\left(\frac{n}{n+1}\right)^{\frac{n}{2}} \nearrow \qquad (n = 0,1,2\dots).$$

The other main assertion of [29], that

$$\frac{(a_1a_2\dots a_n)^{\frac{1}{n}}}{n+1} \searrow \qquad (n = 1,2,\dots) \tag{34}$$

holds under the hypothesis

$$\frac{a_{n+2}}{a_{n+1}} \leqslant \left(\frac{n+3}{n+2}\right)^2\left(\frac{(n+1)(n+3)}{(n+2)^2}\right)^n \qquad (n = 0,1,2,\dots), \tag{35}$$

may be proved by the same method.

## 3. Martins

The following result was announced without proof as Theorem 13 of [5].

THEOREM 5. *If the sequences*

$$a_n \quad \textit{and} \quad \frac{a_{n+1}}{(a_1a_2\dots a_n)^{\frac{1}{n}}} \tag{36}$$

*both increase with* $n$ *(or both decrease), then the sequence*

$$\frac{a_1^p + a_2^p + \cdots + a_n^p}{n(a_1a_2\dots a_n)^{\frac{p}{n}}} \tag{37}$$

*increases with* $n$*, no matter what the value of* $p \in \mathbb{R}$.

Taking $\boldsymbol{a} = (1, 2, 3, \dots)$, we see that Theorem 5 contains Martins' inequality (Theorem 2), hypothesis (36) then being satisfied courtesy of Proposition 2. Alzer's version, [2], of Martins' inequality, asserting that the sequence

$$\frac{(n!)^{\frac{r}{n}}\left(\frac{1}{1^r} + \frac{1}{2^r} + \cdots + \frac{1}{n^r}\right)}{n} \tag{38}$$

increases with $n$ whenever $r > 0$, is covered as well, by taking $p = -r$ in (37). (Alzer's result has recently been re-discovered by Chen, Qi and Dragomir [12].)

Chan, Gao and Qi ([8], Theorem 1) give a pair of conditions, which, taken together, are sufficient for an increasing sequence $\boldsymbol{a}$ to satisfy (37), at any rate, when $p > 0$:

$$\frac{a_{n+1}}{a_n} \searrow \quad \text{and} \quad \left(\frac{a_{n+1}}{a_n}\right)^n \nearrow . \tag{39}$$

Theorem 5 shows that their first condition is superfluous, and that their second is too strong. The implication

$$\left(\frac{a_{n+1}}{a_n}\right)^n \nearrow \implies \frac{a_{n+1}}{(a_1a_2\dots a_n)^{\frac{1}{n}}} \nearrow \tag{40}$$

is readily confirmed by viewing the terms on the right of (40) as the geometric means of the terms on the left:

$$\frac{a_{n+1}}{(a_1a_2\dots a_n)^{\frac{1}{n}}} = \left(\left(\frac{a_2}{a_1}\right)^1 \left(\frac{a_3}{a_2}\right)^2 \cdots \left(\frac{a_{n+1}}{a_n}\right)^n\right)^{\frac{1}{n}}. \tag{41}$$

Moreover, it is easy to see that the implication (40) cannot be reversed.

Our proof of Theorem 5 uses three lemmas.

LEMMA 1. (Ratio Principle) *If* $\boldsymbol{a}$ *is a sequence of positive terms, then*

$$\frac{b_n}{a_n} \nearrow \implies \frac{b_1 + b_2 + \cdots + b_n}{a_1 + a_2 + \cdots + a_n} \nearrow . \tag{42}$$

*Proof.* The identity,

$$\frac{b_1+b_2+\cdots+b_n}{a_1+a_2+\cdots+a_n}=\frac{a_1\left(\frac{b_1}{a_1}\right)+a_2\left(\frac{b_2}{b_3}\right)+\cdots+a_n\left(\frac{b_n}{a_n}\right)}{a_1+a_2+\cdots+a_n}, \tag{43}$$

expresses the ratios of the sums as averages of the ratios. □

LEMMA 2. *Suppose that* $\alpha,\beta,\gamma,\delta\geqslant 0$ *and* $a<b\leqslant c<d$. *Then*

$$\beta\varphi(b)+\gamma\varphi(c)\leqslant\alpha\varphi(a)+\delta\varphi(d) \tag{44}$$

*for all convex functions* $\varphi$ *(defined on* $[a,d]$ *) if and only if*

$$\beta+\gamma=\alpha+\delta \tag{45}$$

*and*

$$\beta b+\gamma c=\alpha a+\delta d. \tag{46}$$

*Proof.* See Lemma 2 of [5]. □

LEMMA 3. *If* $\boldsymbol{a}$ *is an increasing sequence of positive terms, then*

$$\frac{a_{n+1}}{(a_1a_2\ldots a_n)^{\frac{1}{n}}}\nearrow \quad\Longrightarrow\quad \frac{a_n}{(a_1a_2\ldots a_n)^{\frac{1}{n}}}\nearrow. \tag{47}$$

*Proof.* We have

$$\begin{aligned}
&\left(\frac{a_{n+2}}{(a_1a_2\ldots a_{n+2})^{\frac{1}{n+2}}}\,\frac{(a_1a_2\ldots a_{n+1})^{\frac{1}{n+1}}}{a_{n+1}}\right)^{n+2}\\
&\qquad=\frac{a_{n+2}}{a_{n+1}}\left(\frac{a_{n+2}}{(a_1a_2\ldots a_{n+1})^{\frac{1}{n+1}}}\,\frac{(a_1a_2\ldots a_n)^{\frac{1}{n}}}{a_{n+1}}\right)^{n}\\
&\qquad\geqslant 1\quad \text{since } a_n\nearrow \text{ and } \frac{a_{n+1}}{(a_1a_2\ldots a_n)^{\frac{1}{n}}}\nearrow.
\end{aligned}$$

This shows that

$$\frac{a_{n+1}}{(a_1a_2\ldots a_{n+1})^{\frac{1}{n+1}}}\nearrow \qquad (n=1,2,\ldots).$$

The "missing" case,

$$\frac{a_1}{a_1^{\frac{1}{1}}}\leqslant\frac{a_2}{(a_1a_2)^{\frac{1}{2}}},$$

is covered as well, because $a_1\leqslant a_2$. □

*Proof of Theorem 5.* (Increasing case). We show that

$$\frac{1(a_1)^{\frac{p}{1}}}{a_1^p}\geqslant\frac{2(a_1a_2)^{\frac{p}{2}}-1(a_1)^{\frac{p}{1}}}{a_2^p}\geqslant\frac{3(a_1a_2a_3)^{\frac{p}{3}}-2(a_1a_2)^{\frac{p}{2}}}{a_3^p}\geqslant\cdots \tag{48}$$

and then apply the Ratio Principle. (We have chosen to work with the reciprocals of sequence (37) so that the denominators in (48) are all positive.)

