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ON CERTAIN CLASSES OF ANALYTIC FUNCTIONS OF COMPLEX ORDER DEFINED BY DZIOK–SRIVASTAVA OPERATOR

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Dedicated to late Professor M. S. Kasi

(*communicated by A. Horwitz*)

Abstract. Making use of the generalized hypergeometric functions, we introduced certain new subclasses of analytic functions defined by Dziok-Srivastava operator in the unit disc. The main object of the present paper is to investigate the various properties and characteristics of analytic functions belonging to the subclasses $S_n(l,m,\lambda,b,\gamma)$ satisfying the inequality

$$\left|\frac{1}{b}\left(\frac{z\left(H_m^l[\alpha_1,\beta_1]f(z)\right)'}{(1-\lambda)H_m^l[\alpha_1,\beta_1]f(z)+\lambda z\left(H_m^l[\alpha_1,\beta_1]f(z)\right)'}-1\right)\right|<\gamma,$$

where $z\in U,\ b\in\mathbb{C}\setminus\{0\},\ 0<\gamma\leqslant 1, 0\leqslant\lambda\leqslant 1$ and $H_m^l[\alpha_1,\beta_1]f(z)$ is Dziok-Srivastava operator. Also let $R_n(l,m,\lambda,b,\gamma)$ be an another subclass satisfying the inequality

$$\left|\frac{1}{b}\left((1-\lambda)\frac{H_m^l[\alpha_1,\beta_1]f(z)}{z}+\lambda\left(H_m^l[\alpha_1,\beta_1]f(z)\right)'-1\right)\right|<\gamma$$

where $z\in U,\ b\in\mathbb{C}\setminus\{0\},\ 0<\gamma\leqslant 1,\ \ 0\leqslant\lambda\leqslant 1$ and $H_m^l[\alpha_1,\beta_1]f(z)$ is given by by Dziok-Srivastava [7]. Apart from deriving a set of coefficient bounds for each of these function classes, we establish several inclusion relationships involving the $(n,\delta)-$ negihborhoods of analytic functions with negative coefficients belonging to these subclasses.

1. Introduction

Let $A(n)$ denote the class of functions of the form

$$f(z)=z-\sum_{k=n+1}^{\infty}a_kz^k,\quad a_k\geqslant 0,\quad n\in\mathbb{N}:=\{1,2,3,\dots\},\tag{1.1}$$

which are analytic and univalent in the open disc $U=\{z:z\in\mathbb{C},\ |z|<1\}$.

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Following Goodman [8], Ruscheweyh [16], Silverman [18] and others [2, 3, 4, 10], we define the $(n,\delta)-$ neighborhood of a function $f \in A(n)$ by

$$N_{n,\delta}(f) := \left\{ f \in A(n) \;:\; g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|a_k - b_k| \leqslant \delta \right\}. \quad (1.2)$$

In particular, for the identity function

$$e(z) = z,$$

we immediately have

$$N_{n,\delta}(e) := \left\{ f \in A(n) \;:\; g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|b_k| \leqslant \delta \right\}. \quad (1.3)$$

A function $f \in A(n)$ is starlike of complex order b $(b \in \mathbb{C} \setminus \{0\})$, that is $S_n^*(b)$, if it also satisfies the following inequality

$$\text{Re} \left\{ 1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0 \quad (z \in U; b \in \mathbb{C} \setminus \{0\}). \quad (1.4)$$

Furthermore, a function $f \in A(n)$ is convex of complex order b $(b \in \mathbb{C} \setminus \{0\})$, that is $C_n(b)$, if it also satisfies the following inequality

$$\text{Re} \left\{ 1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)} \right) \right\} > 0 \quad (z \in U; b \in \mathbb{C} \setminus \{0\}). \quad (1.5)$$

The classes $S_n^*(b)$ and $C_n(b)$ stem essentially from the classes of starlike and convex functions of complex order, which were considered earlier by Nasr et.al., [11] and Wiatrowski [20], respectively.

