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ON SOME DIFFERENTIAL OPERATORS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Abstract. Let $f(z) = D(F(z))$ where D is a differential operator defined separately in every result. Let F be analytic and $F(0) = 0, F'(0) = 1$. We shall find out the disc in which operator D transforms some classes of analytic functions into the same.

1. Introduction

Let $\mathscr{A}(n)$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad (n \in N = \{1, 2, 3, \dots, \}), \tag{1.1}$$

analytic in the unit disc $E = \{z : |z| < 1\}$. Let $P(n, \beta)$ be the class of functions $h(z)$ of the form

$$h(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \dots, \tag{1.2}$$

which are analytic in E and satisfy $\operatorname{Re}\{h(z)\} > \beta$, $0 \leqslant \beta < 1$, $z \in E$. We note that $P(1,0) \equiv P$ is the class of functions with positive real part, $P(1, \beta) \equiv P(\beta)$, and $P(n, 0) \equiv P(n)$.

Let $P_k(n, \beta)$, $k \geqslant 2$, $0 \leqslant \beta < 1$, be the class of functions p, analytic in E, such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z)$$

if and only if $p_1, p_2 \in P(n, \beta)$ for $z \in E$. It is easy to see that $P_2(1, \beta) \equiv P(\beta)$ and $P(1, 0) \equiv P$. The class $P_k(1, 0) \equiv P_k$ was introduced by Pinchuk [6], where he has generalized the concept of functions of bounded boundary rotation. It is worth mentioning that, for $k > 2$, functions in P_k need not be with the positive real part. It is easy to see that $p \in P_k(n, \beta)$ if and only if there exists $h \in P_k(n, 0)$ such that

$$p(z) = (1 - \beta)h(z) + \beta.$$

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Let f and g be analytic in E with $f(z)$ given by (1.1) and $g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k$. Then the convolution (or Hadamard product) of f and g is defined by

$$(f \star g)(z) = z + \sum_{k=n+1}^{\infty} a_k b_k z^k.$$

DEFINITION 1.1. A function $f \in \mathscr{A}(n)$ is said to belong to the class $R_k(n,\beta)$, $k \geqslant 2$, $0 \leqslant \beta < 1$, if and only if $\frac{zf'}{f} \in P_k(n,\beta)$ for $z \in E$.

We note that $R_k(1,0) \equiv R_k$ is the class of functions with bounded radius rotation, first discussed by Tammi, see [1] and $R_2(1,0)$ consists of starlike univalent functions.

Similarly $f \in \mathscr{A}(n)$ belongs to $V_k(n,\beta)$ for $z \in E$ if and only if $\frac{(f')'}{f'} \in P_k(n,\beta)$. It is obvious that

$$f \in V_k(n,\beta) \quad \text{if and only if} \quad zf' \in R_k(n,\beta). \tag{1.3}$$

It may be observed that $V_2(1,0) \equiv C$, the class of convex univalent functions and $V_k(1,0) \equiv V_k$ is the class of functions with bounded boundary rotation first discussed by Paatero, see [1].

2. Preliminary Results

We extend a result proved in [7] as follows.

LEMMA 2.1. *Let $p \in P_k$ and be given by (1.3). Let $s > 0$ and $u \neq -1$ (complex). Then*

$$\left\{p(z) + \frac{szp'(z)}{p(z)+\mu}\right\} \in P_k, \quad for \quad |z| < R_0,$$

where R_0 is given by

$$\begin{aligned} R_0 &= \frac{|\mu+1|}{\sqrt{A + \sqrt{A^2 - |\mu^2 - 1|^2}}}, \qquad (2.1)\\ A &= \left\{2(s+1)^2 + |\mu|^2 - 1\right\}. \end{aligned}$$

The radius R_0 is best possible.

