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DYNAMIC DOUBLE INTEGRAL INEQUALITIES IN TWO INDEPENDENT VARIABLES ON TIME SCALES

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Abstract. First, we establish some new nonlinear dynamic inequalities in two independent variables of Pachpatte type, that might be useful tools in the study of qualitative properties of solutions of certain classes of dynamic equations on time scales. These results extend recent inequalities for difference equations to the general time-scale setting. Then, after establishing a nabla Jensen's inequality, we relate several inequalities of Hilbert-Pachpatte type that extend and unify recent continuous and discrete inequalities of this type.

1. Introduction

The unification and extension of differential equations, difference equations, q-difference equations, and so on to the encompassing theory of dynamic equations on time scales was first accomplished by Hilger in his Ph. D. thesis [10]. Since then, time-scale calculus has made steady inroads in explaining the interconnections that exist among the various differential and difference theories, and in extending our understanding to a new, more general, robust, and overarching theory. The purpose of this note is to illustrate this new understanding by extending some discrete inequalities by Ma and Cheung [12], and continuous and discrete inequalities of Pachpatte [15], to arbitrary time scales; see also some related time-scale inequalities in Agarwal, Bohner, and Peterson [1], and Akin-Bohner, Bohner, and Akin [2]. In particular, in the first part of the paper we establish some general nonlinear dynamic inequalities on general time scales involving functions of two independent variables; these inequalities may be of use in the analysis of certain classes of partial dynamic equations on time scales, introduced by Jackson [11]. Next, we extend double sum and integral inequalities of Hilbert-Pachpatte type to general dynamic double integral inequalities on time scales.

Throughout this work a knowledge and understanding of time scales and time-scale notation is assumed; for an excellent introduction to the calculus on time scales, see Bohner and Peterson [7, 8] and the paper introducing nabla derivatives by Atici and Guseinov [4].

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2. Nonlinear inequalities on time scales

Let $\mathbb{T}$ be an arbitrary, unbounded time scale, and let $t_0 \in \mathbb{T}$. Before we arrive at the main results in this section, we need the following lemmas. Note that if $\mathbb{T} = \mathbb{Z}$, then Lemma 2.1 (I) below is a discrete inequality from Bainov and Simeonov [5, p. 161], and Lemma 2.1 (II) below is a discrete inequality from Pachpatte [17]. Furthermore if again $\mathbb{T} = \mathbb{Z}$, then Lemma 2.2 reduces to [12, Lemma 2.2]. Proceeding from the foundational lemmas, all of the resulting theorems and corresponding proofs in this section are modelled after those given in the special case of $\mathbb{T} = \mathbb{Z}$ recently presented by Ma and Cheung [12].

LEMMA 2.1. *Let $\mathbb{T}$ be an unbounded time scale with $t, t_0 \in \mathbb{T}$.*
(I) *Suppose $u, a, b, p, q \in C_{rd}$ and $b, p \geqslant 0$. If*

$$u(t) \leqslant a(t) + p(t) \int_{t_0}^{t} [b(\tau)u(\tau) + q(\tau)]\Delta\tau \tag{2.1}$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$, then

$$u(t) \leqslant a(t) + p(t) \int_{t_0}^{t} [a(\tau)b(\tau) + q(\tau)]e_{bp}(t, \sigma(\tau))\Delta\tau \tag{2.2}$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$.
(II) *Suppose $u, a, b, p, q \in C_{ld}$ and $b, p \geqslant 0$. If*

$$u(t) \leqslant a(t) + p(t) \int_{t}^{t_0} [b(\tau)u(\tau) + q(\tau)]\,\nabla\tau \tag{2.3}$$

for all $t \in (-\infty, t_0]_{\mathbb{T}}$, then

$$u(t) \leqslant a(t) + p(t) \int_{t}^{t_0} [a(\tau)b(\tau) + q(\tau)]\hat{e}_{-bp}(t, \rho(\tau))\nabla\tau \tag{2.4}$$

for all $t \in (-\infty, t_0]_{\mathbb{T}}$, where $\hat{e}$ is the nabla exponential function [3, Def 9].

Proof. The statement and proof of (I) are from [2, Theorem 3.1], so we focus on (II). Define

$$w(t) := -\int_{t}^{t_0} [b(\tau)u(\tau) + q(\tau)]\nabla\tau.$$

Then $w^{\nabla}(t) = b(t)u(t) + q(t)$ implies from assumption (2.3) that

$$w^{\nabla}(t) \leqslant b(t)a(t) - b(t)p(t)w(t) + q(t).$$

Since by assumption $b, p \geqslant 0$ we have that $-bp \in \mathscr{R}_{\nu}^{+}$ and $\hat{e}_{-bp}(t, t_0) > 0$ for all $t \in \mathbb{T}$ by [8, Thm 3.22 (i)]. Employing the nabla quotient rule, we have

$$\left[w/\hat{e}_{-bp}(\cdot, t_0)\right]^{\nabla}(t) = \left[w^{\nabla}(t) + b(t)p(t)w(t)\right]/\hat{e}_{-bp}(\rho(t), t_0);$$

integrating both sides from t to t_0 and using $w(t_0) = 0$, we obtain

$$\begin{aligned} -w(t)/\hat{e}_{-bp}(t,t_0) &= \int_t^{t_0} \left[w^{\nabla}(\tau) + b(\tau)p(\tau)w(\tau)\right] / \hat{e}_{-bp}(\rho(\tau),t_0)\nabla\tau \\ &\leqslant \int_t^{t_0} \left[b(\tau)a(\tau) + q(\tau)\right] / \hat{e}_{-bp}(\rho(\tau),t_0)\nabla\tau. \end{aligned}$$

Thus, using [8, Thm 3.15 (iv) & (v)], we have

$$\int_t^{t_0} [b(\tau)u(\tau) + q(\tau)]\nabla\tau \leqslant \int_t^{t_0} [b(\tau)a(\tau) + q(\tau)]\, \hat{e}_{-bp}(t,\rho(\tau))\nabla\tau,$$

and the result follows. □

EXAMPLE 1. If $\mathbb{T} = \mathbb{R}$, then $\rho(\tau) = \tau$, $\hat{e}_{-bp}(t,\tau) = \exp\left(-\int_\tau^t b(\gamma)p(\gamma)d\gamma\right)$, and Lemma 2.1 (II) reads as follows: Suppose $u,a,b,p,q \in \mathrm{C}$ and $b,p \geqslant 0$. If

$$u(t) \leqslant a(t) + p(t)\int_t^{t_0} [b(\tau)u(\tau) + q(\tau)]\, d\tau$$

for all $t \in (-\infty, t_0]$, then

$$u(t) \leqslant a(t) + p(t)\int_t^{t_0} [a(\tau)b(\tau) + q(\tau)] \exp\left(\int_t^{\tau} b(\gamma)p(\gamma)d\gamma\right) d\tau$$

for all $t \in (-\infty, t_0]$.

EXAMPLE 2. Consider the q-difference equations case. Let $q > 1$, and take

$$\mathbb{T} = q^{\mathbb{Z}} := \{0\} \cup \{q^n\}_{n\in\mathbb{Z}}.$$

Replace t by q^t, t_0 by q^{t_0}, and τ by q^τ. Then $\rho(\tau) = q^{\tau-1}$,

$$\hat{e}_{-bp}(t,\rho(\tau)) = \prod_{m=t+1}^{\tau-1} \left[1 + (q-1)q^{m-1}b(q^m)p(q^m)\right],$$

and Lemma 2.1 (II) reads as follows: Suppose u,a,b,p,r are functions and $b,p \geqslant 0$. If

$$u(q^t) \leqslant a(q^t) + (q-1)p(q^t)\sum_{\tau=t+1}^{t_0} q^\tau \left[b(q^\tau)u(q^\tau) + r(q^\tau)\right]$$

for all integers $t \in \{\cdots, t_0-2, t_0-1, t_0\}$, then

$$u(q^t) \leqslant a(q^t) + (q-1)p(q^t)\sum_{\tau=t+1}^{t_0} q^\tau[a(q^\tau)b(q^\tau) + r(q^\tau)] \prod_{m=t+1}^{\tau-1} \left[1 + (q-1)q^{m-1}b(q^m)p(q^m)\right]$$

for all integers $t \in \{\cdots, t_0-2, t_0-1, t_0\}$.

LEMMA 2.2. *Let $u(t,s)$, $a(t,s)$, $c(t,s)$, and $d(t,s)$ be nonnegative continuous functions defined for $(t,s) \in \mathbb{T}\times\mathbb{T}$, and let $w:\mathbb{R}\to\mathbb{R}$ be a nonnegative, nondecreasing continuous function on $[0,\infty)$, with $w(x)>0$ for $x>0$.*

(I) *Assume that $a(t,s)$ and $c(t,s)$ are nondecreasing in t and nonincreasing in s for $(t,s)\in\mathbb{T}\times\mathbb{T}$. If for $(t,s)\in[t_0,\infty)_{\mathbb{T}}\times[t_0,\infty)_{\mathbb{T}}$ we have*

$$u(t,s) \leqslant a(t,s) + c(t,s)\int_{t_0}^{t}\int_{s}^{\infty} d(\tau,\eta)w(u(\tau,\eta))\nabla\eta\Delta\tau, \tag{2.5}$$

then

$$u(t,s) \leqslant G^{-1}\left\{G(a(t,s)) + c(t,s)\int_{t_0}^{t}\int_{s}^{\infty} d(\tau,\eta)\nabla\eta\Delta\tau\right\} \tag{2.6}$$

for $(t,s)\in[t_0,t_1]_{\mathbb{T}}\times[s_1,\infty)_{\mathbb{T}}$, where $t_1,s_1\in[t_0,\infty)_{\mathbb{T}}$, $G(x)=\int_{x_0}^{x} dr/w(r)$ for $x_0,x>0$, G^{-1} is the inverse of G, and for $(t,s)\in[t_0,t_1]_{\mathbb{T}}\times[s_1,\infty)_{\mathbb{T}}$,

$$G(a(t,s)) + c(t,s)\int_{t_0}^{t}\int_{s}^{\infty} d(\tau,\eta)\nabla\eta\Delta\tau \in \mathrm{Dom}\,(G^{-1}),$$

where $\mathrm{Dom}\,(G^{-1})$ is the domain of G^{-1}.

