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THE INTEGRAL OPERATOR ON THE CLASSES $\mathscr{S}_{\alpha}^{*}(b)$ AND $\mathscr{C}_{\alpha}(b)$

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Abstract. In this paper we present some properties for two general integral operators on the classes $\mathscr{S}_{\alpha}^{*}(b)$ and $\mathscr{C}_{\alpha}(b)$.

1. Introduction

Let $\mathscr{U} = \{z \in \mathbb{C}, |z| < 1\}$ be the open unit disc of the complex plane. Denote by $\mathscr{H}(U)$ and $\mathscr{A}$, the class of the holomorphic functions in $\mathscr{U}$ and the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathscr{U}$, respectively.

In the paper [3], Frasin studied the classes $\mathscr{S}_{\alpha}^{*}(b)$ and $\mathscr{C}_{\alpha}(b)$.

A function $f(z) \in \mathscr{A}$ is said to be a starlike of complex order $b, (b \in \mathbb{C} - \{0\})$ and type $\alpha, (0 \leqslant \alpha < 1)$, that is $f \in \mathscr{S}_{\alpha}^{*}(b)$, if and only if

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > \alpha \tag{1}$$

for all $z \in \mathscr{U}$.

A function $f(z) \in \mathscr{A}$ is said to be convex of complex order $b, (b \in \mathbb{C} - \{0\})$ and type $\alpha, (0 \leqslant \alpha < 1)$, that is $f \in \mathscr{C}_{\alpha}(b)$, if and only if

$$\operatorname{Re}\left\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{2}$$

for all $z \in \mathscr{U}$.

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Recently, the first author and N. Breaz in [1] introduced and studied the integral operator

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{3}$$

and the integral operator

$$F_{\alpha_1,\ldots,\alpha_n}(z) = \int_0^z \left(f_1'(t)\right)^{\alpha_1} \ldots \left(f_n'(t)\right)^{\alpha_n} dt \tag{4}$$

for $\alpha_i > 0$, was is introduced by the first author et al. in [2].

In the present paper, we consider two integral operators in above and study their properties on the classes $\mathscr{S}_\alpha^*(b)$ and $\mathscr{C}_\alpha(b)$.

2. Main results

THEOREM 1. *Let* $\alpha_i, i \in \{1,\ldots,n\}$ *the real numbers with the properties* $\alpha_i > 0$ *for* $i \in \{1,\ldots,n\}$, α *the real number,* $0 \leqslant \alpha < 1$ *and*

$$0 \leqslant (\alpha - 1)\sum_{i=1}^{n} \alpha_i + 1 < 1. \tag{5}$$

If $f_i \in \mathscr{S}_\alpha^*(b)$ *for* $i = \{1,\ldots,n\}$ *and* $b \in \mathbb{C} - \{0\}$, *then* $F_n \in \mathscr{C}_\gamma(b)$, *where*

$$\gamma = (\alpha - 1)\sum_{i=1}^{n} \alpha_i + 1.$$

Proof. We calculate the derivatives of the first and second order for F_n. From (3), we obtain

$$F_n'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(z)}{z}\right)^{\alpha_n}$$

and

$$F_n''(z) = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z) - f_i(z)}{zf_i(z)}\right) F_n'(z).$$

From the above equalities, we have

$$\frac{F_n''(z)}{F_n'(z)} = \alpha_1 \left(\frac{zf_1'(z) - f_1(z)}{zf_1(z)}\right) + \ldots + \alpha_n \left(\frac{zf_n'(z) - f_n(z)}{zf_n(z)}\right)$$

and

$$\frac{F_n''(z)}{F_n'(z)} = \alpha_1 \left(\frac{f_1'(z)}{f_1(z)} - \frac{1}{z}\right) + \ldots + \alpha_n \left(\frac{f_n'(z)}{f_n(z)} - \frac{1}{z}\right). \tag{6}$$

By multiplying the relation (6) with z, we obtain

$$\frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1\right). \tag{7}$$

Then by multiplying the relation (7) with $\frac{1}{b}$, we obtain

$$\frac{1}{b}\frac{zF_n''(z)}{F_n'(z)} = \frac{1}{b}\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)} - 1\right) = \sum_{i=1}^{n}\alpha_i\left[1+\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right] - \sum_{i=1}^{n}\alpha_i \quad (8)$$

The relation (8) is equivalent to

$$1+\frac{1}{b}\frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n}\alpha_i\left[1+\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right] - \sum_{i=1}^{n}\alpha_i + 1 \quad (9)$$

