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ON A NEW GENERALIZATION OF MARTINS' INEQUALITY

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(*communicated by N. Ujević*)

Abstract. Let $n,m \in \mathbb{N}$ and $\{a_i\}_{i=1}^{n+m}$ be an increasing, logarithmically concave, positive, and nonconstant sequence such that the sequence $\left\{i\left[\frac{a_{i+1}}{a_i}-1\right]\right\}_{i=1}^{n+m-1}$ is increasing. Then the following inequality between ratios of the power means and of the geometric means holds:

$$\left(\frac{1}{n}\sum_{i=1}^{n}a_i^r \Big/ \frac{1}{n+m}\sum_{i=1}^{n+m}a_i^r\right)^{1/r} < \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}},$$

where r is a positive number, $a_n!$ denotes the sequence factorial defined by $\prod_{i=1}^{n}a_i$. The upper bound is the best possible.

1. Introduction

It is well-known that the following inequality

$$\frac{n}{n+1} < \left(\frac{1}{n}\sum_{i=1}^{n}i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1}i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \tag{1}$$

holds for $r>0$ and $n\in\mathbb{N}$. The lower and upper bounds in (1) are both sharp. We call the left-hand side of this inequality Alzer's inequality [1], and the right-hand side Martins' inequality [14].

The first easy proof of Alzer's inequality is due to J. Sándor who used Cauchy mean value theorem and mathematical induction in his proof, see [31]. Also the method of Lagrange mean value theorem and mathematical induction has been used by J. Sándor in [32].

Also by induction, N. Elezović and J. Pečarić [6] generalized inequality (1) and showed that, if the positive sequence $\{a_n\}_{n=1}^{\infty}$ satisfies

$$1 \leqslant \left(\frac{a_{n+2}}{a_{n+1}}\right)^r\left[\frac{a_{n+2}}{a_{n+1}}-1+\left(\frac{a_n}{a_{n+1}}\right)^{r+1}\right], \quad n\geqslant 0, \quad a_0=0, \tag{2}$$

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then, for $r > 0$, we have

$$\frac{a_n}{a_{n+1}} \leqslant \left(\frac{1}{a_n} \sum_{i=1}^{n} a_i^r \Big/ \frac{1}{a_{n+1}} \sum_{i=1}^{n+1} a_i^r \right)^{1/r}. \tag{3}$$

In [22], F. Qi and L. Debnath proved that: Let $n,m \in \mathbb{N}$ and $\{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive real numbers satisfying

$$\frac{(k+2)a_{k+2}^r - (k+1)a_{k+1}^r}{(k+1)a_{k+1}^r - ka_k^r} \geqslant \left(\frac{a_{k+2}}{a_{k+1}} \right)^r \tag{4}$$

for a given positive real number r and $k \in \mathbb{N}$. Then

$$\frac{a_n}{a_{n+m}} \leqslant \left(\frac{(1/n)\sum_{i=1}^{n} a_i^r}{(1/(n+m))\sum_{i=1}^{n+m} a_i^r} \right)^{1/r}. \tag{5}$$

The lower bound of (5) is the best possible.

In [9, 10, 19, 20, 25, 26, 27], the following inequalities and other more general results are proved:

$$\frac{n+k+1}{n+m+k+1} < \left(\prod_{i=k+1}^{n+k} i \right)^{1/n} \Big/ \left(\prod_{i=k+1}^{n+m+k} i \right)^{1/(n+m)} \leqslant \sqrt{\frac{n+k}{n+m+k}}, \tag{6}$$

$$\frac{a(n+k+1)+b}{a(n+m+k+1)+b} < \frac{\left[\prod_{i=k+1}^{n+k} (ai+b) \right]^{\frac{1}{n}}}{\left[\prod_{i=k+1}^{n+m+k} (ai+b) \right]^{\frac{1}{n+m}}} \leqslant \sqrt{\frac{a(n+k)+b}{a(n+m+k)+b}}, \tag{7}$$

where $n,m \in \mathbb{N}$, k is a nonnegative integer, a a positive constant, and b a nonnegative constant. The equalities in (6) and (7) is valid for $n = 1$ and $m = 1$.

