Journal of
Mathematical
Inequalities
Volume 2, Number 2 (2008), 229–238

SUBORDINATION AND SUPERORDINATION–PRESERVING PROPERTIES OF CERTAIN INTEGRAL OPERATOR

T. N. SHANMUGAM AND M. P. JEYARAMAN

(*communicated by S. Owa*)

Abstract. In the present investigation, we obtain subordination and superordination – preserving properties of certain integral operator with the sandwich type theorem.

1. Introduction

Let $\mathscr{H} = \mathscr{H}(\Delta)$ denote the class of analytic functions in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$, let

$$\mathscr{H}[a,n] = \{f \in \mathscr{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}. \tag{1.1}$$

Let $\mathscr{A}$ denote the subclass of $\mathscr{H}[a,1]$ with the normalization

$$f(0) = f'(0) - 1 = 0.$$

Let f and F be members of $\mathscr{H}$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in Δ, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case, we write $f \prec F$ or $f(z) \prec F(z)$. If the function F is univalent in Δ, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\Delta) \subset F(\Delta)$.

Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in Δ. If p is analytic in Δ and satisfies the differential subordination

$$\phi(p(z), zp'(z)) \prec h(z) \quad (z \in \Delta), \tag{1.2}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or simply a dominant if $p \prec q$ for all p satisfying (1.2). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.2) is said to be the best dominant.

Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be analytic in Δ. If p and $\phi(p(z), zp'(z))$ are univalent in Δ and satisfy the differential superordination

$$h(z) \prec \phi(p(z), zp'(z)) \quad (z \in \Delta), \tag{1.3}$$

Mathematics subject classification (2000): 30C45.
Key words and phrases: Subordination, superordination, best dominant and best subordinant.

© ELEMENT, Zagreb
Paper JMI-02-20

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or simply a subordinant if $q \prec p$ for all p satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.3) is said to be the best subordinant.

For $f \in \mathscr{A}$, Ruscheweyh [9], considered the following generalized integral operator

$$I_{\beta,\gamma}(f)(z) = \left[\frac{\beta+\gamma}{z^{\gamma}}\int_0^z t^{\gamma-1}f^{\beta}(t)dt\right]^{\frac{1}{\beta}} \quad (\beta > 0, \operatorname{Re}\gamma > 0). \tag{1.4}$$

For a function $f \in \mathscr{A}$, Shanmugam [10] introduced and studied the following integral operator $I_{\beta,\gamma}$ defined by

$$I_{\beta,\gamma}(f)(z) := \left[\frac{\gamma+\frac{1}{\beta}}{g^{\gamma}(z)}\int_0^z g^{\gamma-1}(t)g'(t)f^{\frac{1}{\beta}}(t)dt\right]^{\beta} \tag{1.5}$$

$$(f \in \mathscr{A}, \beta > 0).$$

which is a generalization of Mocanu's [7] integral operator.

For $g(z) = z$ in (1.5), we have

$$I_{\beta,\gamma}(f)(z) := \left[\frac{\gamma+\frac{1}{\beta}}{z^{\gamma}}\int_0^z t^{\gamma-1}f^{\frac{1}{\beta}}(t)dt\right]^{\beta}. \tag{1.6}$$

Miller et al. [6] obtained some subordination theorems involving certain integral operators for analytic function in Δ. Also, Bulboaca [1] considered superordination – preserving properties of certain integral operator as the dual problem of subordination. Recently Cho and Owa [2] obtained double subordination – preserving properties for certian integral operator.

Motivated by the aforementioned works, in the present investigation, using the technique in [1, 2], we obtain sandwich type theorems which contains the subordination and superordination – preserving properties for certain integral operator defined in the open unit disk.

2. Definitions and preliminaries

To prove our main results, we shall need the following definitions and Lemmas.