For functions f_j $(j = 1,2)$ given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \quad (j = 1,2),$$

let $f_1 * f_2$ denote the Hadamard product (or Convolution) of f_1 and f_2 defined by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k, \quad z \in U. \quad (1.6)$$

For complex parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ $(\beta_j \neq 0, -1, \ldots; \quad j = 1, 2, \ldots, m)$ the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$${}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \;:=\; \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_l)_k}{(\beta_1)_k \ldots (\beta_m)_k} \frac{z^k}{k!} \quad (1.7)$$
$$(l \leqslant m+1;\; l, m \in \mathbb{N}_0 \;:=\; \mathbb{N} \cup \{0\}; z \in U)$$

where $\mathbb{N}$ denotes the set of all positive integers and $(x)_k$ is the Pochhammer symbol defined by

$$(x)_k = \begin{cases} 1, & k = 0 \\ x(x+1)(x+2)\dots(x+k-1), & k \in N. \end{cases} \tag{1.8}$$

The notation ${}_lF_m$ is quite useful for representing many well-known functions such as the exponential, the Binomial, the Bessel, the Laguerre polynomial, and others; for example see [5] and [14].

For positive real values of $\alpha_1, \dots, \alpha_l$ and $\beta_1, \dots, \beta_m$ $(\beta_j \neq 0, -1, \dots; j = 1, 2, \dots, m)$, let $H(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m) : A(n) \to A(n)$ be a linear operator defined by

$$\begin{aligned} [(H(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m))(f)](z) &:= z\, {}_lF_m(\alpha_1, \alpha_2, \dots \alpha_l; \beta_1, \beta_2 \dots, \beta_m; z) * f(z) \\ &= z - \sum_{k=n+1}^{\infty} \Gamma_k\, a_k\, z^k \end{aligned} \tag{1.9}$$

where

$$\Gamma_k = \frac{(\alpha_1)_{k-1} \dots (\alpha_l)_{k-1}}{(\beta_1)_{k-1} \dots (\beta_m)_{k-1}} \, \frac{1}{(k-1)!} \, . \tag{1.10}$$

For notational simplicity, we can use a shorter notation $H_m^l[\alpha_1, \beta_1]$ for $H(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m)$ in the sequel. It follows from (1.9) that

$$H_0^1[1,1] = f(z), \quad H_0^1[2,1] = zf'(z)$$

The linear operator $H_m^l[\alpha_1, \beta_1]$ is called Dziok-Srivastava operator (see [7]), includes (as its special cases) various other linear operators introduced and studied by Carlson and Shaffer [6], Owa [12], Ruscheweyh [15] and Srivastava-Owa [19].

For $0 \leqslant \lambda \leqslant 1$, we let $S_n(l, m, \lambda, b, \gamma)$ be the subclass of $A(n)$ consisting of functions of the form (1.1) and satisfying the inequality

$$\left| \frac{1}{b} \left(\frac{z(H_m^l[\alpha_1, \beta_1]f(z))'}{(1-\lambda)H_m^l[\alpha_1, \beta_1]f(z) + \lambda z(H_m^l[\alpha_1, \beta_1]f(z))'} - 1 \right) \right| < \gamma, \tag{1.11}$$

where $z \in U$, $b \in \mathbb{C} \setminus \{0\}$, $0 < \gamma \leqslant 1$ and $H_m^l[\alpha_1, \beta_1]f(z)$ is given by (1.9).

Also let $R_n(l, m, \lambda, b, \gamma)$ be the subclass of $A(n)$ consisting of functions of the form (1.1) and satisfying the inequality

$$\left| \frac{1}{b} \left((1-\lambda) \frac{H_m^l[\alpha_1, \beta_1]f(z)}{z} + \lambda (H_m^l[\alpha_1, \beta_1]f(z))' - 1 \right) \right| < \gamma \tag{1.12}$$

where $z \in U$, $b \in \mathbb{C} \setminus \{0\}$, $0 < \gamma \leqslant 1$, $0 \leqslant \lambda \leqslant 1$ and $H_m^l[\alpha_1, \beta_1]f(z)$ is given by (1.9).

We deem it proper to mention some of the function classes which emerge from the function class $S_n(l, m, \lambda, b, \gamma)$ defined above.

EXAMPLE 1. we observe that if we specialize that $l = 1$ and $m = 0$ with $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$, in (1.11) the class $S_n(l,m,\lambda,b,\gamma)$ reduces to the class $\mathscr{S}_n^{\lambda}(b,\gamma)$ subclass of $A(n)$ consisting of functions of the form (1.1) and satisfying the inequality

$$\left|\frac{1}{b}\left(\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-1\right)\right| < \gamma,$$

where $z \in U$, $b \in \mathbb{C}\setminus\{0\}$, $0 < \gamma \leqslant 1$ and $0 \leqslant \lambda \leqslant 1$.