In [8], the following monotonicity results for the gamma function were obtained: The function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{x+y+1} \tag{8}$$

is decreasing in $x \geqslant 1$ for fixed $y \geqslant 0$. Then, for positive real numbers x and y, we have

$$\frac{x+y+1}{x+y+2} \leqslant \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}. \tag{9}$$

In [18, 22], it is proved that: Let n and m be natural numbers, k a nonnegative integer. Then

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n} \sum_{i=k+1}^{n+k} i^r \Big/ \frac{1}{n+m} \sum_{i=k+1}^{n+m+k} i^r \right)^{1/r}, \tag{10}$$

where r is a given positive real number. The lower bound is the best possible.

In [5, 24, 28], some more general results for the lower bound of ratio of power means $\left(\frac{1}{n} \sum_{i=1}^{n} a_i^r \big/ \frac{1}{n+m} \sum_{i=1}^{n+m} a_i^r \right)^{1/r}$ for positive sequence $\{a_i\}_{i \in \mathbb{N}}$ were obtained.

An open problem in [17, 18] asked for the validity of the following inequality:

$$\left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r \bigg/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r} < \frac{\sqrt[n]{(n+k)!/k!}}{\sqrt[n+m]{(n+m+k)!/k!}}, \tag{11}$$

where $r > 0$, $n, m \in \mathbb{N}$, $k \in \mathbb{Z}^+$.

Let $\{a_i\}_{i\in\mathbb{N}}$ be a positive sequence. If $a_{i+1}a_{i-1} \geqslant a_i^2$ for $i \geqslant 2$, we call $\{a_i\}_{i\in\mathbb{N}}$ a logarithmically convex sequence; if $a_{i+1}a_{i-1} \leqslant a_i^2$ for $i \geqslant 2$, we call $\{a_i\}_{i\in\mathbb{N}}$ a logarithmically concave sequence. See [15, p. 284].

In [4], the open problem mentioned above was solved and generalized affirmatively: Let $\{a_i\}_{i=1}^{n+m}$ be an increasing, logarithmically concave, positive, and nonconstant sequence satisfying

$$\left(\frac{a_{\ell+1}}{a_\ell}\right)^{\ell} \geqslant \left(\frac{a_\ell}{a_{\ell-1}}\right)^{\ell-1} \tag{12}$$

for any positive integer $\ell > 1$, then inequality (13) holds for r being a positive number, $n, m \in \mathbb{N}$, and $a_n!$ denoting the sequence factorial $\prod_{i=1}^n a_i$. The upper bound in (13) is best possible.

On generalizations of Alzer's inequality and Martins' inequality (1) have invoked the interest of several mathematicians and there is a rich literature. For more detailed information, we refer the reader to [3, 7, 12, 13, 16, 21, 28, 23, 24, 31, 34, 35, 36] and the references therein.

The purpose of this paper is to give a new generalization of inequality (11) as follows.

THEOREM 1. *Let $n, m \in \mathbb{N}$ and $\{a_i\}_{i=1}^{n+m}$ be an increasing, logarithmically concave, positive, and nonconstant sequence such that the sequence $\left\{i\left[\frac{a_{i+1}}{a_i}-1\right]\right\}_{i=1}^{n+m-1}$ is increasing. Then the following inequality between ratios of the power means and of the geometric means holds:*

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \bigg/ \frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r\right)^{1/r} < \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}}, \tag{13}$$

where r is a positive number and $a_n!$ denotes the sequence factorial $\prod_{i=1}^n a_i$. The upper bound is the best possible.