Proof. Define

$$\Phi(z) = \left\{\frac{1}{1+\mu}\frac{z}{(1-z)^{s+1}} + \frac{\mu}{1+\mu}\frac{z}{(1-z)^{s+2}}\right\}. \tag{2.2}$$

Then

$$\begin{aligned} \left(p(z) \star \frac{\Phi(z)}{z}\right) &= p(z) + \frac{szp'(z)}{p(z)+\mu} \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left[p_1(z) + \frac{szp_1'(z)}{p_1(z)+\mu}\right] \\ &\quad - \left(\frac{k}{4}-\frac{1}{2}\right)\left[p_2(z) + \frac{szp_2'(z)}{p_2(z)+\mu}\right]. \end{aligned}$$

Now, for $i = 1, 2$, it is shown [7] that

$$\operatorname{Re}\left[p_i(z) + \frac{szp_i'(z)}{p_i(z)+\mu}\right] > 0 \quad \text{for} \quad |z| < r_0,$$

where the best possible value of r_0 is given by (2.1). Using this along with (1.3), we obtain the required result. □

LEMMA 2.2. *Let* $h \in P(n)$. *Then, for* $z \in E$,

(i) $\dfrac{1-r^n}{1+r^n} \leqslant \operatorname{Re} h(z) \leqslant |h(z)| \leqslant \dfrac{1+r^n}{1-r^n}$,

(ii) $|zh'(z)| \leqslant \dfrac{2nr^n \operatorname{Re} h(z)}{1-r^{2n}}$,

(iii) $\left|\dfrac{zh'(z)}{h(z)}\right| \leqslant \dfrac{2nr^n}{1-r^{2n}}$.

For (i), we refer to [8], (ii) may be found in [1] and for (iii), see [5].

3. Main Results

THEOREM 3.1. *Let* $F \in R_k(1,\gamma) \equiv R_k(\gamma)$, $0 \leqslant \gamma < 1$ *and let the function* f *be defined by the differential operator*

$$D_{\alpha,\beta}(F(z)) = f^{\beta}(z) = \frac{1}{\alpha\beta-\alpha+1}\left[(1-\alpha)F^{\beta}(z) + \alpha z\left(F^{\beta}(z)\right)'\right], \quad (\alpha,\beta > 0). \quad (3.1)$$

Then $f \in R_k(\gamma)$ *for* $|z| < R_0$, *where* R_0 *is given by (2.1) with*

$$s = \frac{1}{\beta(1-\gamma)}, \quad \mu = \frac{1-\alpha+\alpha\gamma}{\alpha(1-\gamma)}.$$

This result is sharp.

Proof. We can write (3.1) as

$$F^{\beta}(z) = \left(\beta + \frac{1}{\alpha} - 1\right) z^{1-\frac{1}{\alpha}} \int_0^z t^{\frac{1}{\alpha}-2} f^{\beta}(t)\,dt. \quad (3.2)$$

Let

$$\frac{zF'(z)}{F(z)} = (1-\gamma)h(z) + \gamma.$$

Differentiating (3.2) logarithmically and with some simple computation, we have

$$\frac{1}{1-\gamma}\left[\frac{zf'(z)}{f(z)} - \gamma\right] = h(z) + \frac{\frac{1}{\beta(1-\gamma)} zh'(z)}{h(z) + \left(\frac{1-\alpha+\alpha\gamma}{\alpha(1-\gamma)}\right)}. \quad (3.3)$$

We now apply Lemma 2.1 to obtain that $p \in P_k$ for $|z| < R_0$, where R_0 is given by (2.1) with $s = \frac{1}{\beta(1-\gamma)}$, $\mu = \frac{1-\alpha+\alpha\gamma}{\alpha(1-\gamma)}$. This implies that $f \in R_k(\gamma)$ for $|z| < r_{\alpha,\beta}$, and proof is complete. □

Special Cases

(i) Let $\beta = 1$, $\gamma = 0$, $\alpha = 1$ and $F \in R_k$. Then $f(z) = zF'(z)$ and it follows from Theorem 3.1 that $f \in R_k$ for $|z| < r_1 = \frac{1}{\sqrt{7+\sqrt{48}}} \approx 2 - \sqrt{3}$ with $s = 1$, $\mu = 0$ and $A = 7$ in (2.1). This sharp result is well known for $k = 2$.

(ii) Let $\beta = 1$, $\gamma = 0$, $\alpha = \frac{1}{2}$ and $F \in R_k$. Then $f(z) = \frac{(zF)'}{2}$ (Livingston's operator [7]) and using Theorem 3.1, we note that $f \in R_k$ for $|z| < r = \frac{1}{2}$, (with $s = 1$, $\mu = 0$, $A = 8$). This sharp result is known [4] for $k = 2$.