The first inequality may be rephrased as

$$a_1^p + a_2^p \geqslant 2(a_1 a_2)^{\frac{p}{2}}, \tag{49}$$

and this is a consequence of the AM/GM inequality.

For the remaining inequalities, we suppose that $n$ is fixed $(n = 2, 3, \dots)$, and we apply Lemma 2 with

$$\begin{aligned} \alpha &= n, & a &= \log\left(a_n (a_1 a_2 \dots a_n)^{\frac{1}{n}}\right) \\ \beta &= n+1, & b &= \log\left(a_n (a_1 a_2 \dots a_{n+1})^{\frac{1}{n+1}}\right) \\ \gamma &= n-1, & c &= \log\left(a_{n+1} (a_1 a_2 \dots a_{n-1})^{\frac{1}{n-1}}\right) \end{aligned}$$

and

$$\delta = n, \qquad d = \log\left(a_{n+1} (a_1 a_2 \dots a_n)^{\frac{1}{n}}\right).$$

It is easy to check that hypotheses (45) and (46) are satisfied. We have $a \leqslant b$ and $c \leqslant d$ because $a_n \nearrow$ and the geometric means of an increasing sequence increase. Furthermore, $a \leqslant c$ by hypothesis (36) and $b \leqslant d$ by Lemma 3. (There is no need to check $b \leqslant c$ because the roles played by them in Lemma 2 are symmetrical.)

Lemma 2 guarantees that inequality (44) holds for every convex function $\varphi : (0, \infty) \to \mathbb{R}$. Setting $\varphi(x) = e^{px}$ $(p \in \mathbb{R}$, fixed), we deduce that

$$\begin{aligned} &(n+1) a_n^p (a_1 a_2 \dots a_{n+1})^{\frac{p}{n+1}} + (n-1) a_{n+1}^p (a_1 a_2 \dots a_{n-1})^{\frac{p}{n-1}} \\ &\qquad\qquad \leqslant \; n a_n^p (a_1 a_2 \dots a_n)^{\frac{p}{n}} + n a_{n+1}^p (a_1 a_2 \dots a_n)^{\frac{p}{n}} \end{aligned}$$

and hence, as desired, that the sequence

$$\frac{n(a_1 a_2 \dots a_n)^{\frac{p}{n}} - (n-1)(a_1 a_2 \dots a_{n-1})^{\frac{p}{n-1}}}{a_n^p} \qquad (n = 2, 3, \dots)$$

decreases with $n$.

(Decreasing case). We apply the above version (increasing case) with the sequence $\boldsymbol{a}$ replaced by $\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots\right)$. □

We shall say that an increasing sequence of positive terms, $\boldsymbol{a}$, satisfies *Martins' condition* if

$$\frac{a_{n+1}}{(a_1 a_2 \dots a_n)^{\frac{1}{n}}} \nearrow \tag{50}$$

and *Martins' inequality* if

$$\frac{a_1^p + a_2^p + \dots + a_n^p}{n(a_1 a_2 \dots a_n)^{\frac{p}{2}}} \nearrow \qquad (\forall\, p > 0). \tag{51}$$

The prototypical result, of course, is Theorem 2, which shows that (51) is valid when $\boldsymbol{a} = (1, 2, 3, \dots)$.

Qi ([23], page 297) asks whether Martins' inequality continues to hold when $\boldsymbol{a} = (n, n+1, n+2, \dots)$, $n$ being an arbitrary positive integer, and the question is repeated on page 148 of [30]. He shows ([25] and [26]) that Martins' condition holds for any arithmetic progression satisfying (28). Absent Theorem 5, however, he makes no connection between (50) and (51). Guo and Qi [15] extend Qi's result to progressions satisfying $a \geqslant d$, but they, too, fail to observe the relevance of their result to Martins' inequality.

Theorems 1–4 all deal exclusively with the sequence $\boldsymbol{a} = (1, 2, 3, \dots)$ and every generalization published thus far, that deals with arithmetic progressions, has $a \geqslant d$, i.e., the first term *always* exceeds the difference. We have already encountered the Minc/Sathre condition (16) and the Martins condition (50) and, not surprisingly, further examples, corresponding to Theorems 3 and 4, will be forthcoming. There is however a *fifth* condition, simple to work with, yet stronger than all the others, namely

$$n\left(\frac{a_{n+1}}{a_n} - 1\right) \nearrow . \tag{52}$$

It has the added virtue of holding precisely for those arithmetic progressions treated in the literature since

$$n\left(\frac{a+nd}{a+(n-1)d} - 1\right) = 1 - \frac{a-d}{a+(n-1)d} \nearrow$$

is valid exactly when $a \geqslant d$.