As a simple consequence of Theorem 1 by taking $\{a_i\}_{i=1}^{m+n} = \{[a(i+k)+b]^\alpha\}_{i=1}^{m+n}$ for positive parameters a and α, we have

COROLLARY 1. *Let $a > 0$, $\alpha > 0$, $m, n \in \mathbb{N}$, and k a nonnegative integer. If $b > -a(1+k)$ and the sequence*

$$\left\{i\left[\left(1+\frac{a}{a(i+k)+b}\right)^{\alpha}-1\right]\right\}_{i=1}^{m+n-1} \tag{14}$$

is increasing, then for any $r > 0$, we have

$$\left(\frac{\frac{1}{n}\sum_{i=k+1}^{k+n}[(ai+b)^\alpha]^r}{\frac{1}{m+n}\sum_{i=k+1}^{k+m+n}[(ai+b)^\alpha]^r}\right)^{1/r} < \frac{\sqrt[n]{\prod_{i=k+1}^{k+n}(ai+b)^\alpha}}{\sqrt[m+n]{\prod_{i=k+1}^{k+m+n}(ai+b)^\alpha}}. \tag{15}$$

The upper bound is the best possible.

If $\alpha = 1$, we have

COROLLARY 2. *Let $a > 0$, $m, n \in \mathbb{N}$, and k a nonnegative integer. Then for any $r > 0$, $b \geqslant -ak$, we have*

$$\left(\frac{\frac{1}{n}\sum_{i=k+1}^{k+n}(ai+b)^r}{\frac{1}{m+n}\sum_{i=k+1}^{k+m+n}(ai+b)^r}\right)^{1/r} < \frac{\sqrt[n]{\prod_{i=k+1}^{k+n}(ai+b)}}{\sqrt[m+n]{\prod_{i=k+1}^{k+m+n}(ai+b)}}. \tag{16}$$

The upper bound is the best possible.

REMARK 1. By letting $a = 1, b = 0$ in 16, we recover inequality (11).

Taking $\alpha = 2$ in Corollary 1 leads to the following

COROLLARY 3. *Let $a > 0, m, n \in \mathbb{N}$, and k a nonnegative integer. Then, for any $r > 0, b \geqslant a(\frac{1}{2} - k)$, we have*

$$\left(\frac{\frac{1}{n}\sum_{i=k+1}^{k+n}[(ai+b)^2]^r}{\frac{1}{m+n}\sum_{i=k+1}^{k+m+n}[(ai+b)^2]^r}\right)^{1/r} < \frac{\sqrt[n]{\prod_{i=k+1}^{k+n}(ai+b)^2}}{\sqrt[m+n]{\prod_{i=k+1}^{k+m+n}(ai+b)^2}}. \tag{17}$$

The upper bound is the best possible.

Considering $\{a_i\}_{i\in\mathbb{N}} = \left\{e^{i^\alpha}\right\}_{i\in\mathbb{N}}$ in Theorem 1, standard argument gives us the following

COROLLARY 4. *Let $m, n \in \mathbb{N}, \alpha \in (0, 1)$ such that*

$$\frac{e^{(1+x)^\alpha} - e^{x^\alpha}}{x^{\alpha-1} - (1+x)^{\alpha-1}} \geqslant \alpha x e^{(1+x)^\alpha}, \quad x \in [1, \infty). \tag{18}$$

Then, for any $r > 0$, we have

$$\left(\frac{\frac{1}{n}\sum_{i=1}^{n} e^{i^\alpha r}}{\frac{1}{m+n}\sum_{i=1}^{m+n} e^{i^\alpha r}}\right)^{1/r} < \exp\left[\frac{1}{n}\sum_{i=1}^{n} i^\alpha - \frac{1}{m+n}\sum_{i=1}^{m+n} i^\alpha\right]. \tag{19}$$

The upper bound is the best possible.

2. Lemmas

To prove our main results, the following lemmas are necessary.

LEMMA 1. *Let* $n, m \in \mathbb{N}$, *and* $\{a_i\}_{i=1}^{n+m+1}$ *a nonconstant positive sequence such that the sequence* $\left\{i\left[\frac{a_{i+1}}{a_i} - 1\right]\right\}_{i=1}^{n+m}$ *is increasing, then the sequence*

$$\left\{\frac{\sqrt[i]{a_i!}}{a_{i+1}}\right\}_{i=1}^{n+m} \tag{20}$$

is decreasing. As a simple consequence, we have the following

$$\frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}} > \frac{a_{n+1}}{a_{n+m+1}}, \tag{21}$$

where $a_n!$ *denotes the sequence factorial defined by* $\prod_{i=1}^{n} a_i$.