(II) *Assume that $a(t,s)$ and $c(t,s)$ are nonincreasing in t and s for $(t,s)\in\mathbb{T}\times\mathbb{T}$. If for $(t,s)\in[t_0,\infty)_{\mathbb{T}}\times[t_0,\infty)_{\mathbb{T}}$ we have*

$$u(t,s) \leqslant a(t,s) + c(t,s)\int_{t}^{\infty}\int_{s}^{\infty} d(\tau,\eta)w(u(\tau,\eta))\nabla\eta\nabla\tau, \tag{2.7}$$

then

$$u(t,s) \leqslant G^{-1}\left\{G(a(t,s)) + c(t,s)\int_{t}^{\infty}\int_{s}^{\infty} d(\tau,\eta)\nabla\eta\nabla\tau\right\} \tag{2.8}$$

for $(t,s)\in[t_2,\infty)_{\mathbb{T}}\times[s_2,\infty)_{\mathbb{T}}$, where $t_2,s_2\in[t_0,\infty)_{\mathbb{T}}$ such that

$$G(a(t,s)) + c(t,s)\int_{t}^{\infty}\int_{s}^{\infty} d(\tau,\eta)\nabla\eta\nabla\tau \in \mathrm{Dom}\,(G^{-1})$$

for $(t,s)\in[t_2,\infty)_{\mathbb{T}}\times[s_2,\infty)_{\mathbb{T}}$.

Proof. First we consider (I). Fix any time-scale points $t^*\in[t_0,t_1]_{\mathbb{T}}$ and $s^*\in[s_1,\infty)_{\mathbb{T}}$. From (2.5) and the assumptions on $a(t,s)$ and $c(t,s)$ in (I) we have

$$u(t,s) \leqslant a(t^*,s^*) + c(t^*,s^*)\int_{t_0}^{t}\int_{s}^{\infty} d(\tau,\eta)w(u(\tau,\eta))\nabla\eta\Delta\tau \tag{2.9}$$

for $(t,s)\in[t_0,t^*]_{\mathbb{T}}\times[s^*,\infty)_{\mathbb{T}}$. Let us define $x_1(t,s)$ to be the right-hand side of inequality (2.9) above; then

$$u(t,s)\leqslant x_1(t,s), \quad \text{with} \quad x_1(t_0,s)=a(t^*,s^*)=x_1(t,\infty).$$

By the definition of $x_1(t,s)$, we get

$$x_1^{\Delta_t}(t,s) = c(t^*,s^*)\int_{s}^{\infty} d(t,\eta)w(u(t,\eta))\nabla\eta \leqslant c(t^*,s^*)w(x_1(t,s))\int_{s}^{\infty} d(t,\eta)\nabla\eta,$$

since w is nondecreasing and $u(t,\eta) \leqslant x_1(t,\eta) \leqslant x_1(t,s)$ for all $\eta \in [s,\infty)_{\mathbb{T}}$. Now $x_1 > 0$ implies $w(x_1(t,s)) > 0$, yielding

$$\frac{x_1^{\Delta_t}(t,s)}{w(x_1(t,s))} \leqslant c(t^*,s^*) \int_s^\infty d(t,\eta)\nabla\eta. \tag{2.10}$$

Take $G(x) := \int_{x_0}^x dr/w(r)$ for $x_0, x > 0$, and let G^{-1} denote the inverse of G. Then

$$\begin{aligned} G^{\Delta_t}(x_1(t,s)) &= \left(\int_{x_0}^{x_1(t,s)} d\gamma/w(\gamma)\right)^{\Delta_t} = \lim_{\theta\to t} \frac{1}{\sigma(t)-\theta} \int_{x_1(\theta,s)}^{x_1(\sigma(t),s)} d\gamma/w(\gamma) \\ &\leqslant \lim_{\theta\to t} \frac{1}{\sigma(t)-\theta} \int_{x_1(\theta,s)}^{x_1(\sigma(t),s)} d\gamma \cdot \frac{1}{w(x_1(\theta,s))} = \frac{x_1^{\Delta_t}(t,s)}{w(x_1(t,s))} \end{aligned}$$

as w is nondecreasing. By (2.10),

$$G^{\Delta_t}(x_1(t,s)) \leqslant c(t^*,s^*) \int_s^\infty d(t,\eta)\nabla\eta. \tag{2.11}$$

For fixed s, delta integrate (2.11) from t_0 to t, and use the fact that $x_1(t_0,s) = a(t^*,s^*)$ to obtain

$$G(x_1(t,s)) \leqslant G(a(t^*,s^*)) + c(t^*,s^*) \int_{t_0}^t \int_s^\infty d(\tau,\eta)\nabla\eta\Delta\tau.$$

Setting $t = t^*$ and $s = s^*$ in (2.9) and the last inequality shows that $u(t^*,s^*) \leqslant x_1(t^*,s^*)$, and

$$G(x_1(t^*,s^*)) \leqslant G(a(t^*,s^*)) + c(t^*,s^*) \int_{t_0}^{t^*} \int_{s^*}^\infty d(\tau,\eta)\nabla\eta\Delta\tau.$$

By the arbitrary nature of t^* and s^*, we have

$$u(t,s) \leqslant x_1(t,s), \quad G(x_1(t,s)) \leqslant G(a(t,s)) + c(t,s) \int_{t_0}^t \int_s^\infty d(\tau,\eta)\nabla\eta\Delta\tau \tag{2.12}$$

for $(t,s) \in [t_0,t_1]_{\mathbb{T}} \times [s_1,\infty)_{\mathbb{T}}$. The desired inequality (2.6) then follows from (2.12).

Next we consider (II). Fix any time-scale points $t^* \in [t_2,\infty)_{\mathbb{T}}$ and $s^* \in [s_2,\infty)_{\mathbb{T}}$. From (2.7) and the assumptions on $a(t,s)$ and $c(t,s)$ in (II) we have

$$u(t,s) \leqslant a(t^*,s^*) + c(t^*,s^*) \int_t^\infty \int_s^\infty d(\tau,\eta) w(u(\tau,\eta))\nabla\eta\nabla\tau \tag{2.13}$$

for $(t,s) \in [t^*,\infty)_{\mathbb{T}} \times [s^*,\infty)_{\mathbb{T}}$. Let us define $x_2(t,s)$ to be the right-hand side of inequality (2.13) above; then

$$u(t,s) \leqslant x_2(t,s), \quad \text{with} \quad x_2(t,\infty) = a(t^*,s^*) = x_2(\infty,s).$$

By the definition of $x_2(t,s)$, we get

$$x_2^{\nabla_t}(t,s) = -c(t^*,s^*) \int_s^\infty d(t,\eta) w(u(t,\eta))\nabla\eta \geqslant -c(t^*,s^*) w(x_2(t,s)) \int_s^\infty d(t,\eta)\nabla\eta,$$

since w is nondecreasing and $u(t,\eta) \leqslant x_2(t,\eta) \leqslant x_2(t,s)$ for all $\eta \in [s,\infty)_{\mathbb{T}}$. Now $x_2 > 0$ implies $w(x_2(t,s)) > 0$, yielding

$$\frac{x_2^{\nabla_t}(t,s)}{w(x_2(t,s))} \geqslant -c(t^*,s^*)\int_s^\infty d(t,\eta)\nabla\eta. \tag{2.14}$$

Consider

$$G^{\nabla_t}(x_2(t,s)) = \left(\int_{x_0}^{x_2(t,s)} d\gamma/w(\gamma)\right)^{\nabla_t};$$

as in the derivation of (2.11) we have

$$G^{\nabla_t}(x_2(t,s)) \geqslant -c(t^*,s^*)\int_s^\infty d(t,\eta)\nabla\eta. \tag{2.15}$$

For fixed s, nabla integrate (2.15) from t to ∞, and use the fact that $x_2(\infty,s) = a(t^*,s^*)$ to obtain

$$G(x_2(t,s)) \leqslant G(a(t^*,s^*)) + c(t^*,s^*)\int_t^\infty\int_s^\infty d(\tau,\eta)\nabla\eta\nabla\tau.$$

Setting $t = t^*$ and $s = s^*$ in (2.13) and the last inequality shows that $u(t^*,s^*) \leqslant x_2(t^*,s^*)$, and

$$G(x_2(t^*,s^*)) \leqslant G(a(t^*,s^*)) + c(t^*,s^*)\int_{t^*}^\infty\int_{s^*}^\infty d(\tau,\eta)\nabla\eta\nabla\tau.$$

By the arbitrary nature of t^* and s^*, we have

$$u(t,s) \leqslant x_2(t,s), \quad G(x_2(t,s)) \leqslant G(a(t,s)) + c(t,s)\int_t^\infty\int_s^\infty d(\tau,\eta)\nabla\eta\nabla\tau \tag{2.16}$$

for $(t,s) \in [t_0,t_1]_{\mathbb{T}} \times [s_1,\infty)_{\mathbb{T}}$. The desired inequality (2.8) then follows from (2.16). □

THEOREM 2.3. *Let $u(t,s)$, $a(t,s)$, $c(t,s)$, $d(t,s)$, and $w(u)$ be as in Lemma 2.2 (I), and let $b(t,s)$ be a nonnegative continuous function for $(t,s) \in [t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}}$. Let $\varphi \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ with $\varphi'(u) > 0$ for $u > 0$, where the prime indicates the traditional derivative. If for $(t,s) \in [t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}}$ we have*

$$\varphi(u(t,s)) \leqslant a(t,s) + c(t,s)\int_{t_0}^t\int_s^\infty \varphi'(u(\tau,\eta))\left[d(\tau,\eta)w(u(\tau,\eta)) + b(\tau,\eta)\right]\nabla\eta\Delta\tau, \tag{2.17}$$

then

$$u(t,s) \leqslant G^{-1}\left\{G\left(\varphi^{-1}(a(t,s)) + B(t,s)\right) + c(t,s)\int_{t_0}^t\int_s^\infty d(\tau,\eta)\nabla\eta\Delta\tau\right\} \tag{2.18}$$

for $B(t,s) := c(t,s)\int_{t_0}^t\int_s^\infty b(\tau,\eta)\nabla\eta\Delta\tau$ and for $(t,s) \in [t_0,t_3]_{\mathbb{T}} \times [s_3,\infty)_{\mathbb{T}}$, where $t_3, s_3 \in [t_0,\infty)_{\mathbb{T}}$ such that

$$G\left(\varphi^{-1}(a(t,s)) + B(t,s)\right) + c(t,s)\int_{t_0}^t\int_s^\infty d(\tau,\eta)\nabla\eta\Delta\tau \in \mathrm{Dom}\,(G^{-1})$$

for all $(t,s) \in [t_0,t_3]_{\mathbb{T}} \times [s_3,\infty)_{\mathbb{T}}$. Here G is as in Lemma 2.2.