REMARK 3.1. By using (1.4), it can easily be seen that Theorem 3.1 holds for the class $K_k(\gamma)$.

THEOREM 3.2. *Let $F \in R_k(\sigma)$, $0 \leqslant \sigma < 1$. Let, for $c \geqslant \alpha > 0$, $s \geqslant \alpha$, f be defined by*

$$[F(z)]^{\alpha} = cz^{\alpha-c} \int_0^z t^{c-\delta-1} (f(t))^{\delta}\, dt. \tag{3.4}$$

Then $f \in R_k(\beta_1)$, $\beta_1 = 1 - \frac{\alpha}{\delta}(1-\sigma)$, for $|z| < R_0$, where R_0 is given by (2.1) with

$$s = \frac{1}{\alpha(1-\sigma)}, \quad \mu = \frac{c - \alpha + \alpha\sigma}{\alpha(1-\sigma)}.$$

This result is sharp.

Proof. Let

$$\frac{zF'(z)}{F(z)} = H(z) = (1-\sigma)h(z) + \sigma, \quad h \in P_k \quad \text{in} \quad E. \tag{3.5}$$

Differentiating (3.4), we have

$$z^{(c-\alpha)-1} (F(z))^{\alpha} [(c-\alpha) + \alpha h(z)] = cz^{c-\delta-1} (f(z))^{\delta},$$

and now logarithmic differentiation, simple computation and (3.5) yield

$$\alpha H(z) + \frac{\alpha z H'(z)}{(c-\alpha) + \alpha H(z)} = (\alpha - \delta) + \delta \frac{zf'(z)}{f(z)}. \tag{3.6}$$

Let

$$\frac{zf'(z)}{f(z)} = (1-\beta_1)p(z) + \beta_1, \tag{3.7}$$

where $p(z)$ is analytic in E with $p(0) = 1$), and $\beta_1 = 1 - \frac{\alpha}{\delta}(1-\sigma)$. Then, from (3.5), (3.6) and (3.7), we have

$$p(z) = \frac{\alpha(1-\sigma)}{\delta(1-\beta_1)} \left\{ h(z) + \frac{\frac{1}{\alpha(1-\sigma)} zh'(z)}{h(z) + \frac{c-\alpha+\alpha\sigma}{\alpha(1-\sigma)}} \right\}. \tag{3.8}$$

Now, using Lemma 2.1 with $s = \frac{1}{\alpha(1-\sigma)} > 0$, $\mu = \frac{c-\alpha+\alpha\sigma}{\alpha(1-\sigma)}$, we obtain the required result. □

REMARK 3.2. By using relation (1.4), we note that similar result holds when $F \in V_k(\sigma)$ implies $f \in V_k(\beta_1)$ for $|z| < R_0$.

For $\sigma = 0$, $\mu = \frac{c-\alpha}{\alpha} \neq -1$ (since $c > 0$) and $s = 1 - \frac{\alpha}{\delta}$, for $|z| < R_0 = \frac{c}{(\alpha+1)\sqrt{(\alpha+1)^2+c^2-2\alpha c}}$. This result has been proved in [2] for $\alpha = \delta$, $k = 2$.

REMARK 3.3. In Theorem 3.2, let $F \in R_k(n,0)$, then, using the convolution technique of Lemma 2.1, we can write (3.8) as

$$p(z) = \frac{\alpha}{\delta(1-\beta_1)}\left\{\left(\frac{k}{4}+\frac{1}{2}\right)\left[h_1(z)+\frac{\frac{1}{\alpha}zh_1'(z)}{h_1(z)+\frac{c-\alpha}{\alpha}}\right]-\left(\frac{k}{4}-\frac{1}{2}\right)\left[h_2(z)+\frac{\frac{1}{\alpha}zh_2'(z)}{h_2(z)+\frac{c-\alpha}{\alpha}}\right]\right\}.$$