We also let $R_n(\lambda,b,\gamma)$ be the subclass of $A(n)$ consisting of functions of the form (1.1) and satisfying the inequality

$$\left|\frac{1}{b}\left((1-\lambda)\frac{f(z)}{z}+\lambda(f'(z)-1\right)\right| < \gamma$$

where $z \in U$, $b \in \mathbb{C}\setminus\{0\}$, $0 < \gamma \leqslant 1$, $0 \leqslant \lambda \leqslant 1$.

EXAMPLE 2. By specializing the parameters as $l = 1$ and $m = 0$ with $\alpha_1 = \eta + 1 (\eta > -1)$, $\alpha_2 = 1$, $\beta_1 = 1$, in (1.11)the class $S_n(l,m,\lambda,b,\gamma)$ reduces to the class $S_n(\eta\lambda,b,\gamma)$ the subclass of $A(n)$ consisting of functions of the form (1.1) and satisfying the inequality

$$\left|\frac{1}{b}\left(\frac{z(D^{\eta}f(z))'}{(1-\lambda)D^{\eta}f(z)+\lambda z(D^{\eta}f(z))'}-1\right)\right| < \gamma,$$

where $z \in U$, $b \in \mathbb{C}\setminus\{0\}$, $0 < \gamma \leqslant 1, 0 \leqslant \lambda \leqslant 1$. We also let $R_n(\eta,\lambda,b,\gamma)$ be the subclass of $A(n)$ consisting of functions of the form (1.1) and satisfying the inequality

$$\left|\frac{1}{b}\left((1-\lambda)\frac{D^{\eta}f(z)}{z}+\lambda(D^{\eta}f(z))'-1\right)\right| < \gamma$$

where $z \in U$, $b \in \mathbb{C}\setminus\{0\}$, $0 < \gamma \leqslant 1$, $0 \leqslant \lambda \leqslant 1$ and

$$D^{\eta}f(z) := (f * g)(z) = z - \sum_{k=n+1}^{\infty}\begin{pmatrix}\eta+k-1\\ \eta-1\end{pmatrix} a_k z^k.$$

Our definitions of function classes $S_n(l,m,\lambda,b,\gamma)$ and $R_n(l,m,\lambda,b,\gamma)$ are motivated essentially by earlier investigations [3] and [10], in each of which further details and references to other closely related subclasses can be found.

The main object of the present paper is to investigate the various properties and characteristics of analytic functions belonging to the subclasses $S_n(l,m,\lambda,b,\gamma)$ and $R_n(l,m,\lambda,b,\gamma)$ introduced. Apart from deriving a set of coefficient bounds for each of these function classes, we establish several inclusion relationships involving the $(n,\delta)-$ negihborhoods of analytic functions with negative and missing coefficients belonging to these subclasses. Also the special cases of some of these inclusion relations are shown to yield known results.

2. A set of coefficient inequalities

In this section we obtain the coefficient inequalities for functions in the subclasses $S_n(l,m,\lambda,b,\gamma)$ and $R_n(l,m,\lambda,b,\gamma)$.

THEOREM 2.1. *Let the function $f \in A(n)$ be defined by (1.1), then $f(z)$ is in the class $S_n(l,m,\lambda,b,\gamma)$ if and only if*

$$\sum_{k=n+1}^{\infty} ([1+\lambda(k-1)](\gamma|b|-1)+k)\,\Gamma_k a_k \leqslant \gamma|b| \tag{2.1}$$

where Γ_k is as defined in (1.10).

Proof. Let a function $f(z)$ of the form (1.1) belong to the class $S_n(l,m,\lambda,b,\gamma)$. Then in view of (1.9) and (1.11), we obtain the following inequality,

$$\left| \frac{\sum\limits_{k=n+1}^{\infty} ([\lambda(k-1)+1]-k)\Gamma_k a_k z^k}{z-\sum\limits_{k=n+1}^{\infty} [\lambda(k-1)+1]\Gamma_k a_k z^k} \right| \leqslant \gamma\ |b|, \quad z\in U.$$

Thus putting $z=r\ (0\leqslant r<0)$,we obtain

$$\frac{\sum\limits_{k=n+1}^{\infty} ([\lambda(k-1)+1]-k)\Gamma_k a_k r^{k-1}}{1-\sum\limits_{k=n+1}^{\infty} [\lambda(k-1)+1]\Gamma_k a_k r^{k-1}} \leqslant \gamma\ |b|, \quad z\in U. \tag{2.2}$$

Hence, we observe that the expression in the denominator on the left-hand side of (2.2) is positive for $r=0$ and also for all $r\ \ (0<r<1)$. Thus, by letting $r\to 1^-$ through real values, (2.2) leads us to the desired assertion (2.1) of Theorem 2.1.