Proof. Fix any time-scale points $t^* \in [t_0, t_3]_{\mathbb{T}}$ and $s^* \in [s_3, \infty)_{\mathbb{T}}$. From (2.17) and the assumptions on $a(t,s)$ and $c(t,s)$ we have

$$\varphi(u(t,s)) \leqslant a(t^*,s^*)+c(t^*,s^*)\int_{t_0}^{t}\int_{s}^{\infty}\varphi'(u(\tau,\eta))\left[d(\tau,\eta)w(u(\tau,\eta))+b(\tau,\eta)\right]\nabla\eta\Delta\tau \tag{2.19}$$

for $(t,s) \in [t_0,t^*]_{\mathbb{T}} \times [s^*,\infty)_{\mathbb{T}}$. Let us define $x_3(t,s)$ to be the right-hand side of inequality (2.19) above; then

$$u(t,s) \leqslant \varphi^{-1}(x_3(t,s)), \quad \text{with} \quad x_3(t_0,s) = a(t^*,s^*) = x_3(t,\infty). \tag{2.20}$$

By the definition of $x_3(t,s)$, we get

$$\begin{aligned} x_3^{\Delta_t}(t,s) &= c(t^*,s^*)\int_s^{\infty}\varphi'(u(t,\eta))\left[d(t,\eta)w(u(t,\eta))+b(t,\eta)\right]\nabla\eta \\ &\leqslant c(t^*,s^*)\varphi'\left(\varphi^{-1}(x_3(t,s))\right)\int_s^{\infty}\left[d(t,\eta)w(\varphi^{-1}(x_3(t,\eta)))+b(t,\eta)\right]\nabla\eta, \end{aligned}$$

since w is nondecreasing and $u(t,\eta) \leqslant \varphi^{-1}(x_3(t,\eta)) \leqslant \varphi^{-1}(x_3(t,s))$ for all $\eta \in [s,\infty)_{\mathbb{T}}$. This yields

$$\frac{x_3^{\Delta_t}(t,s)}{\varphi'\left(\varphi^{-1}(x_3(t,s))\right)} \leqslant c(t^*,s^*)\int_s^{\infty}\left[d(t,\eta)w(\varphi^{-1}(x_3(t,\eta)))+b(t,\eta)\right]\nabla\eta. \tag{2.21}$$

Consider $\left[\varphi^{-1}(x_3(t,s))\right]^{\Delta_t}$. Then

$$\left[\varphi^{-1}(x_3(t,s))\right]^{\Delta_t} = \frac{x_3^{\Delta_t}(t,s)}{\varphi'(\varphi^{-1}(\theta))} \leqslant \frac{x_3^{\Delta_t}(t,s)}{\varphi'\left(\varphi^{-1}(x_3(t,s))\right)},$$

where we have used the differential Mean Value Theorem on φ, for some $\theta \in \mathbb{R}$ between $x_3(t,s)$ and $x_3(\sigma(t),s)$. By (2.21),

$$\left[\varphi^{-1}(x_3(t,s))\right]^{\Delta_t} \leqslant c(t^*,s^*)\int_s^{\infty}\left[d(t,\eta)w(\varphi^{-1}(x_3(t,\eta)))+b(t,\eta)\right]\nabla\eta. \tag{2.22}$$

For fixed s, delta integrate (2.22) from t_0 to t, and use the fact that $x_3(t_0,s) = a(t^*,s^*)$ to obtain

$$\begin{aligned} \varphi^{-1}(x_3(t,s)) \leqslant{}& \varphi^{-1}(a(t^*,s^*)) + c(t^*,s^*) \\ &\times \int_{t_0}^{t}\int_s^{\infty}\left[d(\tau,\eta)w(\varphi^{-1}(x_3(\tau,\eta)))+b(\tau,\eta)\right]\nabla\eta\Delta\tau. \end{aligned}$$

Applying Lemma 2.2 (I) to this inequality we obtain

$$\begin{aligned} \varphi^{-1}(x_3(t,s)) \leqslant{}& G^{-1}\left\{G\left[\varphi^{-1}(a(t^*,s^*)) + c(t^*,s^*)\int_{t_0}^{t}\int_s^{\infty}b(\tau,\eta)\nabla\eta\Delta\tau\right]\right. \\ &\left.+c(t^*,s^*)\int_{t_0}^{t}\int_s^{\infty}d(\tau,\eta)\nabla\eta\Delta\tau\right\} \end{aligned} \tag{2.23}$$

for all $(t,s) \in [t_0,t^*]_{\mathbb{T}} \times [s^*,\infty)_{\mathbb{T}}$. In particular, (2.23) holds for $(t,s) = (t^*,s^*)$. By the arbitrary nature of t^* and s^*, we have from (2.20) that (2.18) follows. □

THEOREM 2.4. *Let $u(t,s)$, $a(t,s)$, $c(t,s)$, $d(t,s)$, and $w(u)$ be as in Lemma 2.2 (II), and let $\varphi(u)$ and $b(t,s)$ be as in Theorem 2.3. If for $(t,s) \in [t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}}$ we have*

$$\varphi(u(t,s)) \leqslant a(t,s) + c(t,s) \int_t^\infty \int_s^\infty \varphi'(u(\tau,\eta))\left[d(\tau,\eta)w(u(\tau,\eta)) + b(\tau,\eta)\right] \nabla\eta\nabla\tau, \tag{2.24}$$

then

$$u(t,s) \leqslant G^{-1}\left\{G\left(\varphi^{-1}(a(t,s)) + \overline{B}(t,s)\right) + c(t,s)\int_t^\infty \int_s^\infty d(\tau,\eta)\nabla\eta\nabla\tau\right\} \tag{2.25}$$

for $\overline{B}(t,s) := c(t,s)\int_t^\infty \int_s^\infty b(\tau,\eta)\nabla\eta\nabla\tau$ and for $(t,s) \in [t_4,\infty)_{\mathbb{T}} \times [s_4,\infty)_{\mathbb{T}}$, where $t_4, s_4 \in [t_0,\infty)_{\mathbb{T}}$ such that

$$G\left(\varphi^{-1}(a(t,s)) + \overline{B}(t,s)\right) + c(t,s)\int_t^\infty \int_s^\infty d(\tau,\eta)\nabla\eta\nabla\tau \in \mathrm{Dom}\left(G^{-1}\right)$$

for all $(t,s) \in [t_4,\infty)_{\mathbb{T}} \times [s_4,\infty)_{\mathbb{T}}$. Here G is as in Lemma 2.2.

Proof. The proof of Theorem 2.4 using Lemma 2.2 (II) is similar to the proof given above of Theorem 2.3 using Lemma 2.2 (I), and thus is omitted. □

THEOREM 2.5. *Let $u(t,s)$, $a(t,s)$, $b(t,s)$, $c(t,s)$, $w(u)$, and $\varphi(u)$ be as in Theorem 2.3. Let $d(t,s)$, $f(t,s)$, and $g(t,s)$ be nonnegative continuous functions defined for $(t,s) \in [t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}}$, with $d(t,s)$ and $g(t,s)$ nondecreasing in t and nonincreasing in s. If for $(t,s) \in [t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}}$ we have*

$$\begin{aligned}\varphi(u(t,s)) \leqslant{}& a(t,s) + g(t,s)\int_{t_0}^t c(\tau,s)\varphi(u(\tau,s))\Delta\tau \\ &+ d(t,s)\int_{t_0}^t \int_s^\infty \varphi'(u(\tau,\eta))\left[f(\tau,\eta)w(u(\tau,\eta)) + b(\tau,\eta)\right]\nabla\eta\Delta\tau,\end{aligned} \tag{2.26}$$

then

$$\begin{aligned}u(t,s) \leqslant{}& G^{-1}\Big\{G\left(\varphi^{-1}(a(t,s)p(t,s)) + p(t,s)B_1(t,s)\right) \\ &+ p(t,s)d(t,s)\int_{t_0}^t \int_s^\infty f(\tau,\eta)\nabla\eta\Delta\tau\Big\}\end{aligned} \tag{2.27}$$

for

$$B_1(t,s) := d(t,s)\int_{t_0}^t \int_s^\infty b(\tau,\eta)\nabla\eta\Delta\tau, \tag{2.28}$$

$$p(t,s) := 1 + g(t,s)\int_{t_0}^t c(\tau,s)e_{cg}(t,\sigma(\tau))\Delta\tau, \tag{2.29}$$

and for $(t,s)\in[t_0,t_5]_{\mathbb{T}}\times[s_5,\infty)_{\mathbb{T}}$, *where* $t_5,s_5\in[t_0,\infty)_{\mathbb{T}}$ *such that*

$$G\left(\varphi^{-1}(a(t,s)p(t,s))+p(t,s)B_1(t,s)\right)+p(t,s)d(t,s)\int_{t_0}^{t}\int_{s}^{\infty}f(\tau,\eta)\nabla\eta\Delta\tau \in \mathrm{Dom}\,(G^{-1})$$

for all $(t,s)\in[t_0,t_5]_{\mathbb{T}}\times[s_5,\infty)_{\mathbb{T}}$. *Here* G *is as in Lemma* 2.2.