Lastly, we calculate the real part of both terms of (9) and obtain

$$\operatorname{Re}\left\{1+\frac{1}{b}\frac{zF_n''(z)}{F_n'(z)}\right\} = \sum_{i=1}^{n}\alpha_i \operatorname{Re}\left[1+\frac{1}{b}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right] - \sum_{i=1}^{n}\alpha_i + 1 \quad (10)$$

Since $f_i \in \mathscr{S}_\alpha^*(b)$ for $i = \{1,...,n\}$, by applying in the relation (10) the inequality (1) we have

$$\operatorname{Re}\left\{1+\frac{1}{b}\frac{zF_n''(z)}{F_n'(z)}\right\} > (\alpha-1)\sum_{i=1}^{n}\alpha_i + 1 \quad (11)$$

Because $0 \leqslant (\alpha-1)\sum_{i=1}^{n}\alpha_i + 1 < 1$, we obtain $F_n \in \mathscr{C}_\gamma(b)$, where $\gamma = (\alpha-1)\sum_{i=1}^{n}\alpha_i + 1$. □

Putting $n = 1$ in Theorem 1, we have

COROLLARY 2. *Let $\alpha_1 > 0$ and α the real number with the property $0 \leqslant \alpha < 1$. If $0 \leqslant (\alpha-1)\alpha_1 + 1 < 1$ and the function $f_1 \in \mathscr{S}_\alpha^*(b)$, then the integral operator $F_1 \in \mathscr{C}_\rho(b)$, where $\rho = (\alpha-1)\alpha_1 + 1$.*

THEOREM 3. *Let $f_i \in \mathscr{C}_\alpha(b)$, $0 \leqslant \alpha < 1$, $\alpha_i > 0$, $b \in \mathbb{C} - \{0\}$ and*

$$0 \leqslant (\alpha-1)\sum_{i=1}^{n}\alpha_i + 1 < 1$$

for all $i \in \{1,...,n\}$. Then the integral operator $F_{\alpha_1,...,\alpha_n} \in \mathscr{C}_\eta(b)$, where

$$\eta = (\alpha-1)\sum_{i=1}^{n}\alpha_i + 1.$$

Proof. If we make the similar operations to the proof of the Theorem 1, we have

$$\frac{F_{\alpha_1,...,\alpha_n}''(z)}{F_{\alpha_1,...,\alpha_n}'(z)} = \alpha_1\frac{f_1''(z)}{f_1'(z)} + ... + \alpha_n\frac{f_n''(z)}{f_n'(z)}. \quad (12)$$

Then by multiplying this relation with $\frac{z}{b}$, we obtain

$$\frac{1}{b}\frac{zF_{\alpha_1,...,\alpha_n}''(z)}{F_{\alpha_1,...,\alpha_n}'(z)} = \alpha_1\frac{1}{b}\frac{zf_1''(z)}{f_1'(z)} + ... + \alpha_n\frac{1}{b}\frac{zf_n''(z)}{f_n'(z)}. \quad (13)$$

From the relation (13), we obtain that

$$\operatorname{Re}\left(\frac{1}{b}\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)}+1\right)=\sum_{i=1}^{n}\alpha_i \operatorname{Re}\left(1+\frac{1}{b}\frac{zf''_i(z)}{f'_i(z)}\right)-\sum_{i=1}^{n}\alpha_i+1. \tag{14}$$

Since $f_i \in \mathscr{C}_\alpha(b)$, we have

$$\operatorname{Re}\left(\frac{1}{b}\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)}+1\right)>(\alpha-1)\sum_{i=1}^{n}\alpha_i+1. \tag{15}$$

Since $0 \leqslant (\alpha-1)\sum_{i=1}^{n}\alpha_i+1<1$, the relation (15) implies that the integral operator $F_{\alpha_1,\ldots,\alpha_n} \in \mathscr{C}_\eta(b)$, where $\eta=(\alpha-1)\sum_{i=1}^{n}\alpha_i+1$. □

Putting $n=1$ in the Theorem 3, we have

COROLLARY 4. *Let $f_1 \in \mathscr{C}_\alpha(b)$, $0 \leqslant \alpha < 1$. Then the integral operator $F_{\alpha_1} \in \mathscr{C}_\sigma(b)$, where $\sigma=(\alpha-1)\alpha_1+1$ and $0 \leqslant (\alpha-1)\alpha_1+1<1$.*

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