PROPOSITION 4. *If $\boldsymbol{a}$ is a sequence of positive terms, then*

$$n\left(\frac{a_{n+1}}{a_n} - 1\right) \nearrow \implies \left(\frac{a_{n+1}}{a_n}\right)^n \nearrow . \tag{53}$$

*Proof.* Our hypothesis may be rephrased as

$$\begin{aligned} \frac{a_{n+2}}{a_{n+1}} &\geqslant \frac{n\left(\frac{a_{n+1}}{a_n} - 1\right)}{n+1} + 1 \\ &= \frac{n}{n+1}\,\frac{a_{n+1}}{a_n} + \frac{1}{n+1} \\ &\geqslant \left(\frac{a_{n+1}}{a_n}\right)^{\frac{n}{n+1}} \end{aligned}$$

by the AM/GM inequality. □

We shall refer to (52) as the *grandfather condition.* Proposition 4, in conjunction with (40) shows that it implies Martins' condition, (50), and hence, via Lemma 3, the Minc/Sathre condition, (16), as well.

The real issue, then, when dealing with arithmetic progressions, is to allow $d > a$, and to ask which satisfy Martins' condition and which satisfy Martins' inequality. The first question has a very satisfactory, and somewhat surprising, solution.

PROPOSITION 5. *An arithmetic progression,*

$$\boldsymbol{a} = (a, a+d, a+2d, \dots) \qquad (a > 0,\ d \geqslant 0), \tag{54}$$

*satisfies Martins' condition, (50), precisely when*

$$d \leqslant \varphi a, \tag{55}$$

*where* $\varphi$ *is the golden mean,* $\varphi = \frac{1+\sqrt{5}}{2}$ .

*Proof.* (Necessity.) If the progression (54) satisfies Martins' condition, we must have

$$\frac{a+d}{a} \leqslant \frac{a+2d}{\sqrt{a(a+d)}}, \tag{56}$$

and this translates to

$$\left(\frac{d}{a}\right)^2 \leqslant 1 + \frac{d}{a}, \tag{57}$$

i.e.,

$$d \leqslant \varphi a. \tag{58}$$

(Sufficiency). We show that

$$\left(\frac{a_2}{a_1}\right)^1 \leqslant \left(\frac{a_3}{a_2}\right)^2 \leqslant \left(\frac{a_4}{a_3}\right)^3 \leqslant \cdots \tag{59}$$

if $d \leqslant \varphi a$. It then follows from the implication (40) that Martins' condition is satisfied.

The first inequality in (59),

$$\frac{a+d}{a} \leqslant \left(\frac{a+2d}{a+d}\right)^2, \tag{60}$$

simplifies to (57), and hence is valid precisely when (58) holds. To complete our proof of (59), we show that

$$\left(\frac{a_{n+1}}{a_n}\right)^n = \left(1 + \frac{d}{a+(n-1)d}\right)^n \nearrow \qquad (n = 2, 3, \dots). \tag{61}$$

Setting

$$f(x) = x \log\left(1 + \frac{d}{a+(x-1)d}\right),$$

we find that

$$f'(x) = \log\left(1 + \frac{d}{a+(x-1)d}\right) - \frac{d^2 x}{(a+xd)(a+(x-1)d)}$$

and

$$f''(x) = \frac{-d^2[2a(a-d) + xd(2a-d)]}{(a+xd)^2(a+(x-1)d)^2}.$$

Since $d \leqslant 2a$, the expression [ ] is non-negative *for all* $x \geqslant 2$ precisely when

$$a(a-d)+d(2a-d) \geqslant 0$$

i.e. when $d \leqslant \varphi a$. Thus, if $d \leqslant \varphi a$ and $x \geqslant 2$, we have $f''(x) \leqslant 0$. But $f'(x) \to 0$ as $x \to \infty$, so that $f'(x) \geqslant 0$ whenever $x \geqslant 2$, and this implies (61). □

PROBLEM 2. Which arithmetic progressions satisfy Martins' inequality (51)?

It can be shown that there is a number, $\delta_M$, which we call the *Martins index,* such that (51) holds whenever $d \leqslant \delta_M a$, and fails whenever $d > \delta_M a$. It is clear that $\delta_M \leqslant \delta_{MS}$ because Martins' inequality implies the Minc/Sathre condition (16). [Take $p^{th}$ roots in (51) and make $p \to \infty$.] This observation, coupled with Proposition 5, gives the bounds

$$1.61803\cdots \leqslant \delta_M < 3. \tag{62}$$

## 4. Meaningful sequences

A sequence $\boldsymbol{a}$, of positive terms, is said to be *meaningful* if the associated sequences,

$$\frac{n^{p-1}(a_1^p + a_2^p + \cdots + a_n^p)}{(a_1 + a_2 + \cdots + a_n)^p} \qquad (n = 1, 2, \ldots), \tag{63}$$

increase with $n$ when $p \geqslant 1$ or $p \leqslant 0$, and decrease when $0 \leqslant p \leqslant 1$. (The directions of the monotonicities, of course, are forced upon us by the Theorem on Means, all in accordance with (13).) Theorem 3 gives us our first example of a meaningful sequence, $\boldsymbol{a} = (1, 2, 3, \ldots)$, and many others are described in [5] and [6].

An equally appealing definition, embracing the full power of the fundamental theorem, would have been to insist that

$$\frac{L_n^p(\boldsymbol{a})}{L_n^q(\boldsymbol{a})} \nearrow \tag{64}$$

whenever $p > q$. This, however, turns out to be too restrictive, so we shall reserve the term *completely meaningful* for sequences satisfying (64).

It is not known whether the sequence $(1, 2, 3, \ldots)$ is completely meaningful. We restate this as

PROBLEM 3. Is the sequence $(1^\alpha, 2^\alpha, 3^\alpha, \ldots)$ meaningful for every $\alpha \in \mathbb{R}$?

A remarkable feature of the problem is that it remains open for positive integer values of $\alpha$ (even though closed-form expressions are available for $a_1 + a_2 + \cdots + a_n$), yet it has been solved, affirmatively, in the seemingly more difficult case, $\alpha \leqslant 1$ ([5], section 7).

The reason for singling out the $L^1$–norm in definition (63) is that it enables us to bring into play the vast resources of the Theory of Majorization. The "flip-flop" behavior, around $p = 0$ and $p = 1$, in (63) is reminiscent of that of the *power functions* ( $\varphi(x) = x^p$ is convex if $p \geqslant 1$ or $p \leqslant 0$, and concave if $0 \leqslant p \leqslant 1$ ), and it suggests that convexity has a key role to play here.