DEFINITION 2.1. [[8] p. 157] A function $L : \Delta \times [0, +\infty) \to \mathbb{C}$ is a subordination (or Loewner) chain if $L(\cdot, t)$ is analytic and univalent in Δ for all $t \geqslant 0$, $L(z, \cdot)$ is continuously differentiable on $[0, +\infty)$ for all $z \in \Delta$, and $L(z; s) \prec L(z, t)$ when $0 \leqslant s \leqslant t$.

The next well-known lemma gives us a necessary and sufficient condition for $L(z; t)$ to be a subordination chain.

LEMMA 2.1. [[8], p. 159] *The function* $L(z;t) = a_1(t)z + a_2(t)z^2 + \cdots$, *with* $a_1(t) \neq 0$ *for* $t \geqslant 0$ *and* $\lim_{t\to+\infty} |a_1(t)| = +\infty$, *is a subordination chain if and only if*

$$\mathrm{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] > 0, \quad z \in \Delta,\ t \geqslant 0.$$

DEFINITION 2.2. [5] We denote by $\mathscr{Q}$ the set of functions h that are analytic and injective on $\Delta \backslash E(h)$ where

$$E(h) = \{\zeta \in \partial\Delta : \lim_{z\to\zeta} h(z) = \infty\},$$

and such that $h'(\zeta) \neq 0$ for $\zeta \in \partial\Delta \backslash E(h)$.

LEMMA 2.2. [3] *Let* p *be analytic in* Δ *and* q *analytic and univalent in* $\bar{\Delta}$ *except for points where* $\lim_{z\to\zeta} p(z) = \infty$ *with* $p(0) = q(0)$. *If* p *is not subordinate to* q, *then there is a point* $z_0 \in \Delta$ *and* $\zeta_0 \in \partial\Delta$ *such that* $p(|z| < |z_0|) \subset q(\Delta)$, $p(z_0) = q(\zeta_0)$, *and* $z_0 p(z_0) = m\zeta_0 q'(\zeta_0)$ *for* $m \geqslant 1$.

LEMMA 2.3. [[5], Thm. 7] *Let* $q \in \mathscr{H}[a,1]$, *let* $\phi : \mathbb{C}^2 \to \mathbb{C}$ *and set* $\phi(q(z), zq'(z)) \equiv h(z)$. *If* $L(z;t) = \phi(q(z), tzq'(z))$ *is a subordination chain and* $p \in \mathscr{H}[a,1] \cap \mathscr{Q}$, *then*

$$h(z) \prec \phi(p(z), zp'(z)) \Rightarrow q(z) \prec p(z).$$

Furthermore, if $\phi(q(z), zq'(z)) = h(z)$ *has a univalent solution* $q \in \mathscr{Q}$, *then* q *is the best subordinant.*

LEMMA 2.4. [4] *Let* $\beta, \gamma \in \mathbb{C}$ *with* $\beta \neq 0$ *and let* $h \in \mathscr{H}(\Delta)$ *with* $h(0) = c$. *If* $\mathrm{Re}\{\beta h(z) + \gamma\} > 0$ $(z \in \Delta)$, *then the solution of the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \quad (z \in \Delta) \tag{2.1}$$

with $q(0) = c$ *is analytic in* Δ *and satisfies* $\mathrm{Re}\{\beta q(z) + \gamma\} > 0$.

LEMMA 2.5. [4] *Let* $\beta > 0$, $\beta + \gamma > 0$ *and let* $I_{\beta,\gamma}$ *be the integral operator defined by (1.4). If* $\alpha \in [-\gamma/\beta, 1)$, *then the order of starlikeness of the class* $I_{\beta,\gamma}(\mathscr{S}^*(\alpha))$, *that is, the largest number* $\delta = \delta(\alpha;\beta,\gamma)$ *such that*

$$I_{\beta,\gamma}(\mathscr{S}^*(\alpha)) \subset \mathscr{S}^*(\delta),$$

is given by the number $\delta(\alpha;\beta,\gamma) = \inf\{\mathrm{Re}\, q(z) : z \in \Delta\}$, *where*

$$q(z) = \frac{1}{\beta Q(z)} - \frac{\gamma}{\beta}, \qquad Q(z) = \int_0^1 \left(\frac{1-z}{1-tz}\right)^{2\beta(1-\alpha)} t^{\beta+\alpha-1}\,dt.$$