Proof. Define the function

$$z(t,s):=a(t,s)+d(t,s)\int_{t_0}^{t}\int_{s}^{\infty}\varphi'(u(\tau,\eta))\left[f(\tau,\eta)w(u(\tau,\eta))+b(\tau,\eta)\right]\nabla\eta\Delta\tau.$$

Then assumption (2.26) becomes

$$\varphi(u(t,s))\leqslant z(t,s)+g(t,s)\int_{t_0}^{t}c(\tau,s)\varphi(u(\tau,s))\Delta\tau. \tag{2.30}$$

By the conditions on the various functions, $z(t,s)$ is nonnegative for $t,s\in[t_0,\infty)_{\mathbb{T}}$. For fixed $s\in[t_0,\infty)_{\mathbb{T}}$, an application of Lemma 2.1 (I) yields

$$\varphi(u(t,s))\leqslant z(t,s)+g(t,s)\int_{t_0}^{t}c(\tau,s)z(\tau,s)e_{cg}(t,\sigma(\tau))\Delta\tau.$$

As $z(t,s)$ is nondecreasing in $t\in[t_0,\infty)_{\mathbb{T}}$, we see from the previous inequality that

$$\varphi(u(t,s))\leqslant z(t,s)p(t,s), \tag{2.31}$$

where $p(t,s)$ is defined in (2.29). After plugging back in the expression for $z(t,s)$, from (2.31) we have that

$$\begin{aligned}\varphi(u(t,s)) \;\leqslant\; & p(t,s)\left(a(t,s)+d(t,s)\int_{t_0}^{t}\int_{s}^{\infty}\varphi'(u(\tau,\eta))[f(\tau,\eta)w(u(\tau,\eta))\right.\\ &\left.+\,b(\tau,\eta)]\nabla\eta\Delta\tau\right).\end{aligned} \tag{2.32}$$

Note that by our assumptions, $p(t,s)$, $a(t,s)$, and $d(t,s)$ are all nondecreasing in t and nonincreasing in s for $t,s\in[t_0,\infty)_{\mathbb{T}}$; likewise with $p(t,s)a(t,s)$ and $p(t,s)d(t,s)$. Applying Theorem 2.3 directly to (2.32), we arrive at the bound for $u(t,s)$ given in (2.27). □

THEOREM 2.6. *Let* $u(t,s)$, $b(t,s)$, $f(t,s)$, $w(u)$, *and* $\varphi(u)$ *be as in Theorem* 2.5. *Let* $a(t,s)$, $c(t,s)$, $d(t,s)$, *and* $g(t,s)$ *be nonnegative continuous functions defined for* $(t,s)\in[t_0,\infty)_{\mathbb{T}}\times[t_0,\infty)_{\mathbb{T}}$ *nonincreasing in both* t *and* s. *If for* $t,s,T\in[t_0,\infty)_{\mathbb{T}}$ *we have*

$$\begin{aligned}\varphi(u(t,s)) \leqslant\; & a(t,s)+g(t,s)\int_{t}^{T}c(\tau,s)\varphi(u(\tau,s))\nabla\tau\\ &+d(t,s)\int_{t}^{\infty}\int_{s}^{\infty}\varphi'(u(\tau,\eta))\left[f(\tau,\eta)w(u(\tau,\eta))+b(\tau,\eta)\right]\nabla\eta\nabla\tau,\end{aligned} \tag{2.33}$$

then

$$u(t,s) \leqslant G^{-1}\Big\{G\left(\varphi^{-1}(a(t,s)\overline{p}(t,s)) + \overline{p}(t,s)\overline{B}_1(t,s)\right) + \overline{p}(t,s)d(t,s)\int_t^\infty\int_s^\infty f(\tau,\eta)\nabla\eta\nabla\tau\Big\} \tag{2.34}$$

for

$$\overline{B}_1(t,s) := d(t,s)\int_t^\infty\int_s^\infty b(\tau,\eta)\nabla\eta\nabla\tau, \tag{2.35}$$

$$\overline{p}(t,s) := 1 + g(t,s)\int_t^T c(\tau,s)\hat{e}_{-cg}(t,\rho(\tau))\nabla\tau, \tag{2.36}$$

and for $(t,s) \in [t_6,T]_{\mathbb{T}} \times [s_6,\infty)_{\mathbb{T}}$, *where* $t_6, s_6 \in [t_0,\infty)_{\mathbb{T}}$ *such that*

$$G\left(\varphi^{-1}(a(t,s)\overline{p}(t,s)) + \overline{p}(t,s)\overline{B}_1(t,s)\right) + \overline{p}(t,s)d(t,s)\int_t^\infty\int_s^\infty f(\tau,\eta)\nabla\eta\nabla\tau \in \mathrm{Dom}\,(G^{-1})$$

for all $(t,s) \in [t_6,\infty)_{\mathbb{T}} \times [s_6,\infty)_{\mathbb{T}}$. *Here* G *is as in Lemma* 2.2.

Proof. The proof of Theorem 2.6 using Lemma 2.1 (II) and Theorem 2.4 is similar to the proof given above of Theorem 2.5 using Lemma 2.1 (I) and Theorem 2.3, and thus is omitted. □

REMARK 1. By choosing suitable functions for φ, new dynamic inequalities in two variables of Gronwall-Ou-Iang [14] and other types can be obtained from Theorems 2.5 and 2.6. The following corollaries illustrate two possibilites.

COROLLARY 2.7. *Let* $a(t,s)$, $b(t,s)$, $c(t,s)$, $d(t,s)$, $f(t,s)$, $g(t,s)$, $u(t,s)$, *and* $w(u)$ *all be as defined in Theorem* 2.5, *with* $\varphi(u) := u^k$ *for any real number* $k \geqslant 1$. *If for* $(t,s) \in [t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}}$ *we have*

$$u^k(t,s) \leqslant a(t,s) + g(t,s)\int_{t_0}^t c(\tau,s)u^k(\tau,s)\Delta\tau + d(t,s)\int_{t_0}^t\int_s^\infty u^{k-1}(\tau,\eta)\left[f(\tau,\eta)w(u(\tau,\eta)) + b(\tau,\eta)\right]\nabla\eta\Delta\tau, \tag{2.37}$$

then for $(t,s) \in [t_0,t_7]_{\mathbb{T}} \times [s_7,\infty)_{\mathbb{T}}$ *we have*

$$u(t,s) \leqslant G^{-1}\Big\{G\left(a^{1/k}(t,s)p^{1/k}(t,s) + \frac{1}{k}p(t,s)B_1(t,s)\right) + \frac{1}{k}p(t,s)d(t,s)\int_{t_0}^t\int_s^\infty f(\tau,\eta)\nabla\eta\Delta\tau\Big\} \tag{2.38}$$

for $B_1(t,s)$ *given in* (2.28) *and* $p(t,s)$ *given in* (2.29)*, where* $t_7, s_7 \in [t_0,\infty)_{\mathbb{T}}$ *such that*

$$G\left(a^{1/k}(t,s)p^{1/k}(t,s)+\frac{1}{k}p(t,s)B_1(t,s)\right)+\frac{1}{k}p(t,s)d(t,s)\int_{t_0}^{t}\int_{s}^{\infty}f(\tau,\eta)\nabla\eta\Delta\tau \in \mathrm{Dom}\,(G^{-1})$$

for all $(t,s)\in[t_0,t_7]_{\mathbb{T}}\times[s_7,\infty)_{\mathbb{T}}$. *Here* G *is as in Lemma* 2.2.

COROLLARY 2.8. *Let* $b(t,s)$, $c(t,s)$, $d(t,s)$, $f(t,s)$, $g(t,s)$, *and* $w(u)$ *all be as defined in Theorem* 2.5*, with* $\varphi(v) := \exp(kv)$ *for any real number* $k>0$. *Suppose* $a(t,s),u(t,s):[t_0,\infty)^2_{\mathbb{T}}\to[1,\infty)_{\mathbb{R}}$. *If for* $(t,s)\in[t_0,\infty)_{\mathbb{T}}\times[t_0,\infty)_{\mathbb{T}}$ *we have*

$$\begin{aligned} u^k(t,s) &\leqslant a(t,s)+g(t,s)\int_{t_0}^{t}c(\tau,s)u^k(\tau,s)\Delta\tau \\ &\quad +d(t,s)\int_{t_0}^{t}\int_{s}^{\infty}u^k(\tau,\eta)\left[f(\tau,\eta)w(\log u(\tau,\eta))+b(\tau,\eta)\right]\nabla\eta\Delta\tau, \end{aligned} \tag{2.39}$$

then for $(t,s)\in[t_0,t_8]_{\mathbb{T}}\times[s_8,\infty)_{\mathbb{T}}$ *we have*

$$\begin{aligned} u(t,s) &\leqslant \exp\left[G^{-1}\left\{G\left(\frac{1}{k}\log(a(t,s)p(t,s))+\frac{1}{k}p(t,s)B_1(t,s)\right)\right.\right. \\ &\quad \left.\left.+\frac{1}{k}p(t,s)d(t,s)\int_{t_0}^{t}\int_{s}^{\infty}f(\tau,\eta)\nabla\eta\Delta\tau\right\}\right] \end{aligned} \tag{2.40}$$

for $B_1(t,s)$ *given in* (2.28) *and* $p(t,s)$ *given in* (2.29)*, where* $t_8, s_8 \in [t_0,\infty)_{\mathbb{T}}$ *such that*

$$G\left(\frac{1}{k}\log(a(t,s)p(t,s))+\frac{1}{k}p(t,s)B_1(t,s)\right)+\frac{1}{k}p(t,s)d(t,s)\int_{t_0}^{t}\int_{s}^{\infty}f(\tau,\eta)\nabla\eta\Delta\tau \in \mathrm{Dom}\,(G^{-1})$$

for all $(t,s)\in[t_0,t_8]_{\mathbb{T}}\times[s_8,\infty)_{\mathbb{T}}$. *Here* G *is as in Lemma* 2.2.