Theorem 3, for example, when spelled out, asserts that

$$\frac{1^p}{1.2^p} \leqslant \frac{1^p+2^p}{2.3^p} \leqslant \frac{1^p+2^p+3^p}{3.4^p} \leqslant \dots \tag{65}$$

when $p \geqslant 1$ or $p \leqslant 0$, with reversal when $0 \leqslant p \leqslant 1$. The first inequality in (65), rephrased as

$$2\varphi(3) \leqslant \varphi(2) + \varphi(4),$$

is certainly satisfied if $\varphi$ is convex. The second,

$$3(\varphi(4) + \varphi(8)) \leqslant 2(\varphi(3) + \varphi(6) + \varphi(9)),$$

is more subtle, but it, too, holds whenever $\varphi$ is convex, courtesy of the *Theory of Majorization* ([17] and [19]). The entire system (65), in fact, can be proved by this method.

The main result of [5], I believe, carries these ideas (proof via majorization) as far as possible. Henceforth we assume that $\boldsymbol{a}$ is a sequence of positive terms and we denote the *partial sums* by

$$A_n = a_1 + a_2 + \dots + a_n. \tag{66}$$

THEOREM 6. *If the sequences*

$$a_n \quad \textit{and} \quad \frac{na_{n+1}}{A_n} \tag{67}$$

*both increase (decrease) with* $n$*, then the sequence* $(a_1^\alpha, a_2^\alpha, \dots)$ *is meaningful whenever* $\alpha \leqslant 1$ $(\alpha \geqslant 1)$.

*Proof.* See Theorem 7 of [5]. □

COROLLARY. *If* $\boldsymbol{a}$ *is an increasing sequence of positive terms, then*

$$\frac{na_{n+1}}{A_n} \nearrow \quad \Longrightarrow \quad \frac{a_{n+1}}{(a_1 a_2 \dots a_n)^{\frac{1}{n}}} \nearrow . \tag{68}$$

*Proof.* We express the sequence on the right as a product of two increasing sequences

$$\frac{na_{n+1}}{A_n} \cdot \frac{A_n}{n(a_1 a_2 \dots a_n)^{\frac{1}{n}}} . \tag{69}$$

The first sequence increases by hypothesis; the second since $\boldsymbol{a}$ is meaningful by Theorem 6. [Make $p \to 0$ in the monotonicity $\frac{L^1}{L^p} \nearrow$.] □

Thus our sufficient condition for an increasing sequence to be meaningful implies Martins' condition (and hence Martins' inequality). The converse implication fails, as may be seen by considering arithmetic progressions $(a, a+d, a+2d, \dots)$. It is easy to check that (67) holds only when $d \leqslant a$, whereas Martins' condition is valid whenever $d \leqslant \varphi a$ (Proposition 5).

The grandfather condition, of course, is stronger still.

PROPOSITION 6. *If $\boldsymbol{a}$ is an increasing sequence of positive terms, then*

$$n\left(\frac{a_{n+1}}{a_n}-1\right)\nearrow \quad\Longrightarrow\quad \frac{na_{n+1}}{A_n}\nearrow\ . \tag{70}$$

*Proof.* By adding 1 to each term of the sequence on the left, we deduce that

$$\frac{a_2}{a_1}\leqslant\frac{2a_3-a_2}{a_2}\leqslant\frac{3a_4-2a_3}{a_3}\leqslant\dots$$

and the Ratio Principle completes the proof. □

Our next result shows that the *meaningful index* (cf. sections 2. and 3.) is 2.

THEOREM 7. *The arithmetic progression $(a,a+d,a+2d,\dots)$ is meaningful if and only if $d\leqslant 2a$.*

*Proof.* See Theorem 1 of [6]. □

Theorem 7 is a very pleasing result because it constitutes our one and only success at fixing an index. (We fail again in Section 5., where we attempt to determine the Alzer index.) Its most important feature, however, is that it shows that Theorem 6 is not definitive, that more remains to be said about meaningful sequences, and that the Theory of Majorization is not up to this task.

Despite these comments, it is intriguing to observe that there is very little room for improvement in Theorem 6.

PROPOSITION 7. *If $\boldsymbol{a}$ is an increasing sequence of positive terms, then*

$$\frac{na_{n+1}}{A_n}\nearrow \quad\Longrightarrow\quad \boldsymbol{a}\ \textit{is meaningful}\quad\Longrightarrow \frac{na_n}{A_n}\nearrow\ . \tag{71}$$

*Proof.* The first implication is Theorem 6; the second follows from definition (63) by making $p\to\infty$ in the associated monotonicity, $\frac{L^p}{L^1}\nearrow$. □

It is easy to see, again by considering arithmetic progressions, that neither implication in (71) can be reversed. The first condition is satisfied only when $d\leqslant a$, the second only when $d\leqslant 2a$ (Theorem 7), yet the third is valid for all $d$. The "gaps" in (71), however, are really quite subtle and they disappear altogether when we come to consider integral analogues of our results. We find, in section 6., that our sufficient condition $\frac{na_{n+1}}{A_n}\nearrow$ coincides with our necessary condition $\frac{na_n}{A_n}\nearrow$ and we thus obtain a complete characterization of the monotonic meaningful functions.

## 5. Alzer

The following generalization of Alzer's inequality is stated without proof as Theorem 14 of [5].

THEOREM 8. *If the sequences*

$$a_n \quad and \quad \left(\frac{a_{n+1}}{a_n}\right)^n \tag{72}$$

*are both monotonic in the same direction, the sequence*

$$\frac{\left(\frac{1}{n}\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}}}{a_n} \qquad (p \neq 0) \tag{73}$$

*is monotonic in the opposite direction.*

Taking $\boldsymbol{a} = (1, 2, 3, \ldots)$, and $p > 0$, we see that Theorem 8 contains Alzer's inequality, (4), hypothesis (72) then being satisfied via Proposition 4. The original derivation of (4) was unduly complicated, [1], and much simpler treatments have been devised by Sándor [31], Ume [32] and Kuang [18]. Dragomir and van der Hoek [13] have re-discovered Alzer's original result, but their proof runs to several pages and it covers only the case $p \geqslant 1$. Theorem $10'$ of [5] offers what may be the simplest of all approaches to Theorem 4, and it has the added virtue of covering negative values of $p$ as well. That case $(p < 0)$ first appeared on page 89 of [4] and it has been re-discovered by Chen and Qi [9].

