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CERTAIN SUBCLASSES OF MULTIVALENT PRESTARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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(*communicated by N. Elezović*)

Abstract. The object of the present paper is to investigate coefficient estimates for functions belonging to the subclasses $R_{\gamma}^{p}[\alpha,\beta]$ and $C_{\gamma}^{p}[\alpha,\beta]$ of p-valent γ-prestarlike functions of order α and type β with negative coefficients. We obtain extreme points, distortion theorems, integral operators and radii of starlikeness and convexity for functions belonging to the classes $R_{\gamma}^{p}[\alpha,\beta]$ and $C_{\gamma}^{p}[\alpha,\beta]$. We also obtain several results for the modified Hadamard products of functions belonging to the classes $R_{\gamma}^{p}[\alpha,\beta]$ and $C_{\gamma}^{p}[\alpha,\beta]$.

1. Introduction

Let $A(p)$ denote the class of functions of the form :

$$f(z)=z^{p}+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}\quad (p\in N=\{1,2,...\}) \tag{1.1}$$

which are analytic and p-valent in the unit disc $U=\{z:|z|<1\}$. A function $f(z)\in A(p)$ is called p-valent starlike of order α and type β if it satisfies

$$\left|\frac{\frac{zf'(z)}{f(z)}-p}{\frac{zf'(z)}{f(z)}+p-2\alpha}\right|<\beta\quad (z\in U), \tag{1.2}$$

where $0\leqslant\alpha<p$, $0<\beta\leqslant 1$ and $p\in N$. We denote by $S^{*}(p,\alpha,\beta)$ the class of p-valent starlike functions of order α and type β. A function $f(z)\in A(p)$ is called p-valent convex of order α and type β if it satisfies

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}-\alpha}{1+\frac{zf''(z)}{f'(z)}+p-2\alpha}\right|<\beta\quad (z\in U), \tag{1.3}$$

where $0\leqslant\alpha<p$, $0<\beta\leqslant 1$ and $p\in N$. Also we denote by $C(p,\alpha,\beta)$ the class of p-valent convex functions of order α and type β. From (1.2) and (1.3), we note that

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$$f(z) \in C(p,\alpha,\beta) \quad \text{if and only if} \quad \frac{zf'(z)}{p} \in S^*(p,\alpha,\beta). \tag{1.4}$$

The classes $S^*(p,\alpha,\beta)$ and $C(p,\alpha,\beta)$ were considered by Aouf [2] and Hossen [7]. For $\beta = 1$, the classes $S^*(p,\alpha,1) = S^*(p,\alpha)$ and $C(p,\alpha,1) = C(p,\alpha)$ were studied by Patil and Thakare [11] and Owa [9], respectively.

The function

$$s_\gamma^p(z) = \frac{z^p}{(1-z)^{2(p-\gamma)}} \quad (0 \leqslant \gamma < p;\ p \in N) \tag{1.5}$$

is the familiar extremal function for the class $S^*(p,\gamma)$. Setting

$$G^p(\gamma,n) = \frac{\prod_{m=2}^{n}[2(p-\gamma)+m-2]}{(n-1)!} \quad (n \in N \setminus \{1\};\ 0 \leqslant \gamma < p), \tag{1.6}$$

$s_\gamma^p(z)$ can be written in the form :

$$s_\gamma^p(z) = z^p + \sum_{n=1}^{\infty} G^p(\gamma,n+1) z^{p+n}. \tag{1.7}$$

Clearly, $s_\gamma^p(z) \in S^*(p,\gamma)$ and $G^p(\gamma,n+1)$ is a decreasing function in $\gamma(0 \leqslant \gamma \leqslant \frac{2p-1}{2};\ p \in N)$ and satisfies

$$\lim_{n\to\infty} G^p(\gamma,n+1) = \begin{cases} \infty & (\gamma < \frac{2p-1}{2}) \\ 1 & (\gamma = \frac{2p-1}{2}) \\ 0 & (\gamma > \frac{2p-1}{2}). \end{cases}$$

Let $(f*g)(z)$ denote the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \tag{1.8}$$

then

$$(f*g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}. \tag{1.9}$$

A function $f(z) \in A(p)$ is said to be p-valent γ-prestarlike function of order α and type $\beta\,(0 \leqslant \gamma < p;\ 0 \leqslant \alpha < p;\ 0 < \beta \leqslant 1;\ p \in N)$ if

$$(f * s_\gamma^p)(z) \in S^*(p,\alpha,\beta), \tag{1.10}$$

where $s_\gamma^p(z)$ is defined by (1.5). We denote by $R_\gamma^p(\alpha,\beta)$ the class of all p-valent γ-prestarlike functions of order α and type β. For $\gamma = \frac{2p-1}{2};\ 0 \leqslant \alpha < p;\ 0 < \beta \leqslant$

1; $p \in N$, $R^p_{\frac{2p-1}{2}}(\alpha,\beta) = S^*(p,\alpha,\beta)$. Further let $C^p_\gamma(\alpha,\beta)$ be the subclass of $A(p)$ consisting of functions $f(z)$ satisfying

$$f(z) \in C^p_\gamma(\alpha,\beta) \quad \text{if and only if} \quad \frac{zf'(z)}{p} \in R^p_\gamma(\alpha,\beta). \tag{1.11}$$

We note that :

(i) $R^p_\gamma(\alpha,1) = R^p(\gamma,\alpha)$, is the class of p-valently γ-prestarlike functions of order α (see Aouf and Silverman [5]) and $C^p_\gamma(\alpha,1) = C^p(\gamma,\alpha)$, consisting of functions $f(z) \in A(p)$ satisfying $\frac{zf'(z)}{p} \in R^p(\gamma,\alpha)$ (see Aouf and Silverman [5]);

(ii) $R^p_\gamma(\gamma,1) = R^p(\gamma)$ $(0 \leqslant \gamma < 1;\ p \in N)$, is the class of p-valently prestarlike functions of order γ (see Kumar and Reddy [8] and Shenen et al. [14]);

(iii) $R^1_\gamma(\alpha,\beta) = R_\gamma(\alpha,\beta)$ $(0 \leqslant \gamma < 1;\ 0 \leqslant \alpha < 1;\ 0 < \beta \leqslant 1)$, is the class of γ-prestarlike functions of order α and type β (see Ahuja and Silverman [1]);

(iv) $C^1_\gamma(\alpha,1) = C(\gamma,\alpha)$ $(0 \leqslant \gamma < 1;\ 0 \leqslant \alpha < 1)$, is the subclass of $A(1) = A$ consisting of functions $f(z) \in A$ satisfying $zf'(z) \in R^1_\gamma(\alpha,1) = R(\gamma,\alpha)$ (see Owa and Uralegaddi [10]).