Proof. For $1 \leqslant i \leqslant n+m-1$, the monotonicity of the sequence (20) is equivalent to the following

$$\frac{\sqrt[i]{a_i!}}{a_{i+1}} \geqslant \frac{\sqrt[i+1]{a_{i+1}!}}{a_{i+2}}, \tag{22}$$

$$\iff \quad \left(\prod_{k=1}^{i} \frac{a_k}{a_{i+1}}\right)^{1/i} \geqslant \left(\prod_{k=1}^{i+1} \frac{a_k}{a_{i+2}}\right)^{1/(i+1)},$$

$$\iff \quad \frac{1}{i}\sum_{k=1}^{i} \ln \frac{a_k}{a_{i+1}} \geqslant \frac{1}{i+1}\sum_{k=1}^{i+1} \ln \frac{a_k}{a_{i+2}},$$

$$\iff \quad \frac{i}{i+1}\sum_{k=1}^{i+1} \ln \frac{a_k}{a_{i+2}} \leqslant \sum_{k=1}^{i} \ln \frac{a_k}{a_{i+1}}. \tag{23}$$

Since $\ln x$ is concave on $(0, \infty)$, by definition of concaveness, it follows that, for $1 \leqslant k \leqslant i$,

$$\begin{aligned} \frac{k}{i+1} \ln \frac{a_{k+1}}{a_{i+2}} &+ \frac{i-k+1}{i+1} \ln \frac{a_k}{a_{i+2}} \\ &\leqslant \ln\left(\frac{k}{i+1} \cdot \frac{a_{k+1}}{a_{i+2}} + \frac{i-k+1}{i+1} \cdot \frac{a_k}{a_{i+2}}\right) \\ &= \ln\left(\frac{ka_{k+1} + (i-k+1)a_k}{(i+1)a_{i+2}}\right). \end{aligned} \tag{24}$$

Since the sequence $\left\{i\left[\frac{a_{i+1}}{a_i}-1\right]\right\}_{i=1}^{n+m}$ is increasing, we have, for $1 \leqslant i \leqslant n+m-1$ and $1 \leqslant k \leqslant i$, the following

$$\begin{aligned}
&\frac{(i+1)a_{i+2}}{a_{i+1}}-(i+1) \geqslant \frac{ia_{i+1}}{a_i}-i,\\
\Longleftrightarrow\quad &\frac{(i+1)a_{i+2}}{a_{i+1}}-(i+1) \geqslant \frac{ka_{k+1}}{a_k}-k,\\
\Longleftrightarrow\quad &\frac{ka_{k+1}+(i-k+1)a_k}{a_k} \leqslant \frac{(i+1)a_{i+2}}{a_{i+1}},\\
\Longleftrightarrow\quad &\frac{ka_{k+1}+(i-k+1)a_k}{(i+1)a_{i+2}} \leqslant \frac{a_k}{a_{i+1}}.
\end{aligned}$$

Combining the last line above with (24) yields

$$\frac{k}{i+1}\ln\frac{a_{k+1}}{a_{i+2}}+\frac{i-k+1}{i+1}\ln\frac{a_k}{a_{i+2}} \leqslant \ln\frac{a_k}{a_{i+1}}. \tag{25}$$

Summing up on both sides of (25) with k from 1 to i and simplifying reveals inequality (23). The monotonicity follows.

Since $\{a_i\}_{i=1}^{n+m+1}$ is a nonconstant positive sequence, there exists at least one number $1 \leqslant i_0 \leqslant n+m-1$ such that $a_{i_0} \neq a_{i_0+1}$. The function $\ln x$ is strictly concave on $(0,\infty)$. Then, for any i such that $i_0 \leqslant i \leqslant n+m-1$, we have