Conversely, by applying (2.1) and setting $|z|=1,$ we find from (1.11) that

$$\begin{aligned}
&\left|z(H_m^l[\alpha_1,\beta_1]f(z))'-(1-\lambda)H_m^l[\alpha_1,\beta_1]f(z)-\lambda z(H_m^l[\alpha_1,\beta_1]f(z))'\right| \\
&\qquad -\gamma|b|\left|(1-\lambda)H_m^l[\alpha_1,\beta_1]f(z)+\lambda z(H_m^l[\alpha_1,\beta_1]f(z))'\right| \\
&\quad = \left|\sum_{k=n+1}^{\infty} ([\lambda(k-1)+1]-k)\Gamma_k a_k z^k\right| - \gamma|b|\left|z-\sum_{k=n+1}^{\infty}[\lambda(k-1)+1]\Gamma_k a_k z^k\right| \\
&\quad < \sum_{k=n+1}^{\infty} ([1+\lambda(k-1)](\gamma|b|-1)+k)\,\Gamma_k a_k - \gamma|b| \leqslant 0.
\end{aligned}$$

Hence, by the maximum modulus principle, we infer that $f(z)\in S_n(l,m,\lambda,b,\gamma)$, which evidently completes the proof of Theorem 2.1. □

Suitably specializing the parameters l,m,α_1,α_2 and β_1 as said in the Examples 1 and 2,Theorem 2.1 yield the coefficient inequalities for the subclasses $\mathscr{S}_n^{\lambda}(b,\gamma)$ and $S_n(\eta,\lambda,b,\gamma)$ as given in the following corollaries respectively.

COROLLARY 2.1. *Let the function* $f \in A(n)$ *be defined by (1.1), then* $f(z)$ *is in the class* $\mathscr{S}_n^{\lambda}(b,\gamma)$ *if and only if*

$$\sum_{k=n+1}^{\infty} ([1+\lambda(k-1)](\gamma|b|-1)+k)\,a_k \leqslant \gamma|b| \tag{2.3}$$

COROLLARY 2.2. *Let the function* $f \in A(n)$ *be defined by (1.1), then* $f(z)$ *is in the class* $S_n(\eta,\lambda,b,\gamma)$ *if and only if*

$$\sum_{k=n+1}^{\infty} \binom{\eta+k-1}{k-1} ([1+\lambda(k-1)](\gamma|b|-1)+k)\,a_k \leqslant \gamma|b| \tag{2.4}$$

REMARK 2.1. Fixing $\lambda = 0$, Corollary 2.2 gives the coefficient inequalities for the class $S_n(\eta,b,\gamma)$ obtained in [10]. Furthermore, if in Corollary 2.2, we set $\eta = 0$ and $\eta = 1$ with $n = 1$, $b = 1$, $\gamma = 1-\alpha$ $(0 \leqslant \alpha < 1)$ and $\lambda = 0$, we shall obtain the familiar results of Silverman [17].

Similarly, we can prove the following theorem.

THEOREM 2.2. *Let the function* $f \in A(n)$ *be defined by (1.1), then* $f(z)$ *is in the class* $R_n(l,m,\lambda,b,\gamma)$ *if and only if*

$$\sum_{k=n+1}^{\infty} [1+\lambda(k-1)]\Gamma_k a_k \leqslant \gamma|b| \tag{2.5}$$

where Γ_k *is as defined in (1.10).*

Suitably specializing the parameters l, m, α_1, α_2 and β_1 as said in the Examples 1 and 2, in the above Theorem 2.2 we have the following corollaries.

COROLLARY 2.3. *Let the function* $f \in A(n)$ *be defined by (1.1), then* $f(z)$ *is in the class* $R_n(\lambda,b,\gamma)$ *if and only if*

$$\sum_{k=n+1}^{\infty} [1+\lambda(k-1)]a_k \leqslant \gamma|b| \tag{2.6}$$

COROLLARY 2.4. *Let the function* $f \in A(n)$ *be defined by (1.1), then* $f(z)$ *is in the class* $R_n(\eta,\lambda,b,\gamma)$ *if and only if*

$$\sum_{k=n+1}^{\infty} \binom{\eta+k-1}{k-1} [1+\lambda(k-1)]a_k \leqslant \gamma|b|. \tag{2.7}$$

Analogous to the result stated in [10].