Denoting by $T(p)$ the subclass of $A(p)$ consisting of functions of the form:

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_{p+n} \geqslant 0;\ p \in N). \tag{1.12}$$

We denote by $S^*[p,\alpha,\beta]$, $C[p,\alpha,\beta]$, $R^p_\gamma[\alpha,\beta]$ and $C^p_\gamma[\alpha,\beta]$ the classes obtained by taking intersections, respectively, of the classes $S^*(p,\alpha,\beta)$, $C(p,\alpha,\beta)$, $R^p_\gamma(\alpha,\beta)$ and $C^p_\gamma(\alpha,\beta)$ with the class $T(p)$. Thus, we have

$$S^*[p,\alpha,\beta] = S^*(p,\alpha,\beta) \cap T(p), \tag{1.13}$$

$$C[p,\alpha,\beta] = C(p,\alpha,\beta) \cap T(p), \tag{1.14}$$

$$R^p_\gamma[\alpha,\beta] = R^p_\gamma(\alpha,\beta) \cap T(p), \tag{1.15}$$

and

$$C^p_\gamma[\alpha,\beta] = C^p_\gamma(\alpha,\beta) \cap T(p). \tag{1.16}$$

The classes $S^*[p,\alpha,\beta]$ and $C[p,\alpha,\beta]$ were studied by Aouf [2] and Hossen [7].

It follows from (1.15) and (1.16) that

$$f(z) \in C^p_\gamma[\alpha,\beta] \quad \text{if and only if} \quad \frac{zf'(z)}{p} \in R^p_\gamma[\alpha,\beta]. \tag{1.17}$$

Also we note that, by specializing the parameters γ,α,β and p, we obtain the following subclasses studied by various authors :

(i) $R^p_\gamma[\alpha,1] = R^p[\gamma,\alpha]$ and $C^p_\gamma[\alpha,1] = C^p[\gamma,\alpha]$ $(0 \leqslant \gamma < 1;\ 0 \leqslant \alpha < p; p \in N)$ (Aouf and Silverman [5]);

(ii) $R_{p\gamma}^{p}[p\gamma,1]=R^{p}[\gamma]$ $(0\leqslant\gamma<1;\ p\in N)$ (Kumar and Reddy [8]);

(iii) $R_{\gamma}^{1}[\alpha,\beta]=R_{\gamma}[\alpha,\beta]$ $(0\leqslant\gamma<1;\ 0\leqslant\alpha<1;\ 0<\beta\leqslant1)$ (Ahuja and Silverman [1]);

(iv) $R_{\gamma}^{1}[\alpha,1]=R[\gamma,\alpha]$ $(0\leqslant\gamma<1;\ 0\leqslant\alpha<1)$ (Aouf et al. [3], Aouf and Salagean [4], Raina and Srivastava [12], Silverman and Silvia [15], Srivastava and Aouf [16] and Uralegaddi and Sarangi [17]);

(v) $C_{\gamma}^{1}[\alpha,1]=C[\gamma,\alpha]$ $(0\leqslant\gamma<1;\ 0\leqslant\alpha<1)$ (Owa and Uralegaddi [10]).

In the present paper we investigate coefficient estimates for functions belonging to the subclasses $R_{\gamma}^{p}[\alpha,\beta]$ and $C_{\gamma}^{p}[\alpha,\beta]$ of p-valent γ-prestarlike functions of order α and type β with negative coefficients. We obtain extreme points, integral operators, radii of starlikeness and convexity and distortion theorems for functions belonging to the classes $R_{\gamma}^{p}[\alpha,\beta]$ and $C_{\gamma}^{p}[\alpha,\beta]$. We also obtain several results for the modified Hadamard products of functions belonging to the classes $R_{\gamma}^{p}[\alpha,\beta]$ and $C_{\gamma}^{p}[\alpha,\beta]$.

2. Coefficient inequalities

We need the following necessary and sufficient coefficient condition for $f(z)$ to be in the class $S^{*}[p,\alpha,\beta]$.

LEMMA 1. ([2]) *If f(z) is given by (1.12), then $f(z)\in S^{*}[p,\alpha,\beta]$ if and only if*

$$\sum_{n=1}^{\infty}[n+\beta(n+2p-2\alpha)]a_{p+n}\leqslant 2\beta(p-\alpha)\,. \tag{2.1}$$

If $f(z)\in T(p)$ and $s_{\gamma}^{p}(z)$ is given by (1.5), then it follows that

$$(f*s_{\gamma}^{p})(z)=z^{p}-\sum_{n=1}^{\infty}G^{p}(\alpha,n+1)a_{p+n}z^{p+n}. \tag{2.2}$$

In view of (2.2), our first result immediately follows from Lemma 1.

THEOREM 1. *Let the function $f(z)$ be defined by (1.12). Then $f(z)$ is in the class $R_{\gamma}^{p}[\alpha,\beta]$ if and only if*

$$\sum_{n=1}^{\infty}[n+\beta(n+2p-2\alpha)]G^{p}(\gamma,n+1)a_{p+n}\leqslant 2\beta(p-\alpha)\,. \tag{2.3}$$

COROLLARY 1. *If $f(z)$ is in the class $R_{\gamma}^{p}[\alpha,\beta]$, then*

$$a_{p+n}\leqslant\frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]G^{p}(\gamma,n+1)}\quad(p,n\in N)\,, \tag{2.4}$$

with equality for

$$f(z)=z^{p}-\frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]G^{p}(\gamma,n+1)}z^{p+n}\quad(p,n\in N)\,. \tag{2.5}$$

In view of (1.17), Theorem 1 yields the following necessary and sufficient condition for $f(z)$ to be in the class $C_\gamma^p[\alpha,\beta]$.