$$\begin{aligned}
&\frac{i_0}{i+1}\ln\frac{a_{i_0+1}}{a_{i+2}}+\frac{i-i_0+1}{i+1}\ln\frac{a_{i_0}}{a_{i+2}}\\
&\qquad< \ln\left(\frac{i_0}{i+1}\cdot\frac{a_{i_0+1}}{a_{i+2}}+\frac{i-i_0+1}{i+1}\cdot\frac{a_{i_0}}{a_{i+2}}\right)\\
&\qquad= \ln\left(\frac{i_0a_{i_0+1}+(i-i_0+1)a_{i_0}}{(i+1)a_{i+2}}\right)\\
&\qquad\leqslant \ln\frac{a_{i_0}}{a_{i+1}},
\end{aligned} \tag{26}$$

notice that the last line follows from the sequence $\left\{i\left[\frac{a_{i+1}}{a_i}-1\right]\right\}_{i=1}^{n+m}$ being increasing. Therefore, for any i such that $i_0 \leqslant i \leqslant n+m-1$, inequality (22) is strict. Inequality (21) is proved. The proof is complete. □

LEMMA 2. *Let $n>1$ be a positive integer and $\{a_i\}_{i=1}^{n}$ an increasing nonconstant positive sequence such that $\left\{i\left[\frac{a_{i+1}}{a_i}-1\right]\right\}_{i=1}^{n-1}$ is increasing. Then the sequence*

$$\left\{\frac{a_i}{(a_i!)^{1/i}}\right\}_{i=1}^{n} \tag{27}$$

is increasing, and, for any positive integer ℓ satisfying $1 \leqslant \ell < n$,

$$\frac{a_\ell}{a_n} < \frac{(a_\ell!)^{1/\ell}}{(a_n!)^{1/n}}, \tag{28}$$

where $a_n!$ denotes the sequence factorial $\prod_{i=1}^{n} a_i$.

Proof. For $1 \leqslant \ell \leqslant n-1$, the monotonicity of the sequence (27) is equivalent to

$$\begin{aligned}
& \frac{a_\ell}{(a_\ell!)^{1/\ell}} \leqslant \frac{a_{\ell+1}}{(a_{\ell+1}!)^{1/(\ell+1)}}, \\
\iff \quad & \left(\prod_{j=1}^{\ell} \frac{a_j}{a_\ell}\right)^{\frac{1}{\ell}} \geqslant \left(\prod_{j=1}^{\ell+1} \frac{a_j}{a_{\ell+1}}\right)^{\frac{1}{\ell+1}}, \\
\iff \quad & \frac{1}{\ell}\sum_{j=1}^{\ell-1} \ln\frac{a_j}{a_\ell} \geqslant \frac{1}{\ell+1}\sum_{j=1}^{\ell} \ln\frac{a_j}{a_{\ell+1}}, \\
\iff \quad & \sum_{j=1}^{\ell-1} \ln\frac{a_j}{a_\ell} \geqslant \frac{\ell}{\ell+1}\sum_{j=1}^{\ell} \ln\frac{a_j}{a_{\ell+1}}. \qquad (29)
\end{aligned}$$

Since $\ln x$ is concave on $(0,\infty)$, by definition of concaveness, it follows that, for $1 \leqslant j \leqslant \ell$,

$$\begin{aligned}
& \frac{j}{\ell+1}\ln\frac{a_{j+1}}{a_{\ell+1}} + \frac{\ell-j+1}{\ell+1}\ln\frac{a_j}{a_{\ell+1}} \\
& \qquad \leqslant \ln\left(\frac{j}{\ell+1}\cdot\frac{a_{j+1}}{a_{\ell+1}} + \frac{\ell-j+1}{\ell+1}\cdot\frac{a_j}{a_{\ell+1}}\right) \qquad (30) \\
& \qquad = \ln\left(\frac{ja_{j+1}+(\ell-j+1)a_j}{(\ell+1)a_{\ell+1}}\right).
\end{aligned}$$