3. Inclusion relations involving the $(n,\delta)-$ neighborhoods

In this section, we establish several inclusion relations for the normalized analytic function classes $S_n(l,m,\lambda,b,\gamma)$ and $R_n(l,m,\lambda,b,\gamma)$ involving the $(n,\delta)-$ neighborhood defined by (1.3).

THEOREM 3.1. *If*

$$\delta := \frac{\gamma|b|(1+n)}{((\gamma|b|-1)[1+n\lambda]+n+1)\,\Gamma_{n+1}}, \quad (\gamma|b|\leqslant 1) \tag{3.1}$$

then

$$S_n(l,m,\lambda,b,\gamma)\subset N_{n,\delta}(e). \tag{3.2}$$

Proof. Let $f(z)\in S_n(l,m,\lambda,b,\gamma)$. Then, in view of the assertion (2.1) of Theorem 2.1, we have

$$\Gamma_{n+1}\left([1+n\lambda](\gamma|b|-1)+n+1\right)\sum_{k=n+1}^{\infty}a_k\leqslant\gamma|b|$$

which readily yields

$$\sum_{k=n+1}^{\infty}a_k\leqslant\frac{\gamma|b|}{([1+n\lambda](\gamma|b|-1)+n+1)\,\Gamma_{n+1}}. \tag{3.3}$$

Making use of (2.1) again, in conjunction with (3.3), we get

$$\begin{aligned}\Gamma_{n+1}\sum_{k=n+1}^{\infty}ka_k &\leqslant \gamma|b|+[1+n\lambda](1-\gamma|b|)\Gamma_{n+1}\sum_{k=n+1}^{\infty}a_k\\ &\leqslant \gamma|b|+[1+n\lambda](1-\gamma|b|)\Gamma_{n+1}\frac{\gamma|b|}{([1+n\lambda](\gamma|b|-1)+n+1)\,\Gamma_{n+1}}\\ &\leqslant \frac{\gamma|b|(1+n)}{[1+n\lambda](\gamma|b|-1)+n+1}.\end{aligned}$$

Hence

$$\sum_{k=n+1}^{\infty}ka_k\leqslant\frac{\gamma|b|(1+n)}{([1+n\lambda](\gamma|b|-1)+n+1)\,\Gamma_{n+1}}=:\delta, \quad (\gamma|b|>1) \tag{3.4}$$

which, by means of the definition (1.3), establishes the inclusion relation (3.2) asserted by Theorem 3.1. □

In a similar manner, by applying the assertion (2.5) of Theorem 2.2 instead of the assertion (2.1) of Theorem 2.1 to functions in the classes $R_n(l,m,\lambda,b,\gamma)$, we can prove the following relationship.

THEOREM 3.2. *If*

$$\delta := \frac{\gamma|b|(n+1)}{[1+n\lambda]\Gamma_{n+1}}, \quad (\lambda\geqslant 1) \tag{3.5}$$

then

$$R_n(l,m,\lambda,b,\gamma)\subset N_{n,\delta}(e). \tag{3.6}$$

REMARK 3.1. By taking $l = 1,\ m = 0,\ \alpha_1 = 2,\ \alpha_2 = 1$ and $\beta_1 = 1$ in Theorem 3.2, we obtain inclusion relation of Altintas et al., [3].

REMARK 3.2. By taking $l = 1,\ m = 0,\ \alpha_1 = \eta + 1\ (\eta > -1),\ \alpha_2 = 1$ and $\beta_1 = 1$ with $\lambda = 0$ in Theorem 3.2, we obtain inclusion relation discussed in [10].

4. Neighborhoods for the classes $S_n^{\alpha}(l, m, \lambda, b, \gamma)$ and $R_n^{\alpha}(l, m, \lambda, b, \gamma)$

In this last section, we determine the neighborhood properties for each the following functions classes $S_n^{\alpha}(l, m, \lambda, b, \gamma)$ and $R_n^{\alpha}(l, m, \lambda, b, \gamma)$. Here the class $S_n^{\alpha}(l, m, \lambda, b, \gamma)$ consists of functions $f(z) \in \mathscr{A}(n)$ for which there exists another function $g(z) \in S_n(l, m, \lambda, b, \gamma)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \alpha \qquad (z \in U; 0 \leqslant \alpha < 1). \tag{4.1}$$

Analogously, the class $R_n^{\alpha}(l, m, \lambda, b, \gamma)$ consists of functions $f(z) \in \mathscr{A}(n)$ for which there exists another function $g(z) \in R_n(l, m, \lambda, b, \gamma)$ satisfying the inequality (4.1).