Qi [23] offers the first generalization of Alzer's inequality (4), but his extension, to sequences of the form $(n, n+1, n+2, \ldots)$, is rather modest, the proof being the same as that of Sándor [31]. Ume, Liu and McDonald [34] give an alternative approach to Qi's result, but whether their treatment is "more elementary", as claimed, is left for the reader to decide.

Qi and Debnath [27] were the first to consider "abstract" versions of Alzer's inequality. Their treatment, however, involves such rigid hypotheses that the only concrete application they give is to Qi's extension described above. They show that (73) holds, when $p > 0$ and $\boldsymbol{a}$ is an increasing sequence, provided that:

$$a_{n+1}^2 \geqslant a_n a_{n+2} \tag{74}$$

$$\frac{a_{n+1} - a_n}{a_{n+1}^2 - a_n a_{n+2}} \geqslant \frac{n+2}{a_{n+2}} \tag{75}$$

and

$$\frac{a_{n+1} - a_n}{a_{n+1}^2 - a_n a_{n+2}} \geqslant \frac{n+1}{a_{n+1}}\,. \tag{76}$$

Theorem 8 shows that hypotheses (74) and (75) are irrelevant, and that (76) is too strong. We leave it as an exercise for the reader to check that

$$(77) \quad \Longrightarrow \quad \left(\frac{a_{n+1}}{a_n}\right)^n \nearrow$$

and that the implication cannot be reversed.

Chen and Qi [10] come close to proving Theorem 8 (increasing case) but their treatment demands the extra hypothesis that $\boldsymbol{a}$ be log-concave, i.e., that (74) holds. Their paper is marred by their claim to be working with negative powers, when, in fact, their analysis deals exclusively with positive powers.

*Proof of Theorem 8.* (Increasing case; $p > 0$). We show that

$$\frac{1a_1^p}{a_1^p} \leqslant \frac{2a_2^p - 1a_1^p}{a_2^p} \leqslant \frac{3a_3^p - 2a_2^p}{a_3^p} \leqslant \cdots \tag{77}$$

and then apply The Ratio Principle (Lemma 1).

The first inequality in (77), alias $a_1 \leqslant a_2$, is guaranteed by our first hypothesis, $a_n \nearrow$. The remaining inequalities follow from our second hypothesis:

$$\left(\frac{a_{n+1}}{a_n}\right)^n \leqslant \left(\frac{a_{n+2}}{a_{n+1}}\right)^{n+1}. \tag{78}$$

To see this, we rephrase (78) as

$$a_{n+1}^2 \leqslant (a_{n+1}a_{n+2})^{\frac{1}{n+1}} (a_n a_{n+2})^{\frac{n}{n+1}}. \tag{79}$$

Recognizing the right-hand side of (79) as a (weighted) geometric mean, we replace it by the larger $L^p$–mean and deduce that

$$a_{n+1}^{2p} \leqslant \frac{(a_{n+1}a_{n+2})^p + n(a_n a_{n+2})^p}{n+1}. \tag{80}$$

Rephrasing once again, (80) becomes

$$\frac{(n+1)a_{n+1}^p - na_n^p}{a_{n+1}^p} \leqslant \frac{(n+2)a_{n+2}^p - (n+1)a_{n+1}^p}{a_{n+2}^p},$$

and this completes the proof of (77).

(Increasing case; $p < 0$). The above proof breaks down when $p < 0$ and we must adopt a different approach. The quickest method is to "factorize" the desired monotonicity

$$\frac{L^p}{L^\infty} \searrow = \left(\frac{L^0}{L^\infty} \searrow\right) \left(\frac{L^p}{L^0} \searrow\right).$$

This amounts to expressing the sequence (73) as a product,

$$\frac{(a_1 a_2 \dots a_n)^{\frac{1}{n}}}{a_n} \cdot \frac{\left(\frac{1}{n}\sum_{k=1}^n a_k^p\right)^{\frac{1}{p}}}{(a_1 a_2 \dots a_n)^{\frac{1}{n}}} \qquad (p < 0),$$

of two decreasing sequences. The first decreases with $n$ by (40) and Lemma 3; the second by (40) and Theorem 5.

(Decreasing case). Replace $\boldsymbol{a}$ above by $\left(\frac{1}{a_1}, \frac{1}{a_2}, \cdots\right)$. □

We say that an increasing sequence $\boldsymbol{a}$, of positive terms, satisfies *Alzer's condition* if

$$\left(\frac{a_{n+1}}{a_n}\right)^n \nearrow \tag{81}$$

and *Alzer's inequality* if

$$\frac{a_1^p + a_2^p + \cdots + a_n^p}{n a_n^p} \searrow \qquad (\forall\, p > 0). \tag{82}$$

Not surprisingly, there is an *Alzer index,* say $\delta_A$, below which $(d \leqslant \delta_A a)$ arithmetic progressions are good, i.e. (82) holds, and above which $(d > \delta_A a)$ they are not. It is obvious that

$$1.61803\ldots \leqslant \delta_A < 3. \tag{83}$$

The first inequality comes from Proposition 5, which shows, incidentally, that Alzer's *condition* is satisfied precisely when $d \leqslant \varphi a$. The second follows from (23) since $\delta_A \leqslant \delta_{MS}$ (see section 2.).