Now, using Lemma 2.2, we have for $i = 1, 2$

$$\begin{aligned} \operatorname{Re}\left\{h_i(z) + \frac{\frac{1}{\alpha}zh_i'(z)}{h_(z) = \frac{c-\alpha}{\alpha}}\right\} &\geqslant \operatorname{Re} h_i(z)\left\{1 - \left(\frac{2nr^m}{1-r^{2n}}\right)\frac{1}{\alpha\frac{1-r^n}{1+r^n} + (c-\alpha)}\right\} \\ &= \operatorname{Re} h_i(z)\left\{\frac{(1-r^n)[c-(2\alpha-c)r^n] - 2nr^n}{(1-r^n)[c-(2\alpha-c)r^n]}\right\}, \end{aligned}$$

and with (1.3), it implies that $f \in R_k(n,\beta_1)$ for $|z| < r_1$, where $\beta_1 = (1-\frac{\alpha}{\delta})$ and

$$r_1 = \left[\frac{c}{(\alpha+n)+\sqrt{(\alpha+n)^2+c^2-2\alpha c}}\right]^{\frac{1}{n}}, \quad (c \neq 2\alpha).$$

We note that $r_1 = R_0$ for $n = 1$ and $c \neq 2\alpha$.

THEOREM 3.3. *Let $F, G_i \in R_k(n,0)$, $i = 1,2$ and $q \in P(n)$. Let f be defined by*

$$F(z) = \frac{2}{z}\int_0^z f(t)\left(\frac{G_1(t)}{G_2(t)}\right)^{\alpha} q(t)dt, \quad \alpha \geqslant 0. \tag{3.9}$$

Then $f \in R_k(n,0)$ for $|z| < R_1$, where $R_1^n \in (0,1)$ is given by

$$R_1 = \left[\frac{2}{(3n+4\alpha+2)+\sqrt{(3n+4\alpha+2)^2-4(n+1)}}\right]^{\frac{1}{n}}. \tag{3.10}$$

This result is sharp.

Proof. Let

$$\begin{aligned} \frac{zF'(z)}{F(z)} &= \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z) \\ \frac{zG_1'(z)}{G_1(z)} &= H(z) = \left(\frac{k}{4}+\frac{1}{2}\right)H_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)H_2(z) \\ \frac{zG_2'(z)}{G_2(z)} &= p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z), \end{aligned}$$

where $h_i, H_i, p_i \in P(n)$ for $i = 1,2$. From(3.9), it follows that

$$zF'(z) + F(z) = 2f(z)\left(\frac{G_1(z)}{G_2(z)}\right)^{\alpha} q(z). \tag{3.11}$$

Differentiating (3.11) logarithmically, we have

$$\frac{zf'(z)}{f(z)} = h(z) + \frac{zh'(z)}{h(z)+1} - \alpha\frac{zG_1'(z)}{G_1(z)} + \alpha\frac{zG_2'(z)}{G_2(z)} - \frac{zq'(z)}{q(z)}.$$

Using convolution technique, we can write

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{h_1(z) + \frac{zh_1'(z)}{h_1(z)+1} - \alpha H_1(z) + \alpha p_1(z) - \frac{zq'(z)}{q(z)}\right\} \\ &\quad - \left(\frac{k}{4}+\frac{1}{2}\right)\left\{h_2(z) + \frac{zh_2'(z)}{h_2(z)+1} - \alpha H_2(z) + \alpha p_2(z) - \frac{zq'(z)}{q(z)}\right\}. \end{aligned} \tag{3.12}$$

Now, by Lemma 2.2, we have

$$\begin{aligned}\operatorname{Re}\left[h_i(z) + \frac{zh'_i(z)}{h_i(z)}\right] &\geqslant \operatorname{Re} h_i(z)\left[1 - \frac{2nr^n}{2-r^{2n}}\left|\frac{1}{h_i(z)+1}\right|\right] \\ &\geqslant \frac{1-r^n}{1+r^n}\left[1 - \frac{2nr^n}{1-r^{2n}}\frac{1}{+\frac{1-r^n}{1+r^n}}\right] \\ &= \frac{1-(n+1)r^n}{1+r^n}. \end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we have

$$\begin{aligned}&\operatorname{Re}\left[\left(h_i(z) + \frac{zh_i'(z)}{h_i(z)+1}\right) - \alpha H_i(z) + \alpha p_i(z) - \frac{zq'(z)}{q(z)}\right] \\ &\geqslant \frac{1-(n+1)r^n}{1+r^n} - \alpha\left(\frac{1+r^n}{1-r^n}\right) + \alpha\left(\frac{1-r^n}{1+r^n}\right) - \frac{2nr^n}{1-r^{2n}}. \\ &= \frac{2\left[(n+1)r^{2n} - (3n+4\alpha+2)r^n + 1\right]}{1-r^{2n}}. \end{aligned} \tag{3.14}$$

The right hand side of (3.14) is positive for $|z| < R_1$ and consequently from (3.12), it follows that $f \in R_k(n,0)$ for $|z| < R_1$ where R_1 is given by (3.10).