Moreover, if $\alpha \in [\alpha_0, 1)$, *where*

$$\alpha_0 := \max\left\{\frac{\beta-\gamma-1}{2\beta}, -\frac{\gamma}{\beta}\right\}$$

and $f \in \mathscr{S}^*(\alpha)$, *then*

$$\operatorname{Re}\left\{\frac{z(I_{\beta,\gamma}(f)(z))'}{I_{\beta,\gamma}(f)(z)}\right\} > \delta(\alpha;\beta,\gamma)$$
$$= \frac{1}{\beta}\left[\frac{\beta+\gamma}{{}_2F_1(1,2\beta(1-\alpha),\beta+\gamma+1;1/2)} - \gamma\right],$$

where ${}_2F_1$ *represents the Gauss hypergeometric function.*

Throughout this paper, we will denote $\mathscr{A}_{\beta,\gamma}$ by

$$\mathscr{A}_{\beta,\gamma} := \left\{f \in \mathscr{A} : \frac{f(z)}{z} \neq 0, \frac{I_{\beta,\gamma}(f)(z)}{z} \neq 0 \quad (z \in \Delta; \beta \neq 1)\right\}.$$

3. Main results

THEOREM 3.1. *Let* $f, g \in \mathscr{A}_{\beta,\gamma}$ *with* $\beta > 0$ *and* $0 < \gamma \leqslant 1$. *Suppose that*

$$\operatorname{Re}\left\{1 + \frac{zg''(z)}{g'(z)} + \left(\frac{1}{\beta} - 1\right)\frac{zg'(z)}{g(z)}\right\} > -\frac{\gamma}{2}. \tag{3.1}$$

Then

$$(f(z))^{1/\beta} \prec (g(z))^{1/\beta} \tag{3.2}$$

implies

$$(I_{\beta,\gamma}(f)(z))^{1/\beta} \prec (I_{\beta,\gamma}(g)(z))^{1/\beta} \tag{3.3}$$

where the Integral operator $I_{\beta,\gamma}$ *is defined by (1.6). Moreover, the function* $(I_{\beta,\gamma}(g)(z))^{1/\beta}$ *is the best dominant.*

Proof. Let

$$F(z) := (I_{\beta,\gamma}(f)(z))^{1/\beta}, \qquad G(z) := (I_{\beta,\gamma}(g)(z))^{1/\beta}. \tag{3.4}$$

Without loss of generality, we can assume that G is analytic and univalent on $\bar{\Delta}$, and $G'(\xi) \neq 0$ for $|\zeta| = 1$.

We first prove that if the function q is defined by

$$q(z) := 1 + \frac{zG''(z)}{G'(z)}, \tag{3.5}$$

then

$$\operatorname{Re}\{q(z)\} > 0 \quad (z \in \Delta). \tag{3.6}$$

From the definition of (1.6), we obtain

$$g^{1/\beta}(z) = (I_{\beta,\gamma}g(z))^{1/\beta}\left(\frac{1}{\beta}\,\frac{z(I_{\beta,\gamma}(g)(z))'}{I_{\beta,\gamma}(g)(z)} + \gamma\right)\frac{1}{\gamma + \frac{1}{\beta}}. \tag{3.7}$$

From (3.4), we have

$$\frac{1}{\beta}\left(\frac{z(I_{\beta,\gamma}(g)(z))'}{I_{\beta,\gamma}(g)(z)}\right) = \frac{zG'(z)}{G(z)} \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\left(\gamma+\frac{1}{\beta}\right)\phi(z)=\gamma G(z)+zG'(z), \tag{3.9}$$

where $\phi(z)=g^{1/\beta}(z)$.

