Journal of Mathematical Inequalities
Volume 1, Number 4 (2007), 529–534

ON THE QUASI–MONOTONE AND ALMOST INCREASING SEQUENCES

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(*communicated by G. Toader*)

Abstract. In this paper, a theorem of Bor and Özarslan [3] dealing with $\mid C,\alpha;\beta\mid_k$ summability factors has been generalized for $\mid C,\alpha,\gamma;\beta\mid_k$ summability methods.

1. Introduction

We will use the following notations and notions in our paper:

If $g>0$, then $f=O(g)$ means that $\mid f\mid< K.g$, for some constant $K>0$ (see [7]). Let (u_n) be a sequence. We write that $\Delta u_n=u_n-u_{n+1}$, $\Delta^0 u_n=u_n$ and $\Delta^k u_n=\Delta\Delta^{k-1}u_n$, for $k=1,2,\ldots$, (see [7]).

Abel's transformation ([8]): Let (a_k), (b_k) be complex sequences, and write $S_n=a_1+a_2+\ldots+a_n$. Then

$$\sum_{k=1}^{n} a_k b_k=\sum_{k=1}^{n-1} S_k\Delta b_k+S_n b_n. \tag{1}$$

Hölder's inequality ([8]): If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $a_1,a_2,a_3,\ldots,a_n\geqslant 0$; $b_1,b_2,b_3,\ldots,b_n\geqslant 0$, then

$$\sum_{k=1}^{n} a_k b_k\leqslant\left(\sum_{k=1}^{n} a_k^p\right)^{1/p}\left(\sum_{k=1}^{n} b_k^q\right)^{1/q}. \tag{2}$$

A sequence (b_n) of positive numbers is said to be δ-quasi-monotone, if $b_n\to 0$, $b_n>0$ ultimately and $\Delta b_n\geqslant-\delta_n$, where (δ_n) is a sequence of positive numbers (see [2]). A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence c_n and two positive constants A and B such that $Ac_n\leqslant b_n\leqslant Bc_n$ (see [1]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by σ_n^α and t_n^α the n-th Cesàro means of order α, with $\alpha>-1$, of the sequence (s_n) and (na_n), respectively, i.e.,

$$\sigma_n^\alpha=\frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1}s_v \tag{3}$$

Mathematics subject classification (2000): 40D15, 40F05, 40G05.
Key words and phrases: Absolute summability, almost increasing and quasi-monotone sequences.

Paper JMI-01-44

$$t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{4}$$

where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad \text{and} \quad A_{-n}^\alpha = 0 \quad for \quad n > 0. \tag{5}$$

The series $\sum a_n$ is said to be summable $\mid C,\alpha \mid_k$, $k \geqslant 1$ and $\alpha > -1$, if (see [5])

$$\sum_{n=1}^{\infty} n^{k-1} \mid \sigma_n^\alpha - \sigma_{n-1}^\alpha \mid^k = \sum_{n=1}^{\infty} \frac{1}{n} \mid t_n^\alpha \mid^k < \infty \tag{6}$$

and it is said to be summable $\mid C,\alpha;\beta \mid_k$, $k \geqslant 1$, $\alpha > -1$ and $\beta \geqslant 0$, if (see [6])

$$\sum_{n=1}^{\infty} n^{\beta k+k-1} \mid \sigma_n^\alpha - \sigma_{n-1}^\alpha \mid^k = \sum_{n=1}^{\infty} n^{\beta k-1} \mid t_n^\alpha \mid^k < \infty. \tag{7}$$

The series $\sum a_n$ is said to be summable $\mid C,\alpha,\gamma;\beta \mid_k$, $k \geqslant 1$ and $\alpha > -1$, $\delta \geqslant 0$ and γ is a real number, if (see [9])

$$\sum_{n=1}^{\infty} n^{\gamma(\beta k+k-1)-k} \mid t_n^\alpha \mid^k < \infty. \tag{8}$$

If we take $\gamma = 1$, then $\mid C,\alpha,\gamma;\beta \mid_k$ summability reduces to $\mid C,\alpha;\beta \mid_k$ summability.
Bor and Özarslan [3] have proved the following theorem for $\mid C,\alpha;\beta \mid_k$ summability factors.