Sharpness follows when we consider

$$h_1(z) = \frac{zs_1'(z)}{s_1(z)}, \quad h_2(z) = \frac{zs_2'(z)}{s_2(z)},$$

with

$$\begin{aligned} s_1(z) &= \frac{z}{(1-z^n)^{\frac{2}{n}}}, \quad s_2(z) = \frac{z}{(1+z^n)^{\frac{2}{n}}}, \\ h_i(z) &= p_i(z) = \frac{1+z^n}{1-z^n} = q(z), \quad i = 1,2, \quad \text{and} \quad z = r^n. \end{aligned}$$

We now consider differential operator D_λ defined by (3.1) with $\alpha = \frac{\lambda}{1+\lambda}, \beta = 1$ and prove the following. □

THEOREM 3.4. *Let* $F \in R_k(n,\gamma)$, $0 \leqslant \gamma < 1$ *and let the function* f *be defined by*

$$D_\lambda(F(z)) = f(z) = \frac{\lambda}{1+\lambda}F(z) + \frac{1}{1+\lambda}zF'(z), \quad (\lambda > -1). \tag{3.15}$$

Then $f \in R_k(n,\gamma)$ *for* $|z| < r_{\lambda,\gamma}$, *where* $r_{\lambda,\gamma}$ *is given by*

$$r_{\lambda,\gamma} = \left[\frac{\lambda+1}{(n+1+\gamma)+\sqrt{(n+1+\gamma)^2+(\lambda+1)(2\gamma+\lambda-1)}}\right]^{\frac{1}{n}}. \tag{3.16}$$

This result is sharp.

Proof. Let

$$\frac{zF'(z)}{F(z)} = H(z) = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z). \tag{3.17}$$

From (3.15), we can write

$$F(z) = \frac{\lambda+1}{z^\lambda}\int_0^z t^{\lambda-1}f(t)dt. \tag{3.18}$$

Differentiating (3.18) logarithmically and using (3.17), we have

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= H(z) + \frac{zH'(z)}{H'(z)+\lambda}\\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left[h_1(z)+\frac{zh_1'(z)}{h_1(z)+\lambda}\right] - \left(\frac{k}{4}-\frac{1}{2}\right)\left[h_2(z)+\frac{zh_2'(z)}{h_2(z)+\lambda}\right]\end{aligned}$$

Since $H \in P_k(n,\gamma)$, $h_i \in P(n,\gamma)$, $i = 1,2$ and we write

$$h_i(z) = (1-\gamma)p_i(z) + \gamma, \quad p_i \in P(n), \quad i = 1,2.$$

Then

$$\begin{aligned}\frac{1}{1-\gamma}\left[\frac{zf'(z)}{f(z)} - \gamma\right] &= \left(\frac{k}{4}+\frac{1}{2}\right)\left[p_1(z) + \frac{\frac{1}{1-\gamma}zp_1'(z)}{p_1(z)+\frac{\gamma+\lambda}{1-\gamma}}\right]\\ &\quad - \left(\frac{k}{4}-\frac{1}{2}\right)\left[p_2(z) + \frac{\frac{1}{1-\gamma}zp_2'(z)}{p_2(z)+\frac{\gamma+\lambda}{1-\gamma}}\right]\end{aligned} \tag{3.19}$$