By differentiating (3.9) and through a little simplification, we obtain

$$\phi'(z)=\frac{\gamma}{\gamma+1/\beta}G'(z)+\frac{1}{(\gamma+1/\beta)}q(z)G'(z) \tag{3.10}$$

where the function $q(z)$ is defined by (3.5).

Logarithmic differentiation of (3.10) and a simplification yields

$$q(z)+\frac{zq'(z)}{q(z)+\gamma}=1+\frac{z\phi''(z)}{\phi'(z)}\equiv h(z). \tag{3.11}$$

From (3.1), we have $\operatorname{Re}\{h(z)+\gamma\}>\frac{\gamma}{2}>0$, and by using Lemma 2.4, we conclude that the differential equation (3.11) has a solution $q\in\mathscr{H}(\Delta)$ with $q(0)=h(0)=1$.

Now, we will use Lemma 2.5 to prove that, under the assumption, the inequality (3.6) holds. Replacing β by 1 in Lemma 2.5, we have

$$\alpha_0=\max\left\{\frac{\beta-\gamma-1}{2\beta},-\frac{\gamma}{\beta}\right\}=-\frac{\gamma}{2}.$$

For the differential equation (3.11), by using Lemma 2.5 in the case $\alpha=\alpha_0=-\frac{\gamma}{2}$, we obtain that

$$\operatorname{Re}\{q(z)\}>\frac{\gamma+1}{{}_2F_1(1,\gamma+2,\gamma+2;1/2)}-\gamma=\frac{1-\gamma}{2}\geqslant 0.$$

Thus, G defined by (3.4) is convex (univalent) in Δ.

Next, we prove that (3.2) implies (3.3). For, consider the function $L(z,t)$ given by

$$L(z,t):=\frac{\gamma}{\gamma+\frac{1}{\beta}}G(z)+\frac{(1+t)}{\gamma+\frac{1}{\beta}}zG'(z)\qquad(0\leqslant t<\infty). \tag{3.12}$$

Since $G(z)$ and $zG'(z)$ are analytic in Δ, the function $L(z,t)$ defined by (3.12) is analytic in Δ for all $t\geqslant 0$, and

$$a_1(t)=\left(\frac{\partial L(z,t)}{\partial z}\right)_{z=0}=G'(0)\left(\frac{\beta(\gamma+1+t)}{\beta\gamma+1}\right)\neq 0.$$

Also

$$\lim_{t\to+\infty}|a_1(t)|=+\infty.$$

A simple computation shows that

$$\operatorname{Re}\left(z\frac{\partial L/\partial z}{\partial L/\partial t}\right)=\left\{\gamma+(1+t)\left(1+z\frac{G''(z)}{G'(z)}\right)\right\},$$

and according to G is convex and $\gamma > 0$, we get

$$\operatorname{Re}\left(z\frac{\partial L/\partial z}{\partial L/\partial t}\right) > 0, \quad z \in \Delta, t \geqslant 0.$$

By Lemma 2.1, we conclude that $L(z,t)$ is a subordination chain. From the definition of a subordination chain, we have

$$\phi(z) = \frac{\gamma}{\gamma+1/\beta}G(z) + \frac{1}{\gamma+1/\beta}zG'(z) = L(z,0), L(z,0) \prec L(z,t)$$

$$(z \in \Delta, 0 \leqslant t < \infty).$$

This implies that

$$L(\xi,t) \notin L(\Delta,0) = \phi(\Delta) \tag{3.13}$$

for $\xi \in \partial\Delta$ and $t \in [0,\infty)$.