Proof. Using the change of variables $v(t,s)=\log u(t,s)$, inequality (2.39) becomes

$$\begin{aligned} \exp(kv(t,s)) &\leqslant a(t,s)+g(t,s)\int_{t_0}^{t}c(\tau,s)\exp(kv(\tau,s))\Delta\tau \\ &\quad +d(t,s)\int_{t_0}^{t}\int_{s}^{\infty}\exp(kv(\tau,\eta))\left[f(\tau,\eta)w(v(\tau,\eta))+b(\tau,\eta)\right]\nabla\eta\Delta\tau, \end{aligned}$$

which is a special case of (2.26). By Theorem 2.5, inequality (2.40) follows. □

REMARK 2. Corollaries 2.7 and 2.8 above extend [12, Corollaries 2.7 & 2.8] to arbitrary time scales. Those corollaries themselves were generalizations of theorems due to Pachpatte [16, Theorems 1 & 2].

THEOREM 2.9. *Let $u(t,s)$, $a(t,s)$, $b(t,s)$, $c(t,s)$, $d(t,s)$, $f(t,s)$, $g(t,s)$, and $\varphi(u)$ be as in Theorem 2.5. Suppose $L,M : [t_0,\infty)_{\mathbb{T}}^2 \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions that for $t,s \in [t_0,\infty)_{\mathbb{T}}$ and $v,w \in \mathbb{R}_+$ satisfy the inequality*

$$0 \leqslant L(t,s,v) - L(t,s,w) \leqslant M(t,s,w)(v-w), \quad w \leqslant v. \tag{2.41}$$

If for $(t,s) \in [t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}}$ we have

$$\begin{aligned}\varphi(u(t,s)) \leqslant\ & a(t,s) + g(t,s)\int_{t_0}^{t} c(\tau,s)\varphi(u(\tau,s))\Delta\tau \\ & + d(t,s)\int_{t_0}^{t}\int_{s}^{\infty}\varphi'(u(\tau,\eta))\left[f(\tau,\eta)L(\tau,\eta,u(\tau,\eta))+b(\tau,\eta)\right]\nabla\eta\Delta\tau, \end{aligned} \tag{2.42}$$

then

$$u(t,s) \leqslant \mathscr{N}(t,s) + p(t,s)d(t,s)\mathscr{L}(t,s)\exp(\mathscr{M}(t,s)) \tag{2.43}$$

for

$$\begin{aligned} \mathscr{N}(t,s) &:= \varphi^{-1}(p(t,s)a(t,s)) + p(t,s)B_1(t,s), \\ \mathscr{L}(t,s) &:= \int_{t_0}^{t}\int_{s}^{\infty} f(\tau,\eta)L(\tau,\eta,\mathscr{N}(\tau,\eta))\nabla\eta\Delta\tau, \\ \mathscr{M}(t,s) &:= \int_{t_0}^{t}\int_{s}^{\infty} f(\tau,\eta)p(\tau,\eta)d(\tau,\eta)M(\tau,\eta,\mathscr{N}(\tau,\eta))\nabla\eta\Delta\tau, \end{aligned}$$

where B_1 and $p(t,s)$ are given by (2.28) *and* (2.29)*, respectively.*

Proof. We proceed with the argument in the manner of the proof of Theorem 2.5. Applying Lemma 2.1 (I) to (2.42), we get that

$$\begin{aligned}\varphi(u(t,s)) \leqslant\ & p(t,s)a(t,s) + p(t,s)d(t,s) \\ & \times \int_{t_0}^{t}\int_{s}^{\infty}\varphi'(u(\tau,\eta))\left[f(\tau,\eta)L(\tau,\eta,u(\tau,\eta))+b(\tau,\eta)\right]\nabla\eta\Delta\tau, \end{aligned} \tag{2.44}$$

for all $t,s \in [t_0,\infty)_{\mathbb{T}}$. After setting the right-hand side of (2.44) to be the continuous function $z(t,s)$, we use a procedure similar to that employed in the proof of Theorem 2.3 to determine that

$$\begin{aligned} u(t,s) &\leqslant \varphi^{-1}(z(t,s)) \qquad (2.45)\\ &\leqslant \mathscr{N}(t,s) + p(t,s)d(t,s)\int_{t_0}^{t}\int_{s}^{\infty} f(\tau,\eta)L\left(\tau,\eta,\varphi^{-1}(z(\tau,\eta))\right)\nabla\eta\Delta\tau \end{aligned}$$

for all $t,s \in [t_0,\infty)_{\mathbb{T}}$, where $\mathscr{N}$ is given above in the statement of the theorem. If we set

$$\xi(t,s) := \int_{t_0}^{t}\int_{s}^{\infty} f(\tau,\eta)L\left(\tau,\eta,\varphi^{-1}(z(\tau,\eta))\right)\nabla\eta\Delta\tau, \tag{2.46}$$

then the inequality (2.45) may be rewritten as

$$\varphi^{-1}(z(t,s)) \leqslant \mathscr{N}(t,s) + p(t,s)d(t,s)\xi(t,s), \quad t,s \in [t_0,\infty)_{\mathbb{T}}. \tag{2.47}$$

Since $L(t,s,v)$ is nondecreasing in v for fixed t and s, by (2.46) and (2.47) with condition (2.41) we see that

$$\begin{aligned}\xi(t,s) &\leqslant \int_{t_0}^{t}\int_{s}^{\infty} f(\tau,\eta)L(\tau,\eta,\mathscr{N}(\tau,\eta)+p(\tau,\eta)d(\tau,\eta)\xi(\tau,\eta))\nabla\eta\Delta\tau \\ &\leqslant \int_{t_0}^{t}\int_{s}^{\infty} f(\tau,\eta)L(\tau,\eta,\mathscr{N}(\tau,\eta))\nabla\eta\Delta\tau \\ &\quad + \int_{t_0}^{t}\int_{s}^{\infty} f(\tau,\eta)p(\tau,\eta)d(\tau,\eta)M(\tau,\eta,\mathscr{N}(\tau,\eta))\,\xi(\tau,\eta)\nabla\eta\Delta\tau.\end{aligned}$$

Applying Lemma 2.2 (I), the case where $w(u)=u$ and $c(t,s)\equiv 1$, to the previous inequality yields

$$\begin{aligned}\xi(t,s) &\leqslant \left(\int_{t_0}^{t}\int_{s}^{\infty} f(\tau,\eta)L(\tau,\eta,\mathscr{N}(\tau,\eta))\nabla\eta\Delta\tau\right) \\ &\quad \times\exp\left(\int_{t_0}^{t}\int_{s}^{\infty} f(\tau,\eta)p(\tau,\eta)d(\tau,\eta)M(\tau,\eta,\mathscr{N}(\tau,\eta))\nabla\eta\Delta\tau\right) \\ &= \mathscr{L}(t,s)\exp(\mathscr{M}(t,s))\end{aligned}$$

where $\mathscr{L}$ and $\mathscr{M}$ are given above in the statement of the theorem. Using (2.45), (2.47), and the previous inequality, we arrive at the desired (2.43). □

THEOREM 2.10. *Let $u(t,s)$, $a(t,s)$, $b(t,s)$, $c(t,s)$, $d(t,s)$, $f(t,s)$, $g(t,s)$, and $\varphi(u)$ be as in Theorem 2.6, with $L(t,s,v)$ and $M(t,s,v)$ as in Theorem 2.9. If for $(t,s)\in[t_0,\infty)_{\mathbb{T}}\times[t_0,\infty)_{\mathbb{T}}$ we have*

$$\begin{aligned}\varphi(u(t,s)) &\leqslant a(t,s)+g(t,s)\int_{t_0}^{t} c(\tau,s)\varphi(u(\tau,s))\Delta\tau \\ &\quad +d(t,s)\int_{t}^{\infty}\int_{s}^{\infty}\varphi'(u(\tau,\eta))\left[f(\tau,\eta)L(\tau,\eta,u(\tau,\eta))+b(\tau,\eta)\right]\nabla\eta\nabla\tau,\end{aligned}$$

then

$$u(t,s)\leqslant\overline{\mathscr{N}}(t,s)+\overline{p}(t,s)d(t,s)\overline{\mathscr{L}}(t,s)\exp(\overline{\mathscr{M}}(t,s))$$

for

$$\begin{aligned}\overline{\mathscr{N}}(t,s) &:= \varphi^{-1}(\overline{p}(t,s)a(t,s))+\overline{p}(t,s)\overline{B}_1(t,s), \\ \overline{\mathscr{L}}(t,s) &:= \int_{t}^{\infty}\int_{s}^{\infty} f(\tau,\eta)L\left(\tau,\eta,\overline{\mathscr{N}}(\tau,\eta)\right)\nabla\eta\nabla\tau, \\ \overline{\mathscr{M}}(t,s) &:= \int_{t}^{\infty}\int_{s}^{\infty} f(\tau,\eta)\overline{p}(\tau,\eta)d(\tau,\eta)M\left(\tau,\eta,\overline{\mathscr{N}}(\tau,\eta)\right)\nabla\eta\nabla\tau,\end{aligned}$$

where $\overline{B}_1(t,s)$ and $\overline{p}(t,s)$ are given by (2.35) and (2.36), respectively.