THEOREM A. *Let (X_n) be an almost increasing sequence such that $\mid \Delta X_n \mid = O(\frac{X_n}{n})$ and $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (B_n) such that it is δ-quasi-monotone with $\sum n X_n \delta_n < \infty$, $\sum B_n X_n$ is convergent and $\mid \Delta\lambda_n \mid \leqslant \mid B_n \mid$ for all n. If the sequence (u_n^α), defined by (see [10])*

$$u_n^\alpha = \begin{cases} \mid t_n^\alpha \mid, & \alpha = 1 \\ \max\limits_{1 \leqslant v \leqslant n} \mid t_v^\alpha \mid, & 0 < \alpha < 1 \end{cases} \tag{9}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{\beta k-1} (u_n^\alpha)^k = O(X_m) \quad as \quad m \to \infty, \tag{10}$$

then the series $\sum a_n \lambda_n$ is summable $\mid C,\alpha;\beta \mid_k$, $k \geqslant 1$ and $0 \leqslant \beta < \alpha \leqslant 1$.

2. The main result

The aim of this paper is to generalize Theorem A for $\mid C,\alpha,\gamma;\beta\mid_k$ summability factors. We shall prove the following theorem.

THEOREM. *Let* (X_n) *be an almost increasing sequence such that* $\mid \Delta X_n \mid = O(\frac{X_n}{n})$ *and* $\lambda_n \to 0$ *as* $n \to \infty$. *Suppose that there exists a sequence of numbers* (B_n) *such that it is* δ*-quasi-monotone with* $\sum nX_n\delta_n < \infty$, $\sum B_nX_n$ *is convergent and* $\mid \Delta\lambda_n \mid \leqslant \mid B_n \mid$ *for all n. If the sequence* (u_n^α), *defined by (9) satisfies the condition*

$$\sum_{n=1}^{m} n^{\gamma(\beta k+k-1)-k}(u_n^\alpha)^k = O(X_m) \quad as \quad m \to \infty, \tag{11}$$

then the series $\sum a_n\lambda_n$ *is summable* $\mid C,\alpha,\gamma;\beta\mid_k$, *where* $k \geqslant 1$, $\beta \geqslant 0$, $0 < \alpha \leqslant 1$ *and* γ *is a real number such that* $k+\alpha k-\gamma(\beta k+k-1) > 1$.

We need the following lemmas for the proof of our theorem.

LEMMA 1. ([4]) *If* $0<\alpha\leqslant 1$ *and* $1\leqslant v\leqslant n$, *then*

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1}a_p \mid \leqslant \max_{1\leqslant m\leqslant v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1}a_p \mid. \tag{12}$$

LEMMA 2. ([3]) *Under the conditions regarding* (λ_n) *and* (X_n) *of the Theorem, we have*

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{13}$$

LEMMA 3. ([3]) *Under the conditions pertaining to* (X_n) *and* (B_n) *of the Theorem, we have that*

$$nB_nX_n = O(1) \tag{14}$$

$$\sum_{n=1}^{\infty} nX_n \mid \Delta B_n \mid < \infty. \tag{15}$$

3. Proof of the Theorem

Let (T_n^α) be the n-th (C,α) mean of the sequence $(na_n\lambda_n)$. Then, by (4) we have

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v. \tag{16}$$

Using Abel's transformation, we get that

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta\lambda_v\sum_{p=1}^{v}A_{n-p}^{\alpha-1}pa_p + \frac{\lambda_n}{A_n^\alpha}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}va_v,$$

so that making use of Lemma 1, we have

$$\begin{aligned} \mid T_n^\alpha \mid &\leqslant \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid\mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^\alpha} \mid \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \mid \\ &\leqslant \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha \mid \Delta\lambda_v \mid + \mid \lambda_n \mid w_n^\alpha \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha, \quad \text{say.} \end{aligned}$$

Since

$$\mid T_{n,1}^\alpha + T_{n,2}^\alpha \mid^k \leqslant 2^k(\mid T_{n,1}^\alpha \mid^k + \mid T_{n,2}^\alpha \mid^k),$$

to complete the proof of the Theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{\gamma(\beta k+k-1)-k} \mid T_{n,r}^\alpha \mid^k < \infty \quad \text{for} \quad r=1,2, \quad \text{by} \quad (8).$$