Now, suppose that F is not subordinate to G. Then, by Lemma 2.2, there exist points $z_0 \in \Delta$ and $\xi_0 \in \partial\Delta$ such that

$$F(z_0) = G(\xi_0), z_0F'(z_0) = (1+t)\xi_0G'(\xi_0) \quad (0 \leqslant t < \infty)$$

Hence, we have

$$\begin{aligned} L(\xi_0,t) &= \frac{\gamma}{\gamma+1/\beta}G(\xi_0) + \frac{(1+t)}{\gamma+1/\beta}\xi_0G'(\xi_0) \\ &= \frac{\gamma}{\gamma+1/\beta}F(z_0) + \frac{1}{\gamma+1/\beta}z_0F'(z_0) = (f(z_0))^{1/\beta} \in \phi(\Delta) \end{aligned}$$

by virtue of the subordination condition (3.2). This contradicts the above observation that $L(\xi,t) \notin \phi(\Delta)$. Therefore we have $F(z) \prec G(z)$. Considering $F(z) = G(z)$, we see that the function G is the best dominant. Therefore, we complete the proof of Theorem 3.1. □

THEOREM 3.2. *Let $f,g \in \mathscr{A}_{\beta,\gamma}$ with $\beta > 0$ and $0 < \gamma \leqslant 1$. Suppose that*

$$\operatorname{Re}\left\{1 + \frac{zg''(z)}{g'(z)} + (\frac{1}{\beta} - 1)\frac{zg'(z)}{g(z)}\right\} > -\frac{\gamma}{2}.$$

If $(f(z))^{1/\beta}$ is univalent in Δ and $(I_{\beta,\gamma}f(z))^{1/\beta} \in \mathscr{Q}$, then

$$(g(z))^{1/\beta} \prec (f(z))^{1/\beta} \tag{3.14}$$

implies

$$(I_{\beta,\gamma}(g)(z))^{1/\beta} \prec (I_{\beta,\gamma}(f)(z))^{1/\beta} \quad (z \in \Delta), \tag{3.15}$$

where the integral operator defined by (1.6). Moreover the function $(I_{\beta,\gamma}(g)(z))^{1/\beta}$ is the best subordinant.

Proof. Let

$$F(z) := (I_{\beta,\gamma}(f)(z))^{1/\beta}, \qquad G(z) := (I_{\beta,\gamma}(g)(z))^{1/\beta} \tag{3.16}$$

From (3.7) and (3.8), we obtain

$$\phi(z) = \frac{\gamma}{\gamma + 1/\beta}G(z) + \frac{1}{\gamma + 1/\beta}zG'(z) = \phi(G(z), zG'(z)). \tag{3.17}$$

By differentiating (3.17) and through a little simplification, we obtain

$$\phi'(z) = \frac{\gamma}{\gamma + 1/\beta}G'(z) + \frac{1}{(\gamma + 1/\beta)}q(z)G'(z). \tag{3.18}$$

Logarithmic differentiation of (3.18) and a simplification yields

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + \gamma},$$

where the function q is defined by (3.5). Then, by using the same technique as in the proof of Theorem 3.1, we observe that G defined by (3.16) is convex (univalent) in Δ.

Consider

$$L(z,t) := \frac{\gamma}{\gamma + 1/\beta}G(z) + \frac{t}{\gamma + 1/\beta}zG'(z) \quad (z \in \Delta, 0 \leqslant t < \infty).$$

Clearly $L(z,t)$ is analytic in Δ and

$$a_1(t) = \frac{\partial L(0,t)}{\partial z} = G'(0)\left(\frac{\beta(\gamma + t)}{\beta\gamma + 1}\right) \neq 0.$$

Also $\lim_{t \to +\infty} |a_1(t)| = +\infty$.

A simple computation shows that

$$\text{Re}\left(z\frac{\partial L/\partial z}{\partial L/\partial t}\right) = \text{Re}\left\{\gamma + t\left(1 + \frac{zG''(z)}{G'(z)}\right)\right\}.$$

Since $\gamma > 0$ and G is convex, we have

$$\text{Re}\left(z\frac{\partial L/\partial z}{\partial L/\partial t}\right) > 0, \quad z \in \Delta, t \geqslant 0,$$

and by Lemma 2.1, we conclude that $L(z,t)$ is a subordination chain.