THEOREM 4.1. *If* $g \in S_n(l, m, \lambda, b, \gamma)$ *and*

$$\alpha = 1 - \frac{\delta\left([1 + n\lambda](\gamma|b| - 1) + n + 1\right)\Gamma_{n+1}}{(n+1)\left[(1 + n\lambda)(\gamma|b| - 1) + n + 1\right]\Gamma_{n+1} - \gamma|b|]}, \quad (\gamma|b| \geqslant 1) \tag{4.2}$$

then

$$N_{n,\delta}(g) \subset S_n^{\alpha}(l, m, \lambda, b, \gamma). \tag{4.3}$$

Proof. Suppose that $f \in N_{n,\delta}(g)$. We then find from the definition (1.2) that

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leqslant \delta$$

which readily implies the coefficient inequality,

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leqslant \frac{\delta}{n+1} \quad (n \in \mathbb{N}).$$

Next, since $g \in S_n(l, m, \lambda, b, \gamma)$, we have

$$\sum_{k=n+1}^{\infty} b_k \leqslant \frac{\gamma|b|}{\left([1 + n\lambda](\gamma|b| - 1) + n + 1\right)\Gamma_{n+1}} \tag{4.4}$$

so that

$$\begin{aligned} \left|\frac{f(z)}{g(z)} - 1\right| &< \frac{\sum_{k=n+1}^{\infty} |a_k - b_k|}{1 - \sum_{k=n+1}^{\infty} b_k} \\ &\leqslant \frac{\delta}{(n+1)} \frac{\left([1 + n\lambda](\gamma|b| - 1) + n + 1\right)\Gamma_{n+1}}{\left[\left[(1 + n\lambda)(\gamma|b| - 1) + n + 1\right]\Gamma_{n+1} - \gamma|b|\right]} \\ &= 1 - \alpha \end{aligned}$$

provided that α is given (4.5). Thus, by definition, $f \in S_n^{\alpha}(l,m,\lambda,b,\gamma)$ for α given by (4.5). This evidently completes our proof of Theorem 4.1. □

By taking $l = 1$, $m = 0$, $\alpha_1 = \eta + 1$ $(\eta > -1)$, $\alpha_2 = 1$ and $\beta_1 = 1$ with $\lambda = 0$ in Theorem 4.1, we state the following corollary

COROLLARY 4.1. *If* $g \in S_n(\eta,\lambda,b,\gamma)$ *and*

$$\alpha = 1 - \frac{\delta\left([1+n\lambda](\gamma|b|-1)+n+1\right)\binom{\eta+n}{n}}{(n+1)\left[\,[(1+n\lambda)(\gamma|b|-1)+n+1]\binom{\eta+n}{n} - \gamma|b|\right]}, \quad (\gamma|b| \geqslant 1)$$

then

$$N_{n,\delta}(g) \subset S_n^{\alpha}(\lambda,b,\gamma,\eta).$$

REMARK 4.1. By taking $\lambda = 0$ in Corollary 4.1, we have neighborhood result as stated in [10].

Our proof of Theorem 4.2 is much akin to that of Theorem 4.1.

THEOREM 4.2. *If* $g \in R_n(l,m,\lambda,b,\gamma)$ *and*

$$\alpha = 1 - \frac{\delta[1+n\lambda]\Gamma_{n+1}}{(n+1)[(1+n\lambda)\Gamma_{n+1} - \gamma|b|]}, (\gamma|b| \geqslant 1) \tag{4.5}$$

then

$$N_{n,\delta}(g) \subset R_n^{\alpha}(l,m,\lambda,b,\gamma). \tag{4.6}$$

REMARK 4.2. By taking the parameters $l = 1$, $m = 0$, with $\alpha_1 = \eta + 1(\eta > -1)$, $\alpha_2 = 1$ and $\beta = 1$ in Theorem 4.2, we get neighborhood result as stated in [10].

REMARK 4.3. By taking the parameters $l = 1$, $m = 0$, with $\alpha_1 = 2$, $\alpha_2 = 1$ and $\beta = 1$ in Theorem 4.2, we obtain neighborhood result as stated in [3].

Concluding Remarks: By suitably specializing the various parameters involved in Theorem 2.1to 4.2, we can derive the corresponding results for many relatively more familiar function classes (see for example $l = 1$, $m = 1$, with $\alpha_1 = a$, $\alpha_2 = 1$ and $\beta = c$ Dziok-Srivastava operator reduces to Carlson-Shaffer operator [5, 6])

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