THEOREM 2. *The function $f(z)$, defined by (1.12), is in the class $C_\gamma^p[\alpha,\beta]$ if and only if*

$$\sum_{n=1}^{\infty}(\frac{p+n}{p})[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)a_{p+n} \leqslant 2\beta(p-\alpha)\,. \tag{2.6}$$

COROLLARY 2. *If $f(z)$ is in the class $C_\gamma^p[\alpha,\beta]$, then*

$$a_{p+n} \leqslant \frac{2\beta p(p-\alpha)}{(p+n)[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)} \quad (p,n\in N)\,, \tag{2.7}$$

with equality for

$$f(z) = z^p - \frac{2\beta p(p-\alpha)}{(p+n)[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}z^{p+n} \quad (p,n\in N)\,. \tag{2.8}$$

3. Extreme points

From Theorem 1 and Theorem 2, we see that both $R_\gamma^p[\alpha,\beta]$ and $C_\gamma^p[\alpha,\beta]$ are closed under convex linear combinations, which enables us to determine the extreme points for these classes.

THEOREM 3. *Let*

$$f_p(z) = z^p \tag{3.1}$$

and

$$f_{p+n}(z) = z^p - \frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}z^{p+n} \quad (p,n\in N). \tag{3.2}$$

Then $f(z)\in R_\gamma^p[\alpha,\beta]$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty}\mu_{p+n}f_{p+n}(z)\,, \tag{3.3}$$

where $\mu_{p+n}\geqslant 0$ and $\sum_{n=0}^{\infty}\mu_{p+n}=1$.

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \mu_{p+n} f_{p+n}(z) \\ &= z^p - \sum_{n=1}^{\infty} \frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)} \mu_{p+n} z^{p+n} \,. \end{aligned} \tag{3.4}$$

Then it follows that

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}{2\beta(p-\alpha)} \cdot \frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)} \mu_{p+n} \\ &= \sum_{n=1}^{\infty} \mu_{p+n} = 1 - \mu_p \leqslant 1 \,. \end{aligned} \tag{3.5}$$

Therefore, by Theorem 1, $f(z) \in R_{\gamma}^{p}[\alpha,\beta]$.

Conversely, assume that the function $f(z)$ defined by (1.12) belongs to the class $R_{\gamma}^{p}[\alpha,\beta]$. Then

$$a_{p+n} \leqslant \frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)} \quad (p,n \in N) \,. \tag{3.6}$$

Setting

$$\mu_{p+n} = \frac{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}{2\beta(p-\alpha)} a_{p+n} \quad (p,n \in N) \tag{3.7}$$

and

$$\mu_p = 1 - \sum_{n=1}^{\infty} \mu_{p+n} \,, \tag{3.8}$$

we see that $f(z)$ can be expressed in the form (3.3). This completes the proof of Theorem 3. □

COROLLARY 3. *The extreme points of the class* $R_{\gamma}^{p}[\alpha,\beta]$ *are the functions* $f_p(z) = z^p$ *and*

$$f_{p+n}(z) = z^p - \frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)} z^{p+n} \quad (p,n \in N) \,.$$

Similarly, we have

THEOREM 4. *Let*

$$f_p(z) = z^p \tag{3.9}$$

and

$$f_{p+n}(z) = z^p - \frac{2\beta(p-\alpha)}{(\frac{p+n}{p})[n+\beta(n+2p-2\alpha)]G^p(\gamma, n+1)} z^{p+n} \quad (p, n \in N)\,. \tag{3.10}$$

Then $f(z) \in C_\gamma^p[\alpha, \beta]$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=0}^{\infty} \mu_{p+n} f_{p+n}(z)\,, \tag{3.11}$$

where $\mu_{p+n} \geqslant 0$ *and* $\sum_{n=0}^{\infty} \mu_{p+n} = 1$.

COROLLARY 4. *The extreme points of the class* $C_\gamma^p[\alpha, \beta]$ *are the functions* $f_p(z) = z^p$ *and*

$$f_{p+n}(z) = z^p - \frac{2\beta(p-\alpha)}{(\frac{p+n}{p})[n+\beta(n+2p-2\alpha)]G^p(\gamma, n+1)} z^{p+n} \quad (p, n \in N)\,.$$

4. Distortion theorems

In view of Theorems 3 and 4, using the technique used earlier by Aouf and Silverman [5], we will obtain distortion theorems for the classes $R_\gamma^p[\alpha, \beta]$ and $C_\gamma^p[\alpha, \beta]$.

LEMMA 2. *For* $0 \leqslant \gamma \leqslant \frac{2p-1}{2}$, $0 \leqslant \alpha < p$, $0 < \beta \leqslant 1$ *and* $p, n \in N$, *then* $[n+\beta(n+2p-2\alpha)]G^p(\gamma, n+1)$ *is an increasing function of* n , *where* $G^p(\gamma, n+1)$ *is defined by* (1.6).

Proof. Let $K(\gamma, \alpha, \beta, n, p) = [n+\beta(n+2p-2\alpha)]G^p(\gamma, n+1)$. Since,

$$G^p(\gamma, n+2) = \frac{2p+n-2\gamma}{n+1}\, G^p(\gamma, n+1), \tag{4.1}$$

we can see that $K(\gamma, \alpha, \beta, n+1, p) \geqslant K(\gamma, \alpha, \beta, n, p)$ if and only if

$$2(p-\gamma)[n+1+\beta(n+1+2p-2\alpha)] - 2\beta(p-\alpha) \geqslant 0\,, \tag{4.2}$$

for $0 \leqslant \gamma \leqslant \frac{2p-1}{2}$, $0 \leqslant \alpha < p$ and $0 < \beta \leqslant 1$ which holds for $p, n \in N$. This completes the proof of Lemma 2. □

In the remainder of this section, we assume that $f(z)$ is defined by (1.12), $0 \leqslant \gamma \leqslant \frac{2p-1}{2}$, $0 \leqslant \alpha < p$, $0 < \beta \leqslant 1$ and $p \in N$.