Therefore, according to Lemma 2.3, we conclude that the superordination condition (3.14) must imply the superordination given by (3.15). Furthermore, since the differential equation has the univalent solution G, it is the best subordinant of the given differential superordination which completes the proof of Theorem 3.2. □

By combining Theorem 3.1 and Theorem 3.2, we get the following sandwich theorem.

THEOREM 3.3. *Let* $f, g_k \in \mathscr{A}_{\beta,\gamma}$, $(k=1,2)$ *with* $\beta > 0$ *and* $0 < \gamma \leqslant 1$. *Suppose that*

$$\operatorname{Re}\left\{1+\frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\frac{\gamma}{2} \qquad (\phi_k(z) := (g_k(z))^{1/\beta}, k=1,2).$$

If $(f(z))^{1/\beta}$ *is univalent in* Δ *and* $(I_{\beta,\gamma}f(z))^{1/\beta} \in \mathscr{Q}$, *then*

$$(g_1(z))^{1/\beta} \prec (f(z))^{1/\beta} \prec (g_2(z))^{1/\beta}$$

implies that

$$(I_{\beta,\gamma}(g_1)(z))^{1/\beta} \prec (I_{\beta,\gamma}(f)(z))^{1/\beta} \prec (I_{\beta,\gamma}(g_2)(z))^{1/\beta},$$

where $I_{\beta,\gamma}$ *is the integral operator defined by (1.6). Moreover, the functions* $(I_{\beta,\gamma}(g_1)(z))^{1/\beta}$ *and* $(I_{\beta,\gamma}(g_2)(z))^{1/\beta}$ *are the best subordinant and the best dominant, respectively.*

THEOREM 3.4. *Let* $f, g \in \mathscr{A}_{\beta,\gamma}$ *with* $\beta > 0$ *and* $0 < \gamma + 1/\beta \leqslant 1$. *Suppose that*

$$\operatorname{Re}\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\frac{1+\beta\gamma}{2\beta} \qquad \left(z \in \Delta; \phi(z) := \left(\frac{g(z)}{z}\right)^{1/\beta}\right).$$

Then

$$\left(\frac{f(z)}{z}\right)^{1/\beta} \prec \left(\frac{g(z)}{z}\right)^{1/\beta}$$

implies that

$$\left(\frac{I_{\beta,\gamma}(f)(z)}{z}\right)^{1/\beta} \prec \left(\frac{I_{\beta,\gamma}(g)(z)}{z}\right)^{1/\beta},$$

where the integral operator $I_{\beta,\gamma}$ *is defined by (1.6). Moreover, the function* $(I_{\beta,\gamma}(g)(z)/z)^{1/\beta}$ *is the best dominant.*

Proof. The proof of Theorem 3.4 is much akin to the proof of Theorem 3.1 and hence can be omitted. □

Since the superordination results are the dual of the subordination, we state the results pertaining to the superordination, using the duality.

THEOREM 3.5. *Let* $f, g \in \mathscr{A}_{\beta,\gamma}$ *with* $\beta > 0$ *and* $0 < \gamma + 1/\beta \leqslant 1$. *Suppose that*

$$\operatorname{Re}\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\frac{1+\beta\gamma}{2\beta} \qquad \left(z \in \Delta; \phi(z) := \left(\frac{g(z)}{z}\right)^{1/\beta}\right).$$

If $(f(z)/z)^{1/\beta}$ *is univalent in* Δ *and* $(I_{\beta,\gamma}(f)(z)/z)^{1/\beta} \in \mathscr{Q}$, *then*

$$\left(\frac{g(z)}{z}\right)^{1/\beta} \prec \left(\frac{f(z)}{z}\right)^{1/\beta}$$

implies that

$$\left(\frac{I_{\beta,\gamma}(g)(z)}{z}\right)^{1/\beta} \prec \left(\frac{I_{\beta,\gamma}(f)(z)}{z}\right)^{1/\beta},$$

where the integral operator defined by (1.6). Moreover, the function $(I_{\beta,\gamma}(g)(z)/z)^{1/\beta}$ *is the best subordinant.*

By combining Theorem 3.4 and Theorem 3.5, we get the following sandwich type theorem.