Straightforward computation gives us

$$\begin{aligned}
& \sum_{j=1}^{\ell}\left[\frac{j}{\ell+1}\ln\frac{a_{j+1}}{a_{\ell+1}} + \frac{\ell-j+1}{\ell+1}\ln\frac{a_j}{a_{\ell+1}}\right] \\
& \quad = \frac{\ell}{\ell+1}\sum_{j=1}^{\ell}\ln\frac{a_j}{a_{\ell+1}} + \sum_{j=1}^{\ell}\left[\frac{j}{\ell+1}\ln\frac{a_{j+1}}{a_{\ell+1}}\right] - \sum_{j=1}^{\ell}\frac{j-1}{\ell+1}\ln\frac{a_j}{a_{\ell+1}} \qquad (31) \\
& \quad = \frac{\ell}{\ell+1}\sum_{j=1}^{\ell}\ln\frac{a_j}{a_{\ell+1}} + \sum_{j=2}^{\ell+1}\left[\frac{j-1}{\ell+1}\ln\frac{a_j}{a_{\ell+1}}\right] - \sum_{j=1}^{\ell}\frac{j-1}{\ell+1}\ln\frac{a_j}{a_{\ell+1}} \\
& \quad = \frac{\ell}{\ell+1}\sum_{j=1}^{\ell}\ln\frac{a_j}{a_{\ell+1}}.
\end{aligned}$$

From combining of (29), (30) and (31), it suffices to prove for $1 \leqslant j \leqslant \ell$

$$\begin{aligned} & \frac{ja_{j+1}+(\ell-j+1)a_j}{(\ell+1)a_{\ell+1}} \leqslant \frac{a_j}{a_\ell}, \\ \Longleftrightarrow \quad & \frac{ja_{j+1}+(\ell-j+1)a_j}{a_j} \leqslant \frac{(\ell+1)a_{\ell+1}}{a_\ell}, \\ \Longleftrightarrow \quad & \frac{ja_{j+1}}{a_j}+\ell-j+1 \leqslant \frac{(\ell+1)a_{\ell+1}}{a_\ell}, \\ \Longleftrightarrow \quad & (\ell+1)\left[\frac{a_{\ell+1}}{a_\ell}-1\right] \geqslant j\left[\frac{a_{j+1}}{a_j}-1\right]. \end{aligned} \tag{32}$$

Since the sequences $\{a_i\}_{i=1}^n$ and $\left\{i\left[\frac{a_{i+1}}{a_i}-1\right]\right\}_{i=1}^{n-1}$ are increasing, the inequality (32) holds.

Moreover, the sequence $\{a_i\}_{i=1}^n$ is nonconstant positive, then there exists at least one number $1 \leqslant i_1 \leqslant n-1$ such that $a_{i_1} \neq a_{i_1+1}$. The function $\ln x$ is strictly concave on $(0,\infty)$. Then, for any ℓ such that $i_1 < \ell \leqslant n-1$, we have

$$\begin{aligned} & \frac{i_1}{\ell+1}\ln\frac{a_{i_1+1}}{a_{\ell+1}}+\frac{\ell-i_1+1}{\ell+1}\ln\frac{a_{i_1}}{a_{\ell+1}} \\ & \quad < \ln\left(\frac{i_1}{\ell+1}\cdot\frac{a_{i_1+1}}{a_{\ell+1}}+\frac{\ell-i_1+1}{\ell+1}\cdot\frac{a_{i_1}}{a_{\ell+1}}\right) \qquad (33) \\ & \quad = \ln\left(\frac{i_1a_{i_1+1}+(\ell-i_1+1)a_{i_1}}{(\ell+1)a_{\ell+1}}\right) \\ & \quad \leqslant \ln\frac{a_{i_1}}{a_\ell}. \end{aligned}$$

Therefore, for any ℓ such that $i_1+1 \leqslant \ell < n$, inequality (29) is strict, and

$$\frac{a_\ell}{a_{\ell+1}} < \frac{(a_\ell!)^{1/\ell}}{(a_{\ell+1}!)^{1/(\ell+1)}}, \tag{34}$$

and then inequality (28) is strict. The proof is complete. □

REMARK 2. Some problems similar to Lemma 1 and Lemma 2 were discussed in [10, 25] by the author and B.-N. Guo.

The methods proving Lemma 1 and Lemma 2 had been used in [24] and others.