THEOREM 5. *If* $f(z)$ *is in the class* $R_\gamma^p[\alpha, \beta]$, *then*

$$\begin{aligned} |z|^p - \frac{\beta(p-\alpha)}{[1+\beta(1+2p-2\alpha)](p-\gamma)}|z|^{p+1} \\ \leqslant |f(z)| \leqslant |z|^p + \frac{\beta(p-\alpha)}{[1+\beta(1+2p-2\alpha)](p-\gamma)}|z|^{p+1} \quad (z \in U)\,. \end{aligned} \tag{4.3}$$

Equality holds for the function $f_{p+1}(z)$ *given by*

$$f_{p+1}(z) = z^p - \frac{\beta(p-\alpha)}{[1+\beta(1+2p-2\alpha)](p-\gamma)} z^{p+1} \quad (z \in U). \tag{4.4}$$

Proof. By virtue of Theorem 3, we note that

$$\begin{aligned} |z|^p - \max_{n \in N} \frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]G^p(\gamma, n+1)} |z|^{p+n} \\ \leqslant |f(z)| \leqslant |z|^p + \max_{n \in N} \frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]G^p(\gamma, n+1)} |z|^{p+n}. \end{aligned} \tag{4.5}$$

From Lemma 2, we see that the max in (4.5) occurs when $n = 1$. This completes the proof of Theorem 5. □

THEOREM 6. *If* $f(z)$ *is in the class* $R^p_\gamma[\alpha, \beta]$, *then*

$$\begin{aligned} p|z|^{p-1} - \frac{\beta(p+1)(p-\alpha)}{[1+\beta(1+2p-2\alpha)](p-\gamma)} |z|^p \\ \leqslant |f'(z)| \leqslant p|z|^{p-1} + \frac{\beta(p+1)(p-\alpha)}{[1+\beta(1+2p-2\alpha)](p-\gamma)} |z|^p \quad (z \in U). \end{aligned} \tag{4.6}$$

Equality holds for $f_{p+1}(z)$ *given by (4.4).*

Proof. We know that

$$\begin{aligned} p|z|^{p-1} - \max_{n \in N} \frac{2\beta(p-\alpha)(p+n)}{[n+\beta(n+2p-2\alpha)]G^p(\gamma, n+1)} |z|^{p+n-1} \leqslant |f'(z)| \\ \leqslant p|z|^{p-1} + \max_{n \in N} \frac{2\beta(p-\alpha)(p+n)}{[n+\beta(n+2p-2\alpha)]G^p(\gamma, n+1)} |z|^{p+n-1} \quad (z \in U). \end{aligned} \tag{4.7}$$

From Lemma 2, we see that the max in (4.7) occurs when $n = 1$. This completes the proof of Theorem 6. □

THEOREM 7. *If* $f(z)$ *is in the class* $C^p_\gamma[\alpha, \beta]$, *then*

$$|f(z)| \geqslant |z|^p - \frac{\beta(p-\alpha)}{(\frac{p+1}{p})[1+\beta(1+2p-2\alpha)](p-\gamma)} |z|^{p+1} \tag{4.8}$$

and

$$|f(z)| \leqslant |z|^p + \frac{\beta(p-\alpha)}{(\frac{p+1}{p})[1+\beta(1+2p-2\alpha)](p-\gamma)} |z|^{p+1} \tag{4.9}$$

for $z \in U$. *The results are sharp for the function* $f(z)$ *given by*

$$f(z) = z^p - \frac{\beta(p-\alpha)}{(\frac{p+1}{p})[1+\beta(1+2p-2\alpha)](p-\gamma)} z^{p+1} \quad (z \in U). \tag{4.10}$$

Proof. From Theorem 4, we have that

$$|f(z)| \geqslant |z|^p - \max_{n\in N} \frac{2\beta(p-\alpha)}{(\frac{p+n}{p})[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}|z|^{p+n} \tag{4.11}$$

and

$$|f(z)| \leqslant |z|^p + \max_{n\in N} \frac{2\beta(p-\alpha)}{(\frac{p+n}{p})[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}|z|^{p+n} \tag{4.12}$$

for $z \in U$. From Lemma 2, we see that the max in (4.11) and (4.12) occur when $n=1$. This completes the proof of Theorem 7. □

COROLLARY 5. *If $f(z)$ is in the class $C^p_\gamma[\alpha,\beta]$. Then $f(z)$ is included in a disc with its center at the origin and radius r given by*

$$r = 1 + \frac{\beta(p-\alpha)}{(\frac{p+1}{p})[1+\beta(1+2p-2\alpha)](p-\gamma)}\ . \tag{4.13}$$

THEOREM 8. *If $f(z)$ is in the class $C^p_\gamma[\alpha,\beta]$, then*

$$|f'(z)| \geqslant p\,|z|^{p-1} - \frac{\beta p(p-\alpha)}{[1+\beta(1+2p-2\alpha)](p-\gamma)}\,|z|^p \tag{4.14}$$

and

$$|f'(z)| \leqslant p\,|z|^{p-1} + \frac{\beta p(p-\alpha)}{[1+\beta(1+2p-2\alpha)](p-\gamma)}\,|z|^p \tag{4.15}$$

for $z \in U$. The bounds for (4.14) and (4.15) are sharp for the function $f(z)$ given by (4.10).

Proof. By means of Theorem 4, we note that

$$|f'(z)| \geqslant p|z|^{p-1} - \max_{n\in N} \frac{2\beta p(p-\alpha)}{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}|z|^{p+n-1} \tag{4.16}$$

and

$$\left|f'(z)\right| \leqslant |z|^{p-1} + \max_{n\in N} \frac{2\beta p(p-\alpha)}{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}|z|^{p+n-1}\ . \tag{4.17}$$

Also by using Lemma 2, we see that the max in (4.16) and (4.17) occur when $n=1$. This completes the proof of Theorem 8. □

REMARK 1. Making use of the relationship (1.17) between the classes $R^p_\gamma[\alpha,\beta]$ and $C^p_\gamma[\alpha,\beta]$, we can deduce Theorem 8 from Theorem 5.