THEOREM 3.6. *Let* $f, g_k \in \mathscr{A}_{\beta,\gamma}$, $(k = 1,2)$ *with* $\beta > 0$ *and* $0 < \gamma + 1/\beta \leqslant 1$. *Suppose that*

$$\mathrm{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\frac{1+\beta\gamma}{2\beta} \qquad \left(\phi_k(z) := \left(\frac{g_k(z)}{z}\right)^{1/\beta},\ k = 1,2\right).$$

If $\left(\frac{f(z)}{z}\right)^{1/\beta}$ *is univalent in* Δ *and* $(I_{\beta,\gamma}f(z)/z)^{1/\beta} \in \mathscr{Q}$, *then*

$$\left(\frac{g_1(z)}{z}\right)^{1/\beta} \prec \left(\frac{f(z)}{z}\right)^{1/\beta} \prec \left(\frac{g_2(z)}{z}\right)^{1/\beta}$$

implies that

$$\left(\frac{I_{\beta,\gamma}(g_1)(z)}{z}\right)^{1/\beta} \prec \left(\frac{I_{\beta,\gamma}(f)(z)}{z}\right)^{1/\beta} \prec \left(\frac{I_{\beta,\gamma}(g_2)(z)}{z}\right)^{1/\beta}.$$

Here $(I_{\beta,\gamma}(g_1)(z)/z)^{1/\beta}$ *and* $(I_{\beta,\gamma}(g_2)(z)/z)^{1/\beta}$ *are the best subordinant and the best dominant, respectively.*

REFERENCES

[1] T. BULBOACA, *A class of superordination – preserving integral opertor*, Indag. Math. 13(3), (2002) 301–311.

[2] N. E. CHO AND S. OWA, *Double subordination – preserving properties for certain integral operators*, J. Inequal. and Appl. Art 83073, Vol. 2007, (2007) pp. 1–10.

[3] S. S. MILLER AND P. T. MOCANU, *Differential subordination and univalent functions*, Michig. Math. J. 28 (1981), 157–171.

[4] S. S. MILLER AND P. T. MOCANU, *Differential subordinations, Theory and Applications, Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, USA, (2000).

[5] S. S. MILLER AND P. T. MOCANU, *Subordinants of differential superordinations*, Complex variables, 48(10) (2003), 815–826.

[6] S. S. MILLER, P. T. MOCANU AND O. READE, *Subordination – preserving integral operators*, Transaction of the American Mathematical Society, 283(2), (1984), 605–615.

[7] P. T. MOCANU, *Convexity and close to convexity preserving integral operators*, Mathematica (cluj), 25 (1983), 177–182.

[8] CH. POMMERENKE, *Univalent Function*, Vanderhoeck and Ruprecht, Gotlingen, 1975.

[9] RUSCHEWEYH, *Eine Invarianzeigenschaft der Bazilevic Functionen*, Math. z., (134) (1973), 215–219.
[10] T. N. SHANMUGAM, *Studies on analytic functions with special reference to integral opertors*, Ph. D., Thesis, Anna University, Chennai, 1987.

(Received February 23, 2008)

T. N. Shanmugam
Department of Mathematics
College of Engineering
Anna University
Chennai – 600025
India
e-mail: drtns2001@yahoo.com

M. P. Jeyaraman
Department of Mathematics
Easwari Engineering College
Chennai – 600089
India
e-mail: jeyaraman_mp@yahoo.co.in