LEMMA 3. (König's inequality [2, p. 149], [11, p. 24] and [14, 30]) *Let* $\{a_i\}_{i=1}^n$ *and* $\{b_i\}_{i=1}^n$ *be decreasing nonnegative* n*-tuples such that*

$$\prod_{i=1}^k b_i \leqslant \prod_{i=1}^k a_i, \quad 1 \leqslant k \leqslant n, \tag{35}$$

then, for $r>0$, *we have*

$$\sum_{i=1}^k b_i^r \leqslant \sum_{i=1}^k a_i^r, \quad 1 \leqslant k \leqslant n. \tag{36}$$

The equality in (36) *is valid if and only if* $a_i = b_i$ *for all* $1 \leqslant i \leqslant n$.

3. Proofs of Theorem 1

Inequality (13) holds for $n = 1$ by the power mean inequality and its case of equality.

For $n \geqslant 2$, inequality (13) is equivalent to

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} a_i^r\right)^{1/r} < \frac{\sqrt[n]{a_n!}}{\sqrt[n+1]{a_{n+1}!}}, \tag{37}$$

which is equivalent to

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{a_i}{\sqrt[n]{a_n!}}\right)^r < \frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{a_i}{\sqrt[n+1]{a_{n+1}!}}\right)^r. \tag{38}$$

Set

$$b_{jn+1} = b_{jn+2} = \cdots = b_{jn+n} = \frac{a_{n+1-j}}{\sqrt[n+1]{a_{n+1}!}}, \quad 0 \leqslant j \leqslant n; \tag{39}$$

$$c_{j(n+1)+1} = c_{j(n+1)+2} = \cdots = c_{j(n+1)+(n+1)} = \frac{a_{n-j}}{\sqrt[n]{a_n!}}, \quad 0 \leqslant j \leqslant n-1. \tag{40}$$

Direct calculation yields

$$\begin{aligned}\sum_{i=1}^{n(n+1)} b_i^r &= \sum_{j=0}^{n}\sum_{k=1}^{n} b_{jn+k}^r \\ &= n\sum_{j=0}^{n}\left(\frac{a_{n+1-j}}{\sqrt[n+1]{a_{n+1}!}}\right)^r \\ &= n\sum_{i=1}^{n+1}\left(\frac{a_i}{\sqrt[n+1]{a_{n+1}!}}\right)^r\end{aligned} \tag{41}$$

and

$$\sum_{i=1}^{n(n+1)} c_i^r = (n+1)\sum_{i=1}^{n}\left(\frac{a_i}{\sqrt[n]{a_n!}}\right)^r. \tag{42}$$

Since $\{a_i\}_{i=1}^{n+1}$ is increasing, the sequence $\{b_i\}_{i=1}^{n(n+1)}$ and $\{c_i\}_{i=1}^{n(n+1)}$ are decreasing. Therefore, by Lemma 3, to obtain inequality (38), it is sufficient to prove inequality

$$b_m! \geqslant c_m! \tag{43}$$

for $1 \leqslant m \leqslant n(n+1)$.

It is easy to see that $b_{n(n+1)}! = c_{n(n+1)}! = 1$. Thus, inequality (43) is equivalent to

$$\prod_{i=m}^{n(n+1)} b_i \leqslant \prod_{i=m}^{n(n+1)} c_i \tag{44}$$

for $2 \leqslant m \leqslant n(n+1)$.

For $0 \leqslant \ell \leqslant n$ and $0 \leqslant j \leqslant n-2$, we have $2 \leqslant (n-\ell)n+(n-j) = (n-\ell)(n+1)+(\ell-j) \leqslant n(n+1)$. Then

$$\prod_{i=(n-\ell)n+(n-j)}^{n(n+1)} b_i = \frac{(a_{\ell+1})^{j+1}(a_\ell!)^n}{(a_{n+1}!)^{\frac{\ell n+j+1}{n+1}}}; \tag{45}$$

$$\prod_{i=(n-\ell)(n+1)+(\ell-j)}^{n(n+1)} c_i = \frac{(a_\ell)^{n-\ell+j+2}(a_{\ell-1}!)^{n+1}}{(a_n!)^{\frac{\ell n+j+1}{n}}}, \quad \ell > j; \tag{46}$$

$$\begin{aligned}\prod_{i=(n-\ell)(n+1)+(\ell-j)}^{n(n+1)} c_i &= \prod_{i=(n-\ell-1)(n+1)+(n+1+\ell-j)}^{n(n+1)} c_i \\ &= \frac{(a_{\ell+1})^{j-\ell+1}(a_\ell!)^{n+1}}{(a_n!)^{\frac{\ell n+j+1}{n}}}, \quad \ell \leqslant j;\end{aligned} \tag{47}$$

where $a_0 = 1$.