Proof. The result follows by an argument similar to that given in the proof of Theorem 2.9, by applying Theorem 2.6 and Lemma 2.2 (II). □

For two examples in the case where $\mathbb{T} = \mathbb{Z}$, please see [12, Section 3]. For an arbitrary time scale $\mathbb{T}$ with $t_0 \in \mathbb{T}$, consider the integral dynamic equation

$$u(t,s) = a(t,s) + \int_{t_0}^{t} g(\tau,s,u(\tau,s))\Delta\tau + \int_{t_0}^{t}\int_{s}^{\infty} h(\tau,\eta,u(\tau,\eta),\log|u(\tau,\eta)|)\nabla\eta\Delta\tau \tag{2.48}$$

where $u : [t_0,\infty)_{\mathbb{T}}^2 \to \mathbb{R}$, $g : [t_0,\infty)_{\mathbb{T}}^2 \times \mathbb{R} \to \mathbb{R}$, and $h : [t_0,\infty)_{\mathbb{T}}^2 \times \mathbb{R}^2 \to \mathbb{R}$, with g and h appropriately integrable. Assume that g and h also satisfy the inequalities

$$|g(t,s,u)| \leqslant c(t,s)|u|, \tag{2.49}$$

$$|h(t,s,u,\log|u|)| \leqslant |u|\big(f(t,s)\log|u| + b(t,s)\big), \tag{2.50}$$

where $b(t,s)$, $c(t,s)$, and $f(t,s)$ are defined as in Corollary 2.8, with $|a(t,s)| \leqslant A$ for some real constant $A \geqslant 0$. Let $u(t,s)$ be a solution of (2.48) defined for $t,s \in [t_0,\infty)_{\mathbb{T}}$. Using (2.48) together with inequalities (2.49) and (2.50), we obtain

$$\begin{aligned} |u(t,s)| + 1 \;&\leqslant\; (A+1) + \int_{t_0}^{t} c(\tau,s)(|u(\tau,s)|+1)\Delta\tau \\ &\quad + \int_{t_0}^{t}\int_{s}^{\infty} [|u(\tau,\eta)|+1]\,[f(\tau,\eta)\log(|u(\tau,\eta)|+1) + b(\tau,\eta)]\,\nabla\eta\Delta\tau \end{aligned}$$

for all $t,s \in [t_0,\infty)_{\mathbb{T}}$. Applying Corollary 2.8 to the previous inequality leads to

$$\begin{aligned} |u(t,s)| \;&\leqslant\; \exp\left\{\exp\left[p(t,s)\int_{t_0}^{t}\int_{s}^{\infty} f(\tau,\eta)\nabla\eta\Delta\tau\right][\log((A+1)p(t,s)) \right. \\ &\qquad \left. + p(t,s)B_1(t,s)]\right\} - 1, \end{aligned}$$

where

$$B_1(t,s) := \int_{t_0}^{t}\int_{s}^{\infty} b(\tau,\eta)\nabla\eta\Delta\tau,$$

$$p(t,s) := 1 + \int_{t_0}^{t} c(\tau,s)e_c(t,\sigma(\tau))\Delta\tau,$$

and $G(x) := \int_1^x dr/r = \log x$; note this requires a final assumption, namely that

$$x = \log((A+1)p(t,s)) + p(t,s)B_1(t,s) \geqslant 1.$$

3. Inequalities of Hilbert-Pachpatte type on time scales

As in the previous section, we will begin with a few foundational results before presenting the main inequalities. Although we are concerned once more with dynamic double integrals, this section is independent of Section 2; the thread of unification and extension, however, continues. Note that if $\mathbb{T} = \mathbb{Z}$, then Lemma 3.1 below is a discrete inequality from [9, 13], and Lemma 3.2 below is a discrete inequality from Németh [13]. All of the theorems and corresponding proofs below are modelled after those given in the special case of $\mathbb{T} = \mathbb{Z}$ presented by Pachpatte [15, 17].

LEMMA 3.1. *Let $z : \mathbb{T} \to \mathbb{R}$ be a left-dense continuous function, with $z \geqslant 0$, and $\alpha \geqslant 1$ a real constant. Then the nabla integral inequality*

$$\left(\int_{t_0}^{t} z(\eta)\nabla\eta\right)^{\alpha} \leqslant \alpha \int_{t_0}^{t} z(\tau)\left(\int_{t_0}^{\tau} z(\eta)\nabla\eta\right)^{\alpha-1}\nabla\tau$$

holds for points $t, t_0 \in \mathbb{T}$ with $t \geqslant t_0$.

Proof. Fix $t \in \mathbb{T}$, $t \geqslant t_0$. If t is a left-dense point, by a modification of the chain rule [7, Theorem 1.87], we have that

$$\left[\left(\int_{t_0}^{t} z(\eta)\nabla\eta\right)^{\alpha}\right]^{\nabla} = \alpha\left(\int_{t_0}^{t} z(\eta)\nabla\eta\right)^{\alpha-1} z(t).$$

If t is a left-scattered point, then $\rho(t) < t$ and $\nu(t) > 0$. Define the nonnegative real numbers $x, y \in \mathbb{R}$ via

$$x := \int_{t_0}^{\rho(t)} z(\eta)\nabla\eta, \qquad y := \int_{\rho(t)}^{t} z(\eta)\nabla\eta = \nu(t)z(t).$$

Then the nabla derivative can be written as

$$\left[\left(\int_{t_0}^{t} z(\eta)\nabla\eta\right)^{\alpha}\right]^{\nabla} = \frac{1}{\nu(t)}\left[(x+y)^{\alpha} - x^{\alpha}\right]. \tag{3.1}$$

Now $f : \mathbb{R} \to \mathbb{R}$ via $f(x) := x^{\alpha}$ is differentiable on $\mathbb{R}$, so by the mean value theorem,

$$(x+y)^{\alpha} - x^{\alpha} = \alpha k^{\alpha-1} y \leqslant \alpha(x+y)^{\alpha-1} y, \quad \text{some real } k \in [x, x+y],$$

since α, x, and y are all nonnegative. Combining this with (3.1), we obtain

$$\left[\left(\int_{t_0}^{t} z(\eta)\nabla\eta\right)^{\alpha}\right]^{\nabla} \leqslant \alpha\left(\int_{t_0}^{t} z(\eta)\nabla\eta\right)^{\alpha-1} z(t). \tag{3.2}$$

In either case, (3.2) holds. If we nabla integrate (3.2) from t_0 to t, the desired inequality follows. □

LEMMA 3.2. *Let $z : \mathbb{T} \to \mathbb{R}$ be a left-dense continuous function. Then the equality that allows interchanging the order of nabla integration given by*

$$\int_{t_0}^{t}\left(\int_{t_0}^{s} z(\eta)\nabla\eta\right)\nabla s = \int_{t_0}^{t}\left(\int_{\rho(\eta)}^{t} z(\eta)\nabla s\right)\nabla\eta = \int_{t_0}^{t}\left[t - \rho(\eta)\right] z(\eta)\nabla\eta \tag{3.3}$$

holds for points $s, t, t_0 \in \mathbb{T}$ with $t, s \geqslant t_0$.

Proof. In a manner similar to [6, Theorem 3.10], we see that all integrals in (3.3) are well-defined. Set

$$Z(t) := \int_{t_0}^{t} [t-\rho(\eta)]\, z(\eta)\nabla\eta - \left(\int_{t_0}^{t}\left(\int_{t_0}^{s} z(\eta)\nabla\eta\right)\nabla s\right).$$

Then $Z(t_0)=0$, and $Z^{\nabla}(t)=\int_{t_0}^{t} z(\eta)\nabla\eta + 0 - \left(\int_{t_0}^{t} z(\eta)\nabla\eta\right) = 0$ for $t \geqslant t_0$, where we have used [7, Theorem 8.50 (iv)]. By the uniqueness of initial value problems, $Z \equiv 0$, and the result follows. □

THEOREM 3.3. *Let $p,q \geqslant 1$ be real numbers, and let $a,b:\mathbb{T}\to\mathbb{R}$ be left-dense continuous nonnegative functions. If for $s,t,t_0 \in \mathbb{T}$ with $s,t \geqslant t_0$ we define*

$$A(s) := \int_{t_0}^{s} a(\tau)\nabla\tau \quad \textit{and} \quad B(t) := \int_{t_0}^{t} b(\tau)\nabla\tau,$$

then for $s_1,t_1 \in \mathbb{T}$ with $s_1,t_1 \geqslant t_0$ we have

$$\int_{t_0}^{s_1}\int_{t_0}^{t_1} \frac{A^p(s)B^q(t)}{s+t-2t_0}\nabla t\nabla s \;\leqslant\; \frac{1}{2}pq\left((s_1-t_0)\int_{t_0}^{s_1}(s_1-\rho(s))\left(A^{p-1}(s)a(s)\right)^2\nabla s\right)^{1/2}$$
$$\times\left((t_1-t_0)\int_{t_0}^{t_1}(t_1-\rho(t))\left(B^{q-1}(t)b(t)\right)^2\nabla t\right)^{1/2}, \quad (3.4)$$

unless $a\equiv 0$ or $b\equiv 0$.