5. Integral operators

THEOREM 9. *Let the function* $f(z)$ *defined by* (1.12) *be in the class* $R_\gamma^p[\alpha,\beta]$ *, and let* c *be a real number such that* $c>-p$*. Then the function* $F(z)$ *defined by*

$$F(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)\,dt \tag{5.1}$$

also belongs to the class $R_\gamma^p[\alpha,\beta]$*.*

Proof. From the representation of $f(z)$, it follows that

$$F(z)=z^p-\sum_{n=1}^{\infty} b_{p+n}\,z^{p+n}, \tag{5.2}$$

where

$$b_{p+n}=(\frac{c+p}{c+p+n})a_{p+n}\ .$$

Therefore

$$\begin{aligned}&\sum_{n=1}^{\infty}[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)b_{p+n}\\ &\qquad= \sum_{n=1}^{\infty}[n+\beta(n+2p-2\alpha)]\left(\frac{c+p}{c+p+n}\right)G^p(\alpha,n+1)a_{p+n}\\ &\qquad\leqslant \sum_{n=1}^{\infty}[n+\beta(n+2p-2\alpha)]G^p(\alpha,n+1)a_{p+n}\leqslant 2\beta(p-\alpha)\,,\end{aligned}$$

since $f(z)\in R_\gamma^p[\alpha,\beta]$. Hence, by Theorem 1, $F(z)\in R_\gamma^p[\alpha,\beta]$.

COROLLARY 6. *Under the same conditions as Theorem 9, a similar proof shows that the function* $F(z)$ *defined by (5.1) is in the class* $C_\gamma^p[\alpha,\beta]$*, whenever* $f(z)$ *is in the class* $C_\gamma^p[\alpha,\beta]$*.*

6. Radii problems

In order to investigate radii problems, we need the following result.

LEMMA 3. ([6]) *Let* $f(z)\in T(p)$ *be defined by* (1.12)*. Then* $f(z)$ *is p-valent in* U *if*

$$\sum_{n=1}^{\infty}(p+n)a_{p+n}\leqslant p. \tag{6.1}$$

In view of Lemma 3 and Theorem 1 we note that $R_{\gamma}^{p}[\alpha,\beta]$ is a subclass of $T(p)$ if $0 \leqslant \gamma \leqslant \dfrac{2p-1}{2}$, $0 \leqslant \alpha < p$, $0 < 0 < \beta \leqslant 1$ and $p \in N$. Also in view of Lemma 3 and Theorem 2 we note that $C_{\gamma}^{p}[\alpha,\beta]$ is a subclass of $T(p)$ if $0 \leqslant \gamma \leqslant \dfrac{\beta(p-\alpha)(2p-1)+(1+\beta)p}{1+\beta(1+2p-2\alpha)}$, $0 \leqslant \alpha < p$, $0 < \beta \leqslant 1$ and $p \in N$.

THEOREM 10. *Let the function $f(z)$ defined by* (1.12) *be in the class $R_{\gamma}^{p}[\alpha,\beta]$, $0 \leqslant \gamma \leqslant \frac{2p-1}{2}$, $0 \leqslant \alpha < p$, $0 < \beta \leqslant 1$, $p \in N$. Then $f(z)$ is p-valently starlike of order δ $(0 \leqslant \delta < p)$ in $|z| < r_1$, where*

$$r_1 = \inf_n \left\{ \frac{(p-\delta)[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}{2\beta(p-\alpha)(n+p-\delta)} \right\}^{\frac{1}{n}} \qquad (n \geqslant 1). \tag{6.2}$$

The result is sharp, with the extremal function $f(z)$ given by (2.5).

Proof. It is sufficient to show that $\left|\dfrac{zf'(z)}{f(z)} - p\right| \leqslant p-\delta$ for $|z| < r_1$. We have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leqslant \frac{\sum\limits_{n=1}^{\infty} n\, a_{p+n}\,|z|^n}{1-\sum\limits_{n=1}^{\infty} a_{p+n}\,|z|^n}.$$

Thus $\left|\dfrac{zf'(z)}{f(z)} - p\right| \leqslant p-\delta$ if

$$\sum_{n=1}^{\infty} \frac{(n+p-\delta)}{(p-\delta)}\, a_{p+n}\,|z|^n \leqslant 1. \tag{6.3}$$

Hence, by Theorem 1, (6.3) will be true if

$$\frac{(n+p-\delta)}{(p-\delta)}\,|z|^n \leqslant \frac{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}{2\beta(p-\alpha)}$$

or if

$$|z| \leqslant \left\{ \frac{(p-\delta)[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}{2\beta(p-\alpha)(n+p-\delta)} \right\}^{\frac{1}{n}} \qquad (n \geqslant 1). \tag{6.4}$$

The theorem follows easily from (6.4).

COROLLARY 7. *Let the function $f(z)$ defined by (1.12) be in the class $R_{\gamma}^{p}[\alpha,\beta]$, $0 \leqslant \gamma \leqslant \frac{2p-1}{2}$, $0 \leqslant \alpha < p$, $0 < \beta \leqslant 1$, $p \in N$. Then $f(z)$ is p-valently convex of order δ $(0 \leqslant \delta < p)$ in $|z| < r_2$, where*

$$r_2 = \inf_n \left\{ \frac{p(p-\delta)[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}{2\beta(p-\alpha)(n+p-\delta)(n+p)} \right\}^{\frac{1}{n}} \qquad (n \geqslant 1). \tag{6.5}$$

The result is sharp, with the extremal function $f(z)$ given by (2.5).