PROPOSITION 8. *If* $\boldsymbol{a}$ *is an increasing sequence of positive terms, then*

$$n\left(1 - \frac{a_n}{a_{n+1}}\right) \nearrow \quad \Longrightarrow \quad \frac{a_1^p + a_2^p + \cdots + a_n^p}{n a_n^p} \searrow \qquad (\forall\, p \geqslant 1). \tag{84}$$

*Proof.* Our hypothesis amounts to the assertion

$$a_{n+1}^2 \leqslant \frac{1}{n+1}\, a_{n+1}a_{n+2} + \frac{n}{n+1}\, a_n a_{n+2}. \tag{85}$$

Replacing the arithmetic mean by the larger $L^p$ –mean, (85) becomes (80) and the proof then follows that of Theorem 8. □

Proposition 8 shows that Alzer's inequality, in case $p \geqslant 1$, is satisfied by any arithmetic progression. Brnetić and Pečarić ([7], Theorem 2.1) come painfully close to making this observation.

Another consequence of Proposition 8 is the following improvement to (83):

$$2 \leqslant \delta_A < 3. \tag{86}$$

At issue here is the assertion that $\frac{L^p}{L^\infty} \searrow$ $(p > 0)$ for arithmetic progressions satisfying $d \leqslant 2a$. Proposition 8 covers all progressions, but only for $p \geqslant 1$. Theorem 7, on the other hand, covers the complementary case, $0 < p \leqslant 1$ (since $d \leqslant 2a$) via the factorization

$$\frac{L^p}{L^\infty} \searrow = \left(\frac{L^p}{L^1} \searrow\right)\left(\frac{L^1}{L^\infty} \searrow\right).$$

Elezović and Pečarić [14] consider the following generalization of Alzer's original inequality:

$$\frac{a_1^p + a_2^p + \cdots + a_n^p}{a_n^{p+1}} \searrow \qquad (\forall\, p > 0). \tag{87}$$

This reduces to Theorem 4 when $a_n = n$, but it is not a meaningful inequality (in the present, technical sense). They come close to proving the following result.

PROPOSITION 9. *If $\boldsymbol{a} = (a_1 a_2, \dots)$ is a sequence of positive terms such that $(0, a_1, a_2, \dots)$ is convex, then (87) holds.*

*Proof.* Our hypotheses guarantee that

$$a_n a_{n+1}(a_{n-1} + a_{n+1} - 2a_n) \geqslant 0 \qquad (n = 1, 2, \dots),$$

which we rephrase as

$$a_n^2 \leqslant \frac{a_n}{a_{n+1}}\,(a_{n-1}a_{n+1}) + \left(1 - \frac{a_n}{a_{n+1}}\right)(a_n a_{n+1}). \tag{88}$$

The right-hand side of (88) is an arithmetic mean with non-negative weights. (The weights, in fact, are positive, since $\frac{a_n}{n} \nearrow$ courtesy of convexity.) Replacing it with the larger $L^{1+p}$–mean, we obtain

$$\left(a_n^2\right)^{p+1} \leqslant \frac{a_n}{a_{n+1}}(a_{n-1}a_{a+1})^{p+1} + \left(1 - \frac{a_n}{a_{n+1}}\right)(a_n a_{n+1})^{p+1}.$$

This, in turn, may be rephrased as

$$\frac{a_n^{p+1} - a_{n-1}^{p+1}}{a_n^p} \nearrow$$

and the Ratio Principle completes the proof. □

Ume [33] considers yet another variant on Alzer's original inequality, but his results are too complicated to be described in detail here. He shows that the sequence

$$\frac{(k+1)^{rp} + (k+2)^{rp} + \cdots + (k+n)^{rp}}{n^r (k+n)^{rp}}$$

decreases with $n$ $(n = 1, 2, \dots)$, where $k$ is a non-negative integer, $p > 0$, and $r = 1$ or $r \geqslant 2$. And he asks whether this assertion continues to hold for all $r \geqslant 1$. A much stronger result,

$$\frac{(k+1)^{rp} + (k+2)^{rp} + \cdots + (k+n)^{rp}}{n(k+n)^{rp}} \searrow$$

is, in fact, available, as is seen by applying Theorem 8 to the sequence $((k+1)^r, (k+2)^r, \dots)$.

## 6. Integral analogues

The Theorem on Means applies to functions ([17], Chapter VI) in the same way as it does to sequences, and so it is reasonable to expect integral analogues of all our results. We shall work with positive, continuous functions defined on finite intervals in order to avoid all difficulties.

We say that such a function, $f : [a, b] \to \mathbb{R}$, is *meaningful* if

$$\frac{\left(\dfrac{1}{x-a}\displaystyle\int_a^x f^p(t)dt\right)^{\frac{1}{p}}}{\dfrac{1}{x-a}\displaystyle\int_a^x f(t)dt} \tag{89}$$

increases with $x$, $a < x \leqslant b$, whenever $p \geqslant 1$, and decreases whenever $p \leqslant 1$. When $p = 0$, the $L^p$–mean is to be replaced, as usual, by the *geometric mean*

$$\exp\left(\frac{1}{x-a}\int_a^x \log f(t)dt\right). \tag{90}$$

Our next result gives a complete description of the monotonic meaningful functions.

THEOREM 9. *With the above conventions, a monotonic function $f$ is meaningful if and only if the function*

$$\frac{f(x)}{\dfrac{1}{x-a}\displaystyle\int_a^x f(t)dt} \tag{91}$$

*is monotonic in the same direction.*

*Proof.* (Necessity; increasing case). Making $p \to \infty$ in (89), we deduce that (91) must hold.