Proof. Using Lemma 3.1 for any $s\in(t_0,s_1]_{\mathbb{T}}$ and $t\in(t_0,t_1]_{\mathbb{T}}$ we obtain

$$A^p(s) \leqslant p\int_{t_0}^{s} a(\tau)A^{p-1}(\tau)\nabla\tau \quad \text{and} \quad B^q(t) \leqslant q\int_{t_0}^{t} b(\tau)B^{q-1}(\tau)\nabla\tau. \qquad (3.5)$$

Using (3.5), the Cauchy-Schwarz inequality [7, Theorem 6.15], and the fact that $(xy)^{1/2} \leqslant \frac{1}{2}(x+y)$ for any real numbers $x,y \geqslant 0$ we observe that

$$\begin{aligned}
A^p(s)B^q(t) &\leqslant pq\left(\int_{t_0}^{s} a(\tau)A^{p-1}(\tau)\nabla\tau\right)\left(\int_{t_0}^{t} b(\tau)B^{q-1}(\tau)\nabla\tau\right)\\
&\leqslant pq(s-t_0)^{1/2}\left(\int_{t_0}^{s}\left(a(\tau)A^{p-1}(\tau)\right)^2\nabla\tau\right)^{1/2}(t-t_0)^{1/2}\\
&\quad\times\left(\int_{t_0}^{t}\left(b(\tau)B^{q-1}(\tau)\right)^2\nabla\tau\right)^{1/2}\\
&\leqslant \frac{1}{2}pq(s+t-2t_0)\left(\int_{t_0}^{s}\left(a(\tau)A^{p-1}(\tau)\right)^2\nabla\tau\right)^{1/2}\\
&\quad\times\left(\int_{t_0}^{t}\left(b(\tau)B^{q-1}(\tau)\right)^2\nabla\tau\right)^{1/2}.
\end{aligned}$$

Dividing both sides by $s+t-2t_0$, nabla integrating over t from t_0 to t_1 and over s from t_0 to s_1, and using the Cauchy-Schwarz inequality again we get

$$\begin{aligned}\int_{t_0}^{s_1}\int_{t_0}^{t_1}\frac{A^p(s)B^q(t)}{s+t-2t_0}\nabla t\nabla s &\leqslant \frac{1}{2}pq\int_{t_0}^{s_1}\left(\int_{t_0}^{s}\left(a(\tau)A^{p-1}(\tau)\right)^2\nabla\tau\right)^{1/2}\nabla s\\ &\quad\times\int_{t_0}^{t_1}\left(\int_{t_0}^{t}\left(b(\tau)B^{q-1}(\tau)\right)^2\nabla\tau\right)^{1/2}\nabla t\\ &\leqslant \frac{1}{2}pq(s_1-t_0)^{1/2}\left[\int_{t_0}^{s_1}\left(\int_{t_0}^{s}\left(a(\tau)A^{p-1}(\tau)\right)^2\nabla\tau\right)\nabla s\right]^{1/2}\\ &\quad\times(t_1-t_0)^{1/2}\left[\int_{t_0}^{t_1}\left(\int_{t_0}^{t}\left(b(\tau)B^{q-1}(\tau)\right)^2\nabla\tau\right)\nabla t\right]^{1/2}.\end{aligned}$$

In the right-hand side of the previous expression we interchange the order of integration as in (3.3) from Lemma 3.2 to get that

$$\begin{aligned}\int_{t_0}^{s_1}\int_{t_0}^{t_1}\frac{A^p(s)B^q(t)}{s+t-2t_0}\nabla t\nabla s &\leqslant \frac{1}{2}pq\left((s_1-t_0)\int_{t_0}^{s_1}(s_1-\rho(s))\left(a(s)A^{p-1}(s)\right)^2\nabla s\right)^{1/2}\\ &\quad\times\left((t_1-t_0)\int_{t_0}^{t_1}(t_1-\rho(t))\left(b(t)B^{q-1}(t)\right)^2\nabla t\right)^{1/2},\end{aligned}$$

that is (3.4). □

REMARK 3. If $\mathbb{T}=\mathbb{Z}$ and $t_0=0$, then $\rho(x)=x-1$ and (3.4) reduces to

$$\begin{aligned}\sum_{s=1}^{s_1}\sum_{t=1}^{t_1}\frac{A_s^pB_t^q}{s+t} &\leqslant \frac{1}{2}pq\left(s_1\sum_{s=1}^{s_1}(s_1-s+1)\left(a_sA_s^{p-1}\right)^2\right)^{1/2}\\ &\quad\times\left(t_1\sum_{t=1}^{t_1}(t_1-t+1)\left(b_tB_t^{q-1}\right)^2\right)^{1/2},\end{aligned}$$

which is [15, Theorem 1]. If $\mathbb{T}=\mathbb{R}$ and $t_0=0$, then $\rho(x)=x$ and (3.4) reduces to

$$\begin{aligned}\int_0^{s_1}\int_0^{t_1}\frac{A^p(s)B^q(t)}{s+t}dtds &\leqslant \frac{1}{2}pq\left(s_1\int_0^{s_1}(s_1-s)\left(a(s)A^{p-1}(s)\right)^2ds\right)^{1/2}\\ &\quad\times\left(t_1\int_0^{t_1}(t_1-t)\left(b(t)B^{q-1}(t)\right)^2dt\right)^{1/2},\end{aligned}$$

which is [15, Theorem 5]. Thus Theorem 3.4 unifies and extends these two theorems to arbitrary time scales.

REMARK 4. If in (3.4) we take $p=q=1$, then the inequality is

$$\begin{aligned}\int_{t_0}^{s_1}\int_{t_0}^{t_1}\frac{A(s)B(t)}{s+t-2t_0}\nabla t\nabla s &\leqslant \frac{1}{2}\left((s_1-t_0)\int_{t_0}^{s_1}(s_1-\rho(s))a^2(s)\nabla s\right)^{1/2}\\ &\quad\times\left((t_1-t_0)\int_{t_0}^{t_1}(t_1-\rho(t))b^2(t)\nabla t\right)^{1/2}. \quad (3.6)\end{aligned}$$

Before continuing, we will need the following key theorem.

THEOREM 3.4. (Jensen's Inequality) *Let* $s,t\in\mathbb{T}$ *and* $x,y\in\mathbb{R}$. *If* $a:\mathbb{T}\to\mathbb{R}$ *and* $b:\mathbb{T}\to(x,y)$ *are nonnegative, left-dense continuous functions with* $\int_s^t a(\tau)\nabla\tau>0$, *and* $\phi:(x,y)\to\mathbb{R}$ *is continuous and convex, then*

$$\phi\left(\frac{\int_s^t a(\tau)b(\tau)\nabla\tau}{\int_s^t a(\tau)\nabla\tau}\right)\leqslant\frac{\int_s^t a(\tau)\phi(b(\tau))\nabla\tau}{\int_s^t a(\tau)\nabla\tau}.$$

Proof. This proof is modelled after those found in Bohner and Peterson [7, Theorem 6.17] and Rudin [18, Theorem 3.3]. Let $z_0\in(x,y)$. By the convexity of ϕ, there exists a $\beta\in\mathbb{R}$ such that $\phi(z)\geqslant\phi(z_0)+\beta(z-z_0)$ for all $z\in(x,y)$. In particular, for $\tau\in[s,t]_{\mathbb{T}}$,

$$\phi(b(\tau))\geqslant\phi(z_0)+\beta(b(\tau)-z_0),\qquad z_0:=\frac{\int_s^t a(\tau)b(\tau)\nabla\tau}{\int_s^t a(\tau)\nabla\tau}.\tag{3.7}$$

Multiplying (3.7) by $a(\tau)\geqslant0$ and nabla integrating from s to t yields

$$\int_s^t a(\tau)\phi(b(\tau))\nabla\tau\geqslant\phi(z_0)\int_s^t a(\tau)\nabla\tau+\beta\int_s^t a(\tau)(b(\tau)-z_0)\nabla\tau;$$

dividing this last expression by $\int_s^t a(\tau)\nabla\tau>0$ leads to

$$\frac{\int_s^t a(\tau)\phi(b(\tau))\nabla\tau}{\int_s^t a(\tau)\nabla\tau}\geqslant\phi(z_0)+\frac{\beta\int_s^t a(\tau)(b(\tau)-z_0)\nabla\tau}{\int_s^t a(\tau)\nabla\tau}=\phi(z_0)+0=\phi(z_0),$$

where z_0 is given in (3.7). □

Our next result deals with a further generalization of the inequality given in (3.6).

THEOREM 3.5. *Let* $a,b:\mathbb{T}\to\mathbb{R}$ *and* $A(s),B(t)$ *be as in Theorem* 3.3. *Let* $p,q:\mathbb{T}\to\mathbb{R}$ *be left-dense continuous positive functions, and for* $s,t,t_0\in\mathbb{T}$ *with* $s,t\geqslant t_0$ *define*

$$P(s):=\int_{t_0}^s p(\tau)\nabla\tau\quad\textit{and}\quad Q(t):=\int_{t_0}^t q(\tau)\nabla\tau.$$

Furthermore, let ϕ *and* ψ *be two real-valued, nonnegative, convex, and submultiplicative functions defined on* $\mathbb{R}_+=[0,\infty)_{\mathbb{R}}$. *Then for* $s_1,t_1\in\mathbb{T}$ *with* $s_1,t_1\geqslant t_0$ *we have*

$$\int_{t_0}^{s_1}\int_{t_0}^{t_1}\frac{\phi(A(s))\psi(B(t))}{s+t-2t_0}\nabla t\nabla s\leqslant M(s_1,t_1)\left(\int_{t_0}^{s_1}(s_1-\rho(s))\left(p(s)\phi\left[\frac{a(s)}{p(s)}\right]\right)^2\nabla s\right)^{1/2}$$
$$\times\left(\int_{t_0}^{t_1}(t_1-\rho(t))\left(q(t)\psi\left[\frac{b(t)}{q(t)}\right]\right)^2\nabla t\right)^{1/2},\tag{3.8}$$

where

$$M(s_1,t_1)=\frac{1}{2}\left(\int_{t_0}^{s_1}\left[\frac{\phi(P(s))}{P(s)}\right]^2\nabla s\right)^{1/2}\left(\int_{t_0}^{t_1}\left[\frac{\psi(Q(t))}{Q(t)}\right]^2\nabla t\right)^{1/2}. \tag{3.9}$$

Proof. From the hypotheses, Jensen's inequality [7, Theorem 6.17], and the Cauchy-Schwarz inequality [7, Theorem 6.15]

$$\begin{aligned}\phi(A(s)) &\overset{\text{hypoth}}{=} \phi\left(\frac{P(s)\int_{t_0}^{s}p(\tau)a(\tau)/p(\tau)\nabla\tau}{\int_{t_0}^{s}p(\tau)\nabla\tau}\right)\\ &\overset{\text{submult}}{\leqslant} \phi(P(s))\phi\left(\frac{\int_{t_0}^{s}p(\tau)a(\tau)/p(\tau)\nabla\tau}{\int_{t_0}^{s}p(\tau)\nabla\tau}\right)\\ &\overset{\text{Jensen's}}{\leqslant} \frac{\phi(P(s))}{P(s)}\int_{t_0}^{s}p(\tau)\phi\left[\frac{a(\tau)}{p(\tau)}\right]\nabla\tau\\ &\overset{\text{C-S ineq}}{\leqslant} \frac{\phi(P(s))}{P(s)}(s-t_0)^{1/2}\left\{\int_{t_0}^{s}\left(p(\tau)\phi\left[\frac{a(\tau)}{p(\tau)}\right]\right)^2\nabla\tau\right\}^{1/2}. \end{aligned}\tag{3.10}$$