THEOREM 11. *Let the function $f(z)$ defined by (1.12) be in the class $C_\gamma^p[\alpha,\beta]$, $0 \leqslant \gamma \leqslant \dfrac{\beta(p-\alpha)(2p-1)+(1+\beta)p}{1+\beta(1+2p-2\alpha)}$, $0 \leqslant \alpha < p$, $0 < \beta \leqslant 1$ and $p \in N$. Then $f(z)$ is p-valently starlike of order δ $(0 \leqslant \delta < p)$ in $|z| < r_3$, where*

$$r_3 = \inf_n \left\{ \frac{(p-\delta)(p+n)[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}{2\beta p(p-\alpha)(n+p-\delta)} \right\}^{\frac{1}{n}} \quad (n \geqslant 1). \quad (6.6)$$

The result is sharp, with the extremal function $f(z)$ given by (2.8).

7. Modified Hadamard products

Let the functions $f_j(z)$ $(j = 1,2)$ be defined by

$$f_j(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,j} z^{p+n} \quad (a_{p+n,j} \geqslant 0;\ j = 1,2;\ p \in N). \quad (7.1)$$

Then the modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 \otimes f_2)(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,1} a_{p+n,2} z^{p+n}. \quad (7.2)$$

Throughout this section, we assume that $0 \leqslant \gamma \leqslant \frac{2p-1}{2}$, $0 \leqslant \alpha < p$, $0 < \beta \leqslant 1$ and $p \in N$.

THEOREM 12. *Let the functions $f_j(z)$ $(j = 1,2)$ defined by (7.1) be in the class $R_\gamma^p[\alpha,\beta]$. Then $(f_1 \otimes f_2)(z) \in R_\gamma^p[\delta(\gamma,\alpha,\beta,p),\beta]$, where*

$$\delta(\gamma,\alpha,\beta,p) = p - \frac{\beta(1+\beta)(p-\alpha)^2}{[1+\beta(1+2p-2\alpha)]^2(p-\gamma) - 2\beta^2(p-\alpha)^2}. \quad (7.3)$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [13], we need to find the largest $\delta = \delta(\gamma,\alpha,\beta,p)$ such that

$$\sum_{n=1}^{\infty} \frac{[n+\beta(n+2p-2\delta)]G^p(\gamma,n+1)}{2\beta(p-\delta)} a_{p+n,1}\ a_{p+n,2} \leqslant 1. \quad (7.4)$$

Since

$$\sum_{n=1}^{\infty} \frac{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}{2\beta(p-\alpha)} a_{p+n,1} \leqslant 1 \quad (7.5)$$

and

$$\sum_{n=1}^{\infty} \frac{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}{2\beta(p-\alpha)} a_{p+n,2} \leqslant 1, \tag{7.6}$$

by the Cauchy - Schwarz inequality we have

$$\sum_{n=1}^{\infty} \frac{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}{2\beta(p-\alpha)} \sqrt{a_{p+n,1}\ a_{p+n,2}} \leqslant 1. \tag{7.7}$$

Thus it is sufficient to show that

$$\begin{aligned} &\frac{[n+\beta(n+2p-2\delta)]G^p(\gamma,n+1)}{(p-\delta)} a_{p+n,1}a_{p+n,2} \\ &\qquad \leqslant \frac{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}{(p-\alpha)} \sqrt{a_{p+n,1}a_{p+n,2}}\,, \end{aligned} \tag{7.8}$$

that is, that

$$\sqrt{a_{p+n,1}\ a_{p+n,2}} \leqslant \frac{[n+\beta(n+2p-2\alpha)](p-\delta)}{[n+\beta(n+2p-2\delta)](p-\alpha)}\,. \tag{7.9}$$

Note that

$$\sqrt{a_{p+n,1}\ a_{p+n,2}} \leqslant \frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)} \qquad (n \geqslant 1)\,. \tag{7.10}$$

Consequently, we need only to prove that

$$\frac{2\beta(p-\alpha)}{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)} \leqslant \frac{[n+\beta(n+2p-2\alpha)](p-\delta)}{[n+\beta(n+2p-2\delta)](p-\alpha)} \qquad (n \geqslant 1) \tag{7.11}$$

or, equivalently , that

$$\delta \leqslant p - \frac{2\beta(1+\beta)n(p-\alpha)^2}{[n+\beta(n+2p-2\alpha)]^2G^p(\gamma,n+1)-4\beta^2(p-\alpha)^2} \qquad (n \geqslant 1). \tag{7.12}$$

Since

$$A(n) = p - \frac{2\beta(1+\beta)n(p-\alpha)^2}{[n+\beta(n+2p-2\alpha)]^2G^p(\gamma,n+1)-4\beta^2(p-\alpha)^2} \tag{7.13}$$

is an increasing function of n $(n \geqslant 1)$ for $0 \leqslant \gamma \leqslant \dfrac{2p-1}{2}$, $0 \leqslant \alpha < p$, $0 < \beta \leqslant 1$ and $p \in N$, letting $n = 1$ in (7.13), we obtain

$$\delta \leqslant A(1) = p - \frac{\beta(1+\beta)(p-\alpha)^2}{[1+\beta(1+2p-2\alpha)]^2(p-\gamma)-2\beta^2(p-\alpha)^2}, \tag{7.14}$$

which completes the proof of Theorem 12. □

Finally, by taking the functions

$$f_j(z) = z^p - \frac{\beta(p-\alpha)}{[1+\beta(1+2p-2\alpha)](p-\gamma)} z^{p+1} \qquad (j=1,2;\ p \in N) \tag{7.15}$$

we can see that the result is sharp.

COROLLARY 8. *For $f_j(z)(j=1,2)$ as in Theorem 12, we have*

$$h(z) = z^p - \sum_{n=1}^{\infty} \sqrt{a_{p+n,1}\ a_{p+n,2}}\, z^{p+n} \tag{7.16}$$

belongs to the class $R_\gamma^p[\alpha,\beta]$.

The result follows from the inequality (7.7). It is sharp for the same functions as in Theorem 12.