(Sufficiency; increasing case). We apply the integral analogue,

$$\frac{\alpha(x)}{\beta(x)} \searrow \quad \Longrightarrow \quad \frac{\displaystyle\int_a^x \alpha(t)dt}{\displaystyle\int_a^x \beta(t)dt} \searrow \tag{92}$$

of the Ratio Principle (Lemma 1), with

$$\alpha(x) = \frac{d}{dx}\left\{(x-a)\left(\frac{1}{x-a}\int_a^x f(t)dt\right)^p\right\}$$

and

$$\beta(x) = (f(x))^p.$$

It turns out that

$$\frac{\alpha(x)}{\beta(x)} = \frac{pu+1-p}{u^p},$$

where

$$u = \frac{(x-a)f(x)}{\displaystyle\int_a^x f(t)dt}.$$

We observe that $u \geqslant 1$ since $f \nearrow$ and that $u$ increases with $x$ by hypothesis (91). On the other hand,

$$\frac{d}{du}\left(\frac{pu+1-p}{u^p}\right) = \frac{-p(p-1)(u-1)}{u^{p+1}}$$

so that $\frac{\alpha(x)}{\beta(x)}$ decreases with $x$ when $p \geqslant 1$ or $p < 0$, and increases when $0 < p \leqslant 1$.

(Decreasing case). This is similar to the above proof. For necessity, we make $p \to -\infty$ in (89); for sufficiency, we observe that $u \leqslant 1$ and that now $u$ decreases with $x$. □

The *integral analogue of Martins' inequality,* that the function

$$\frac{\dfrac{1}{x-a}\displaystyle\int_a^x f^p(t)dt}{\exp\left(\dfrac{p}{x-a}\displaystyle\int_a^x \log f(t)dt\right)} \tag{93}$$

increases with $x$, no matter what the value of $p \in \mathbb{R}$, may be handled similarly.

THEOREM 10. *Under the above conventions, a monotonic function $f$ satisfies Martins' inequality (93) if and only if the function*

$$\frac{f(x)}{\exp\left(\dfrac{1}{x-a}\displaystyle\int_a^x \log f(t)dt\right)} \tag{94}$$

*is monotonic in the same direction.*

*Proof.* (Necessity; increasing case). Take $p^{th}$ roots in (93) and make $p \to \infty$.

(Sufficiency; increasing case). We again apply the Ratio Principle, (92), this time with

$$\alpha(x) = \frac{d}{dx}(x-a)\exp\left(\frac{p}{x-a}\int_a^x \log f(t)dt\right)$$

and

$$\beta(x) = (f(x))^p.$$

It transpires that

$$\frac{\alpha(x)}{\beta(x)} = \frac{p\log u + 1}{u^p}$$

where

$$u = \frac{f(x)}{\exp\left(\dfrac{1}{x-a}\displaystyle\int_a^x \log f(t)dt\right)}.$$

We observe that $u \geqslant 1$ since $f \nearrow$ and that $u$ increases with $x$ by hypothesis (94). Furthermore, since

$$\frac{d}{du}\frac{p\log u + 1}{u^p} = \frac{-p^2\log u}{u^{p+1}},$$

it follows that $\frac{\alpha(x)}{\beta(x)}$ decreases with $x$ regardless of the value of $p \in \mathbb{R}$.

(Decreasing case). This is similar. □

Mascioni [21] gives an interesting sufficient condition for the validity of (93) for increasing functions, namely:

$$x(\log f(x))'' + (\log f(x))' \geqslant 0 \qquad (x > 0). \tag{95}$$

(He works on $(0, \infty)$ and assumes that $f > 0$, $f \nearrow$ and $f''$ exists.)

But his condition is too strong, being equivalent, in fact, to the integral version of our grandfather condition (52)

$$\frac{xf'(x)}{f(x)} \nearrow . \tag{96}$$

It is not difficult to see that any increasing function satisfying (96) must be *completely meaningful,* i.e.

$$\frac{\left(\frac{1}{x}\int_0^x f^p(t)dt\right)^{\frac{1}{p}}}{\left(\frac{1}{x}\int_0^x f^q(t)dt\right)^{\frac{1}{q}}} \nearrow \qquad (p \geqslant q > 0). \tag{97}$$

First, we observe that

$$\frac{xf'(x)}{f(x)} \nearrow \implies \frac{xf(x)}{\int_0^x f(t)dt} \nearrow . \tag{98}$$

This implication follows from the Ratio Principle, (92), since

$$\frac{xf'(x)}{f(x)} \nearrow \iff \frac{\frac{d}{dx}(xf(x))}{f(x)} \nearrow . \tag{99}$$

Second, we notice that hypothesis (96) remains true if $f(x)$ is replaced by $(f(x))^p$, with $p > 0$. It thus follows from (96), and Theorem 9, that the function $(f(x))^p$ is meaningful, so that (97) holds.

The implication (98) cannot be reversed. To see this, we take

$$f(x) = 1 + 2x^2 - x^3.$$

It can be shown that both

$$f(x) \quad \text{and} \quad \frac{xf(x)}{\int_0^x f(t)dt}$$

increase on $[0,1]$, whereas (96) fails.

I have been unable to characterize the increasing functions that satisfy the *integral version of Alzer's inequality*

$$\frac{(x-a)f(x)}{\int_a^x f^p(t)dt} \nearrow \qquad (\forall\, p > 0). \tag{100}$$

It is clear that (96) $\implies$ (100) $\implies$ (94), but there is a gap between (96) and (94). Qi [24] and Zhang, Chen and Qi [36] study (100) in the "classical" case, $f(x) = x$, but the result is then a calculus problem. Qi and Guo [28] claim that Theorem 10 and inequality (100) hold for all positive, increasing functions, but their claim is obviously false.

We close with the observation that the fundamental theorem remains valid for *weighted* means. There is thus the possibility of obtaining weighted analogues of all our results.

I am grateful to the referee for bringing references [11], [16] and [35] to my attention. Proposition 9 appears in [11] and Proposition 3 in [16], both with longer proofs than the ones given here. Theorem 5 is proved in [35] under the additional hypotheses: $p > 0$, $\boldsymbol{a}$ increasing and log-concave.