In a similar way, we likewise obtain

$$\psi(B(t))\leqslant\frac{\psi(Q(t))}{Q(t)}(t-t_0)^{1/2}\left\{\int_{t_0}^{t}\left(q(\eta)\psi\left[\frac{b(\eta)}{q(\eta)}\right]\right)^2\nabla\eta\right\}^{1/2}. \tag{3.11}$$

From (3.10) and (3.11) and the fact that $(xy)^{1/2}\leqslant\frac{1}{2}(x+y)$ for any real numbers $x,y\geqslant 0$ we observe that

$$\begin{aligned}\phi(A(s))\psi(B(t)) &\leqslant \frac{1}{2}(s+t-2t_0)\left(\frac{\phi(P(s))}{P(s)}\left\{\int_{t_0}^{s}\left(p(\tau)\phi\left[\frac{a(\tau)}{p(\tau)}\right]\right)^2\nabla\tau\right\}^{1/2}\right)\\ &\quad\times\left(\frac{\psi(Q(t))}{Q(t)}\left\{\int_{t_0}^{t}\left(q(\eta)\psi\left[\frac{b(\eta)}{q(\eta)}\right]\right)^2\nabla\eta\right\}^{1/2}\right). \end{aligned}\tag{3.12}$$

Dividing both sides by $s+t-2t_0$, nabla integrating over t from t_0 to t_1 and over s from t_0 to s_1, and using the Cauchy-Schwarz inequality again we get

$$\begin{aligned}
&\int_{t_0}^{s_1}\int_{t_0}^{t_1}\frac{\phi(A(s))\psi(B(t))}{s+t-2t_0}\nabla t\nabla s\\
&\quad\leqslant\quad \frac{1}{2}\int_{t_0}^{s_1}\left(\frac{\phi(P(s))}{P(s)}\left\{\int_{t_0}^{s}\left(p(\tau)\phi\left[\frac{a(\tau)}{p(\tau)}\right]\right)^2\nabla\tau\right\}^{1/2}\right)\nabla s\\
&\qquad\quad \times\int_{t_0}^{t_1}\left(\frac{\psi(Q(t))}{Q(t)}\left\{\int_{t_0}^{t}\left(q(\eta)\psi\left[\frac{b(\eta)}{q(\eta)}\right]\right)^2\nabla\eta\right\}^{1/2}\right)\nabla t\\
&\quad\overset{\text{C-S ineq}}{\leqslant}\quad \frac{1}{2}\left\{\int_{t_0}^{s_1}\left(\frac{\phi(P(s))}{P(s)}\right)^2\nabla s\right\}^{1/2}\left\{\int_{t_0}^{s_1}\int_{t_0}^{s}\left(p(\tau)\phi\left[\frac{a(\tau)}{p(\tau)}\right]\right)^2\nabla\tau\nabla s\right\}^{1/2}\\
&\qquad\quad \times\left\{\int_{t_0}^{t_1}\left(\frac{\psi(Q(t))}{Q(t)}\right)^2\nabla t\right\}^{1/2}\left\{\int_{t_0}^{t_1}\int_{t_0}^{t}\left(q(\eta)\psi\left[\frac{b(\eta)}{q(\eta)}\right]\right)^2\nabla\eta\nabla t\right\}^{1/2}.
\end{aligned}$$

Recall from the notation from (3.9); in the right-hand side of the previous expression we interchange the order of integration as in (3.3) from Lemma 3.2 to get that

$$\begin{aligned}
\int_{t_0}^{s_1}\int_{t_0}^{t_1}\frac{\phi(A(s))\psi(B(t))}{s+t-2t_0}\nabla t\nabla s &\leqslant M(s_1,t_1)\left\{\int_{t_0}^{s_1}(s_1-\rho(s))\left(p(s)\phi\left[\frac{a(s)}{p(s)}\right]\right)^2\nabla s\right\}^{1/2}\\
&\quad\times\left\{\int_{t_0}^{t_1}(t_1-\rho(t))\left(q(t)\psi\left[\frac{b(t)}{q(t)}\right]\right)^2\nabla t\right\}^{1/2}.
\end{aligned}$$

This is (3.8), so this completes the proof. □

The next two theorems deal with minor variants of the inequality given in Theorem 3.5.

THEOREM 3.6. *Let $a,b:\mathbb{T}\to\mathbb{R}$ be as in Theorem 3.3, and define for $s,t,t_0\in\mathbb{T}$ with $s,t\geqslant t_0$*

$$A(s)=\frac{1}{s-t_0}\int_{t_0}^{s}a(\tau)\nabla\tau,\qquad B(t)=\frac{1}{t-t_0}\int_{t_0}^{t}b(\eta)\nabla\eta.$$

Furthermore, let ϕ and ψ be two real-valued, nonnegative, and convex functions defined on $\mathbb{R}_+=[0,\infty)_{\mathbb{R}}$. Then for $s_1,t_1\in\mathbb{T}$ with $s_1,t_1\geqslant t_0$ we have

$$\begin{aligned}
&\int_{t_0}^{s_1}\int_{t_0}^{t_1}\frac{(s-t_0)(t-t_0)}{s+t-2t_0}\phi(A(s))\psi(B(t))\nabla t\nabla s\\
&\qquad\leqslant\frac{1}{2}\left((s_1-t_0)\int_{t_0}^{s_1}(s_1-\rho(s))\left(\phi\left[a(s)\right]\right)^2\nabla s\right)^{1/2}\\
&\qquad\quad\times\left((t_1-t_0)\int_{t_0}^{t_1}(t_1-\rho(t))\left(\psi\left[b(t)\right]\right)^2\nabla t\right)^{1/2}.
\end{aligned}$$

Proof. Using the hypotheses and Jensen's inequality, Theorem 3.4, we see that

$$\phi(A(s)) \overset{\text{hypoth}}{=} \phi\left(\frac{1}{s-t_0}\int_{t_0}^{s} a(\tau)\nabla\tau\right) \overset{\text{Jensen's}}{\leqslant} \frac{1}{s-t_0}\int_{t_0}^{s} \phi\left[a(\tau)\right]\nabla\tau$$
$$\overset{\text{C-S ineq}}{\leqslant} \frac{(s-t_0)^{1/2}}{s-t_0}\left\{\int_{t_0}^{s} \left(\phi\left[a(\tau)\right]\right)^2\nabla\tau\right\}^{1/2}.$$

In a similar way, we likewise obtain

$$\psi(B(t)) \leqslant \frac{(t-t_0)^{1/2}}{t-t_0}\left\{\int_{t_0}^{t} \left(\psi\left[b(\eta)\right]\right)^2\nabla\eta\right\}^{1/2}.$$

The rest of the proof can be completed by following the same procedure employed in the proofs of Theorems 3.4 and 3.6, respectively, making suitable adjustments along the way; we omit the details. □

THEOREM 3.7. *Let a,b,p,q,P,Q be as in Theorem 3.5, and define for $s,t,t_0\in\mathbb{T}$ with $s,t\geqslant t_0$*

$$A(s)=\frac{1}{P(s)}\int_{t_0}^{s} p(\tau)a(\tau)\nabla\tau, \qquad B(t)=\frac{1}{Q(t)}\int_{t_0}^{t} q(\eta)b(\eta)\nabla\eta.$$

Furthermore, let ϕ and ψ be defined as in Theorem 3.6. Then for $s_1,t_1\in\mathbb{T}$ with $s_1,t_1\geqslant t_0$ we have

$$\int_{t_0}^{s_1}\int_{t_0}^{t_1} \frac{P(s)Q(t)\phi(A(s))\psi(B(t))}{s+t-2t_0}\nabla t\nabla s$$
$$\leqslant \frac{1}{2}\left((s_1-t_0)\int_{t_0}^{s_1}(s_1-\rho(s))\left(p(s)\phi\left[a(s)\right]\right)^2\nabla s\right)^{1/2}$$
$$\times\left((t_1-t_0)\int_{t_0}^{t_1}(t_1-\rho(t))\left(q(t)\psi\left[b(t)\right]\right)^2\nabla t\right)^{1/2}.$$

Proof. Using the hypotheses and Jensen's inequality, Theorem 3.4, we see that

$$\phi(A(s)) \overset{\text{hypoth}}{=} \phi\left(\frac{1}{P(s)}\int_{t_0}^{s} p(\tau)a(\tau)\nabla\tau\right) \overset{\text{Jensen's}}{\leqslant} \frac{1}{P(s)}\int_{t_0}^{s} p(\tau)\phi\left[a(\tau)\right]\nabla\tau$$
$$\overset{\text{C-S ineq}}{\leqslant} \frac{(s-t_0)^{1/2}}{P(s)}\left\{\int_{t_0}^{s} \left(p(\tau)\phi\left[a(\tau)\right]\right)^2\nabla\tau\right\}^{1/2}.$$

In a similar way, we likewise obtain

$$\psi(B(t)) \leqslant \frac{(t-t_0)^{1/2}}{Q(t)}\left\{\int_{t_0}^{t} \left(q(\eta)\psi\left[b(\eta)\right]\right)^2\nabla\eta\right\}^{1/2}.$$

As before, the rest of the proof can be completed by following the same procedure employed in the proofs of Theorems 3.4 and 3.6, respectively, making suitable adjustments along the way; we omit the details. □

REMARK 5. Established over arbitrary time scales, including the special cases of $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{R}$, Theorem 3.5 unifies and extends [15, Theorem 2] and [15, Theorem 6]. Likewise Theorems 3.6 and 3.7 combine and generalize under one theory the previously distinct results [15, Theorem 3] and [15, Theorem 7], and [15, Theorem 4] and [15, Theorem 8], respectively.

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