Now, when $k>1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get that

$$\begin{aligned} \sum_{n=2}^{m+1} n^{\gamma(\beta k+k-1)-k} \mid T_{n,1}^\alpha \mid^k &\leqslant \sum_{n=2}^{m+1} n^{\gamma(\beta k+k-1)-k} \left\{ \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} A_v^\alpha u_v^\alpha \mid \Delta\lambda_v \mid \right\}^k \\ &= O(1) \sum_{n=2}^{m+1} n^{\gamma(\beta k+k-1)-k-\alpha k} \left\{ \sum_{v=1}^{n-1} v^{\alpha k} (u_v^\alpha)^k \mid B_v \mid \right\} \\ &\quad \times \left\{ \sum_{v=1}^{n-1} \mid B_v \mid \right\}^{k-1} \\ &= O(1) \sum_{v=1}^{m} v^{\alpha k} (u_v^\alpha)^k \mid B_v \mid \sum_{n=v+1}^{m+1} \frac{1}{n^{k+\alpha k-\gamma(\beta k+k-1)}} \\ &= O(1) \sum_{v=1}^{m} v^{\alpha k} (u_v^\alpha)^k \mid B_v \mid \int_v^\infty \frac{dx}{x^{\alpha k+k-\gamma(\beta k+k-1)}} \\ &= O(1) \sum_{v=1}^{m} v \mid B_v \mid v^{\gamma(\beta k+k-1)-k} (u_v^\alpha)^k \\ &= O(1) \sum_{v=1}^{m-1} \Delta(v \mid B_v \mid) \sum_{r=1}^{v} r^{\gamma(\beta k+k-1)-k} (u_r^\alpha)^k \\ &\quad + O(1) m \mid B_m \mid \sum_{v=1}^{m} v^{\gamma(\beta k+k-1)-k} (u_v^\alpha)^k \\ &= O(1) \sum_{v=1}^{m-1} \mid \Delta(v \mid B_v \mid) \mid X_v + O(1) m \mid B_m \mid X_m \end{aligned}$$

$$= O(1)\sum_{v=1}^{m-1} v \mid \Delta B_v \mid X_v + O(1)\sum_{v=1}^{m-1} \mid B_{v+1} \mid X_{v+1}$$
$$+O(1)m \mid B_m \mid X_m = O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of the hypotheses of the Theorem and Lemma 3.

Again, since $\mid \lambda_n \mid = O(1/X_n) = O(1)$ by (13), we have that

$$\begin{aligned}
\sum_{n=1}^{m} n^{\gamma(\beta k+k-1)-k} \mid T_{n,2}^{\alpha} \mid^k &= \sum_{n=1}^{m} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid n^{\gamma(\beta k+k-1)-k} (u_n^{\alpha})^k \\
&= O(1)\sum_{n=1}^{m} \mid \lambda_n \mid n^{\gamma(\beta k+k-1)-k} (u_n^{\alpha})^k \\
&= O(1)\sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} v^{\gamma(\beta k+k-1)-k} (u_v^{\alpha})^k \\
&\quad +O(1) \mid \lambda_m \mid \sum_{n=1}^{m} n^{\gamma(\beta k+k-1)-k} (u_n^{\alpha})^k \\
&= O(1)\sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m \\
&= O(1)\sum_{n=1}^{m-1} \mid B_n \mid X_n + O(1) \mid \lambda_m \mid X_m = O(1) \\
&\qquad \text{as} \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem, Lemma 2 and Lemma 3.

Therefore, we get that

$$\sum_{n=1}^{m} n^{\gamma(\beta k+k-1)-k} \mid T_{n,r}^{\alpha} \mid^k = O(1) \quad \text{as} \quad m \to \infty, \quad \text{for} \quad r = 1, 2.$$

This completes the proof of the Theorem.

If we take $\gamma = 1$, then we get Theorem A. In this case condition (11) reduces to condition (10). Also if we take $\gamma = 1$ and $\beta = 0$, then we have a new result concerning $\mid C, \alpha \mid_k$ summability factors. Finally if we take $\gamma = 1$, $\beta = 0$ and $\alpha = 1$, then we obtain a new result related to $\mid C, 1 \mid_k$ summability factors.

Acknowledgement. The authors are very grateful to the referee for his/her invaluable suggestions.

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(Received July 28, 2007)

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