The last term in (47) is bigger than the right term in (46), so, without loss of generality, we can assume $j < \ell$. Therefore, from formulae (45) and (46), inequality (44) is reduced to

$$\frac{(a_{\ell+1})^{j+1}(a_\ell!)^n(a_{n+1}!)^{\frac{\ell-j-1}{n+1}}}{(a_{n+1}!)^\ell} \leqslant \frac{(a_\ell)^{n-\ell+j+2}(a_{\ell-1}!)^{n+1}}{(a_n!)^\ell(a_n!)^{\frac{j+1}{n}}}, \tag{48}$$

that is

$$\frac{(a_{\ell+1})^{j+1}(a_{n+1}!)^{\frac{\ell-j-1}{n+1}}}{(a_\ell!)(a_\ell)^{j-\ell+1}} \leqslant \frac{(a_{n+1})^\ell(a_n!)^{\frac{-\ell}{n}}}{(a_n!)^{\frac{j-\ell+1}{n}}}, \tag{49}$$

this is further equivalent to

$$\frac{(a_{\ell+1})^{j+1}(a_{n+1}!)^{\frac{\ell-j-1}{n+1}}}{a_\ell!(a_\ell)^{j-\ell+1}(a_n!)^{\frac{\ell-j-1}{n}}} \leqslant \frac{(a_{n+1})^\ell}{(a_n!)^{\frac{\ell}{n}}}, \tag{50}$$

which can be rearranged as

$$\left(\frac{a_{\ell+1}}{a_\ell} \cdot \frac{\sqrt[n]{a_n!}}{\sqrt[n+1]{a_{n+1}!}}\right)^{\frac{j+1}{\ell}} \leqslant \frac{\sqrt[\ell]{a_\ell!}}{a_\ell} \cdot \frac{a_{n+1}}{\sqrt[n+1]{a_{n+1}!}}, \quad j+1 \leqslant \ell \leqslant n. \tag{51}$$

Utilizing Lemma 2 and the logarithmical concaveness of the sequence $\{a_i\}_{i=1}^{n+1}$ yields

$$\frac{\sqrt[n]{a_n!}}{\sqrt[n+1]{a_{n+1}!}} > \frac{a_n}{a_{n+1}} \geqslant \frac{a_\ell}{a_{\ell+1}}. \tag{52}$$

Since $\frac{j+1}{\ell} \leqslant 1$ and $\frac{a_{\ell+1}}{a_\ell} \cdot \frac{\sqrt[n]{a_n!}}{\sqrt[n+1]{a_{n+1}!}} > 1$ by (52), thus, to obtain (51), it suffices to prove

$$a_{\ell+1}\sqrt[n]{a_n!} < a_{n+1}\sqrt[\ell]{a_\ell!}, \tag{53}$$

this follows from Lemma 1.

Since the sequence $\{a_i\}_{i=1}^{n+1}$ is increasing and nonconstant, then it is a well known fact that $\sqrt[n]{a_n!} < \sqrt[n+1]{a_{n+1}!}$ which is equivalent to $b_{n(n+1)} < c_{n(n+1)}$. This implies that inequality (37), and then inequality (13), from Lemma 3, is strict.

By L'Hospital rule, easy calculation produces

$$\lim_{r\to 0}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \Big/ \frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r\right)^{1/r} = \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}}, \tag{54}$$

thus, the upper bound is the best possible. The proof is complete.

REMARK 3. Recently, some new inequalities for the ratios of the mean values of functions were established in [33].

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