Now, using Lemma 2.2, we have for $i = 1,2$

$$\begin{aligned}&\mathrm{Re}\left[p_i(z) + \frac{\frac{1}{1-\gamma}zp_i'(z)}{p_i(z)+\frac{\gamma+\lambda}{1-\gamma}}\right]\\ &\geqslant \mathrm{Re}\,p_i(z) - \frac{\frac{1}{1-\gamma}\left(\frac{2nr^n\mathrm{Re}\,p_i(z)}{1-r^{2n}}\right)}{\frac{1-r^n}{1+r^n}+\left(\frac{\gamma+\lambda}{1-\gamma}\right)}\\ &= \mathrm{Re}\,p_i(z)\left[1 - \frac{2nr^n}{(1-r^n)[(1-\gamma)(1-r^n)+(\gamma+\lambda)(1+r^n)]}\right]\\ &= \mathrm{Re}\,p_i(z)\frac{(1+\lambda)+2(\gamma-n)r^n-(2\gamma+\lambda-1)r^{2n}}{(1-r^n)\{(1-\gamma)(1-r^n)+(\gamma+\lambda)(1+r^n)\}}.\end{aligned}$$

The right hand side is positive for $r \leqslant r_{\lambda,\gamma}$ where $r_{\lambda,\gamma}$ is given by (3.16).

Sharpness follows when we take

$$h_1(z) = \frac{zs_1'(z)}{s_1(z)}, \quad h_2(z) = \frac{zs_2'(z)}{s_2(z)},$$

with

$$s_1(z) = \frac{z}{(1-z^n)^{\frac{2}{n}(1-\gamma)}}, \quad s_2(z) = \frac{z}{(1+z^n)^{\frac{2}{n}(1-\gamma)}}$$

in (3.17). From (3.19), we conclude that $f \in R_k(n,\gamma)$ for $|z| < r_{\lambda,\gamma}$ and the proof is complete. □

THEOREM 3.5. *Let $F \in \mathscr{A}(n)$ and $F' \in P_k(n,\beta)$, $k \geqslant 2$, $0 \leqslant \beta < 1$ and let f be defined by (3.15). Then $f' \in P_k(n,\beta)$ for $|z| < R_n$, where*

$$R_n = \left\{ \frac{(\lambda+1)}{n+\sqrt{n^2+(\lambda+1)}} \right\}^{\frac{1}{n}}. \tag{3.20}$$

This result is sharp.

Proof. Let

$$F'(z) = H(z) = \left(\frac{k}{4}+\frac{1}{2}\right) H_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) H_2(z).$$

From (3.15), we have

$$\begin{aligned}(1+\lambda)\left[f'(z)-\beta\right] \;=\; & \left(\frac{k}{4}+\frac{1}{2}\right)\left[(1+\lambda)\{H_1(z)-\beta\}+zH_1'(z)\right] \\ & -\left(\frac{k}{4}-\frac{1}{2}\right)\left[(1+\lambda)\{H_2(z)-\beta\}+zH_2'(z)\right] \end{aligned} \tag{3.21}$$

Since $H_1, H_2 \in P(n,\beta)$, we use Lemma 2.2 and have for $i = 1,2$

$$\begin{aligned}\left[(1+\lambda)\{H_i(z)-\beta\}+zH_i'(z)\right] &= \text{Re}\;\{H_i(z)-\beta\}\left[(1+\lambda)-\frac{2nr^n}{1-r^{2n}}\right] \\ &= \text{Re}\;\{H_i(z)-\beta\}\left[\frac{(1+\lambda)-2nr^n-(1+\lambda)r^{2n}}{1-r^{2n}}\right].\end{aligned}$$

The right hand side is positive for $r < R_n$ and from (3.21) it follows that $f \in P_k(n,\beta)$ for $r < R_n$ where R_n is given by (3.20).

Sharpness follows by taking $H_i(z)$ in (3.21) as

$$H_i^o(z) = \frac{1-(2\beta-1)z^n}{1-z^n}.$$

It is clear that $\operatorname{Re}\{H_i^o(z)\} > \beta$ in E. Therefore, with some computation, we have

$$(1+\lambda)\{H_i^o(z)-\beta\}+z\,(H_i^o(z))' = \frac{(1-\beta)}{1+\lambda}\left[\frac{(1+\lambda)+2nz^n-(\lambda+1)z^{2n}}{(1-z^n)^2}\right].$$

This gives us

$$H_i^o(z)-\beta = 0 \quad \text{for} \quad z^n = -(R_n)$$

and it implies that $\operatorname{Re}[H_i^o(z)]$ is not greater than β in any disc $|z|<r$ if $r>R_n$. □

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