COROLLARY 9. *Let the functions $f_j(z)$ $(j=1,2)$ defined by (7.1) be in the class $C_\gamma^p[\alpha,\beta]$. Then $(f_1 \otimes f_2)(z) \in C_\gamma^p[\lambda(\gamma,\alpha,\beta,p),\beta]$, where*

$$\lambda(\gamma,\alpha,\beta,p) = p - \frac{\beta(1+\beta)(p-\alpha)^2}{(\frac{p+1}{p})[1+\beta(1+2p-2\alpha)]^2(p-\gamma) - 2\beta^2(p-\alpha)^2}. \tag{7.17}$$

The result is sharp for the functions

$$f_j(z) = z^p - \frac{\beta(p-\alpha)}{(\frac{p+1}{p})[1+\beta(1+2p-2\alpha)](p-\gamma)} z^{p+1} \qquad (j=1,2;\ p \in N). \tag{7.18}$$

THEOREM 13. *Let the function $f_1(z)$ defined by (7.1) be in the class $R_\gamma^p[\alpha,\beta]$ and the function $f_2(z)$ defined by (7.1) be in the class $R_\gamma^p[\eta,\beta]$. Then $(f_1 \otimes f_2)(z) \in R_\gamma^p[\xi(\gamma,\alpha,\beta,\eta,p),\beta]$, where*

$$\begin{aligned}&\xi(\gamma,\alpha,\beta,\eta,p)\\ &\quad = p - \frac{\beta(1+\beta)(p-\alpha)(p-\eta)}{[1+\beta(1+2p-2\alpha)][1+\beta(1+2p-2\eta)](p-\gamma)-2\beta^2(p-\alpha)(p-\eta)}.\end{aligned} \tag{7.19}$$

The result is sharp.

Proof. Proceeding as in the proof of Theorem 12, we get

$$\begin{aligned}&\xi \leqslant B(n)\\ &\quad = p - \frac{2\beta(1+\beta)n(p-\alpha)(p-\eta)}{[n+\beta(n+2p-2\alpha)][n+\beta(n+2p-2\eta)]G^p(\gamma,n+1)-4\beta^2(p-\alpha)(p-\eta)}.\end{aligned} \tag{7.20}$$

Since the function $B(n)$ is an increasing function of n $(n \geqslant 1)$ for $0 \leqslant \gamma \leqslant \frac{2p-1}{2}$, $0 \leqslant \alpha < p$, $0 < \eta < p$, $0 < \beta \leqslant 1$ and $p \in N$, letting $n = 1$ in (7.20), we obtain

$$\begin{aligned} \xi &\leqslant B(1) \\ &= p - \frac{\beta(1+\beta)(p-\alpha)(p-\eta)}{[1+\beta(1+2p-2\alpha)][1+\beta(1+2p-2\eta)](p-\gamma)-2\beta^2(p-\alpha)(p-\eta)}. \end{aligned} \tag{7.21}$$

which evidently proves Theorem 13. □

Finally the result is the best possible for the functions

$$f_1(z) = z^p - \frac{\beta(p-\alpha)}{[1+\beta(1+2p-2\alpha)](p-\gamma)} z^{p+1} \qquad (p \in N) \tag{7.22}$$

and

$$f_2(z) = z^p - \frac{\beta(p-\eta)}{[1+\beta(1+2p-2\eta)](p-\gamma)} z^{p+1} \qquad (p \in N). \tag{7.23}$$

In the same way, we can prove the following theorem using Corollary 9 instead of Theorem 12.

THEOREM 14. *Let the function* $f_1(z)$ *defined by (7.1) be in the class* $C_\gamma^p[\alpha,\beta]$ *and the function* $f_2(z)$ *defined by (7.1) be in the class* $C_\gamma^p[\eta,\beta]$. *Then* $(f_1 \otimes f_2)(z) \in C_\gamma^p[\zeta(\gamma,\alpha,\beta,\eta),\beta]$, *where*

$$\begin{aligned} &\zeta(\gamma,\alpha,\beta,\eta,p) \\ &= p - \frac{\beta(1+\beta)(p-\alpha)(p-\eta)}{(\frac{p+1}{p})[1+\beta(1+2p-2\alpha)][1+\beta(1+2p-2\eta)](p-\gamma)-2\beta^2(p-\alpha)(p-\eta)}. \end{aligned} \tag{7.24}$$

The result is sharp for the functions

$$f_1(z) = z^p - \frac{\beta(p-\alpha)}{(\frac{p+1}{p})[1+\beta(1+2p-2\alpha)](p-\gamma)} z^{p+1} \qquad (p \in N) \tag{7.25}$$

and

$$f_2(z) = z^p - \frac{\beta(p-\eta)}{(\frac{p+1}{p})[1+\beta(1+2p-2\eta)](p-\gamma)} z^{p+1} \qquad (p \in N). \tag{7.26}$$

COROLLARY 10. *Let the functions* $f_j\ (z)(j = 1,2,3)$ *defined by* (7.1) *be in the class* $R_\gamma^p[\alpha,\beta]$. *Then* $(f_1 \otimes f_2 \otimes f_3)(z) \in R_\gamma^p[\eta(\gamma,\alpha,\beta,p),\beta]$, *where*

$$\eta(\gamma,\alpha,\beta,p) = p - \frac{\beta^2(1+\beta)(p-\alpha)^3}{[1+\beta(1+2p-2\alpha)]^3(p-\gamma)^2 - 2\beta^3(p-\alpha)^3}. \tag{7.27}$$

The result is the best possible for the functions

$$f_j(z) = z^p - \frac{\beta(p-\alpha)}{[1+\beta(1+2p-2\alpha)](p-\gamma)}\ z^{p+1} \qquad (j=1,2,3;\ p\in N)\,. \quad (7.28)$$

Proof. From Theorem 12, we have $(f_1 \otimes f_2)(z) \in R_\gamma^p[\delta(\gamma,\alpha,\beta,p),\beta]$, where $\delta(\gamma,\alpha,\beta,p)$ is given by (7.3). We use now Theorem 13, we get $(f_1 \otimes f_2 \otimes f_3)(z) \in R_\gamma^p[\eta(\gamma,\alpha,\beta,p),\beta]$, where

$$\begin{aligned}\eta(\gamma,\alpha,\beta,p) &= p - \frac{\beta(1+\beta)(p-\alpha)(p-\delta)}{[1+\beta(1+2p-2\alpha)][1+\beta(1+2p-2\delta)](p-\gamma)-2\beta^2(p-\alpha)(p-\delta)} \\ &= p - \frac{\beta^2(1+\beta)(p-\alpha)^3}{[1+\beta(1+2p-2\alpha)]^3(p-\gamma)^2-2\beta^3(p-\alpha)^3}\,.\end{aligned}$$