## REFERENCES

[1] H. ALZER, *On an inequality of Minc and Sathre*, J. Math. Anal. Appl. **179** (1993), no. 2, 396-402.
[2] H. ALZER, *Refinement of an inequality of G. Bennett*, Discrete Math. **135** (1994), no. 1-3, 39-46.
[3] G. BENNETT, *Lower bounds for matrices, II*, Canad. J. Math. **44** (1992), 54-74.
[4] G. BENNETT, *Factorizing the classical inequalities*, Mem. Amer. Math. Soc. **120** (1996), no. 576, viii + 130 pp.
[5] G. BENNETT, *Meaningful sequences*, Houston J. Math. **33** (2007), no. 2, 555-580.
[6] G. BENNETT, *Meaningful sequences and the theory of majorization*, to appear in Houston J. Math.
[7] I. BRNETIĆ AND J. PEČARIĆ, *Comments on some analytic inequalities*, J. Inequal. Pure Appl. Math. **4** (2003), no. 1, Article 20, 5 pp. (electronic).
[8] T.H. CHAN, P. GAO AND F. QI, *On a generalization of Martins' inequality*, Monatsh. Math. **138** (2003), no. 3, 179-187.
[9] C.-P. CHEN AND F. QI, *Extension of an inequality of H. Alzer for negative powers*, Tamkang J. Math. **36** (2005), no. 1, 69-72.
[10] C.-P. CHEN AND F. QI, *Generalizations of an inequality of Alzer for negative powers*, Tamkang J. Math. **36** (2005), no. 3, 219-222.
[11] C.-P. CHEN AND F. QI, *Note on Alzer's inequality*, Tamkang J. Math. **37** (2006), no. 1, 11-14.
[12] C.-P. CHEN, F. QI AMD S.S. DRAGOMIR. *Reverse of Martin's inequality*. Aust. J. Math. Anal. Appl. **2** (2005), no. 1, Article 2, 5 pp. (electronic).
[13] S.S. DRAGOMIR AND J. VAN DER HOEK, *Some new analytic inequalities and their applications in guessing theory*, J. Math. Anal. Appl. **225** (1998), no. 2, 542-556.
[14] N. ELEZOVIĆ AND J. PEČARIĆ, *On Alzer's inequality*, J. Math. Anal. Appl. **223** (1998), 366-369.
[15] B.-N. GUO AND F. QI, *Inequalities and monotonicity of the ratio for the geometric means of a positive arithmetic sequence with arbitrary difference*, Tamkang J. Math. **34** (2003), no. 3, 261-270.
[16] B.-N. GUO AND F. QI, *Monotonicity of sequences involving geometric means of positive sequences with monotonicity and logarithmical convexity*, Math. Inequal. Appl. **9**, (2006), no. 1, 1-9.
[17] G.H. HARDY, J.E. LITTLEWOOD AND G. PÓLYA, *Inequalities,* 2nd edition, Cambridge University Press, Cambridge, 1967.
[18] J. KUANG, *Some extensions and refinements of Minc–Sathre inequality*, Math. Gaz. **83** (1999), 123-127.
[19] A.W. MARSHALL AND I. OLKIN, *Inequalities: Theory of Majorization and its Applications,* Academic Press, New York, 1979.
[20] J.S. MARTINS, *Arithmetic and geometric means, an application to Lorentz sequence spaces*, Math. Nachr. **139** (1988), 281-288.
[21] V. MASCIONI, *A sufficient condition for the integral version of Martins' inequality*, J. Inequal. Pure Appl. Math. **5** (2004), no. 2, Article 32, 3 pp. (electronic).
[22] H. MINC AND L. SATHRE, *Some inequalities involving* $(r!)^{\frac{1}{r}}$, Proc. Edinburgh Math. Soc. (2) **14** (1964/5), 41-46.
[23] F. QI, *Generalization of H. Alzer's inequality*, J. Math. Anal. Appl. **240** (1999), no. 1, 294-297.
[24] F. QI, AN ALGEBRAIC INEQUALITY, J. Inequal. Pure Appl. Math. **2** (2001), no. 1, Article 13, 3 pp. (electronic).
[25] F. QI, *Inequalities and monotonicity of the ratio of the geometric means of a positive arithmetic sequence with unit difference*, Internat. J. Math. Ed. Sci. Tech. **34** (2003), no. 4, 601-607.
[26] F. QI, *Inequalities and monotonicity of the ratio for the geometric means of a positive arithmetic sequence with unit difference*, Austral. Math. Soc. Gaz. **30** (2003), no. 3, 142-147.
[27] F. QI AND L. DEBNATH, *On a new generalization of Alzer's inequality*, Internat. J. Math. Math. Sci. **23** (2000), no. 12, 815-818.
[28] F. QI AND B.-N. GUO, *An inequality between ratio of the extended logarithmic means and ratio of the exponential means*, Taiwanese J. Math. **7** (2003), no. 2, 229-237.
[29] F. QI AND B.-N. GUO, *Monotonicity of sequences involving geometric means of positive sequences*, Nonlinear Funct. Anal. Appl. **8** (2003), no. 4, 507-517.

[30] F. QI AND Q.-M. LUO, *Generalization of H. Minc and L. Sathre's inequality*, Tamkang J. Math. **31** (2000), no. 2, 145-148.
[31] J. SÁNDOR, *On an inequality of Alzer*, J. Math. Anal. Appl. **192** (1995), no. 3, 1034-1035.
[32] J.S. UME, *An elementary proof of H. Alzer's inequality*, Math. Japonica **44** (1996), 521-522.
[33] J.S. UME, *An inequality for a positive real function*, Math. Inequal. Appl. **5** (2002), no. 4, 693-696.
[34] J.S. UME, Z. LIU AND J.N. MCDONALD, *A simple proof of generalized Alzer's inequality*, Indian J. Pure Appl. Math. **35** (8) (2004), 969-971.
[35] Z. XU AND D. XU, *A general form of Alzer's inequality*, Comput. Math. Appl. **44** (2002), no. 3-4, 365-373.
[36] S.-L. ZHANG, C.-P. CHEN AND F. QI, *Continuous analogue of Alzer's inequality*, Tamkang J. Math. **37** (2006), no. 2, 105-108.

(Received June 3, 2007)

*Grahame Bennett*
*Department of Mathematics*
*Indiana University*
*Bloomington*
*Indiana 47405*
*U.S.A.*
*e-mail:* bennettg@indiana.edu