This completes the proof of Corollary 10. □

COROLLARY 11. *Let the functions $f_j(z)$ $(j=1,2,3)$ defined by (7.1) be in the class $C_\gamma^p[\alpha,\beta]$. Then $(f_1 \otimes f_2 \otimes f_3)(z) \in C_\gamma^p[\theta(\gamma,\alpha,\beta,p),\beta]$, where*

$$\theta(\gamma,\alpha,\beta,p) = p - \frac{\beta^2(1+\beta)(p-\alpha)^3}{(\frac{p+1}{p})^2[1+\beta(1+2p-2\alpha)]^3(p-\gamma)^2-2\beta^3(p-\alpha)^3}. \quad (7.29)$$

The result is the best possible for the functions

$$f_j(z) = z^p - \frac{\beta(p-\alpha)}{(\frac{p+1}{p})[1+\beta(1+2p-2\alpha)](p-\gamma)}\ z^{p+1} \quad (j=1,2,3;p\in N)\,. \quad (7.30)$$

THEOREM 15. *Let the functions $f_j(z)$ $(j=1,2)$ defined by (7.1) be in the class $R_\gamma^p[\alpha,\beta]$. Then the function*

$$h(z) = z^p - \sum_{n=1}^{\infty}(a_{p+n,1}^2 + a_{p+n,2}^2)z^{p+n} \quad (7.31)$$

belong to the class $R_\gamma^p[\phi(\gamma,\alpha,\beta,p),\beta]$, where

$$\phi(\gamma,\alpha,\beta,p) = p - \frac{2\beta(1+\beta)(p-\alpha)^2}{[1+\beta(1+2p-2\alpha)]^2(p-\gamma)-4\beta^2(p-\alpha)^2}\,. \quad (7.32)$$

The result is sharp for the functions $f_j(z)$ $(j=1,2)$ defined by (7.15).

Proof. By virture of Theorem 1, we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty}\left\{\frac{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}{2\beta(p-\alpha)}\right\}^2 a_{p+n,1}^2 \\ &\qquad \leqslant \left\{\sum_{n=1}^{\infty}\frac{[n+\beta(n+2p-2\alpha)]G^p(\gamma,n+1)}{2\beta(p-\alpha)}a_{p+n,1}\right\}^2 \leqslant 1\end{aligned} \quad (7.33)$$

and

$$\sum_{n=1}^{\infty}\left\{\frac{[n+\beta(n+2p-2\alpha)]G^{p}(\gamma,n+1)}{2\beta(p-\alpha)}\right\}^{2}a_{p+n,2}^{2} \leqslant \left\{\sum_{n=1}^{\infty}\frac{[n+\beta(n+2p-2\alpha)]G^{p}(\gamma,n+1)}{2\beta(p-\alpha)}a_{p+n,2}\right\}^{2} \leqslant 1\,. \tag{7.34}$$

It follows from (7.33) and (7.34) that

$$\sum_{n=1}^{\infty}\frac{1}{2}\left\{\frac{[n+\beta(n+2p-2\alpha)]G^{p}(\gamma,n+1)}{2\beta(p-\alpha)}\right\}^{2}\left(a_{p+n,1}^{2}+a_{p+n,2}^{2}\right) \leqslant 1\,. \tag{7.35}$$

Therefore, we need to find the largest $\varphi(\gamma,\alpha,\beta,p)$ such that

$$\frac{[n+\beta(n+2p-2\varphi)]G^{p}(\gamma,n+1)}{2\beta(p-\varphi)} \leqslant \frac{1}{2}\left\{\frac{[n+\beta(n+2p-2\alpha)]G^{p}(\gamma,n+1)}{2\beta(p-\alpha)}\right\}^{2} \quad (n \geqslant 1) \tag{7.36}$$

that is, that

$$\varphi \leqslant p-\frac{4\beta(1+\beta)n(p-\alpha)^{2}}{[n+\beta(n+2p-2\alpha)]^{2}G^{p}(\gamma,n+1)-8\beta^{2}(p-\alpha)^{2}} \qquad (n \geqslant 1). \tag{7.37}$$

Since

$$D(n)=p-\frac{4\beta(1+\beta)n(p-\alpha)^{2}}{[n+\beta(n+2p-2\alpha)]^{2}G^{p}(\gamma,n+1)-8\beta^{2}(p-\alpha)^{2}} \tag{7.38}$$

is an increasing function of n $(n \geqslant 1)$ for $0 \leqslant \gamma \leqslant \dfrac{2p-1}{2}$, $0 \leqslant \alpha < p$, $0 < \beta \leqslant 1$ and $p \in N$, we readily have

$$\varphi \leqslant p-\frac{2\beta(1+\beta)(p-\alpha)^{2}}{[1+\beta(1+2p-2\alpha)]^{2}(p-\gamma)-4\beta^{2}(p-\alpha)^{2}}\,, \tag{7.39}$$

which completes the proof of Theorem 15. □

THEOREM 16. *Let the functions* $f_j(z)$ $(j=1,2)$ *defined by* (7.1) *be in the class* $C_{\gamma}^{p}[\alpha,\beta]$. *Then the function* $h(z)$ *defined by (7.31) belongs to the class* $C_{\gamma}^{p}[\rho(\gamma,\alpha,\beta,p),\beta]$, *where*

$$\rho(\gamma,\alpha,\beta,p)=p-\frac{2\beta(1+\beta)(p-\alpha)^{2}}{(\frac{p+1}{p})[1+\beta(1+2p-2\alpha)]^{2}(p-\gamma)-4\beta^{2}(p-\alpha)^{2}}\,. \tag{7.40}$$

The result is sharp for the functions $f_j(z)$ $(j=1,2)$ *defined by (7.18).*

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