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MULTIVARIATE MOMENT TYPE OPERATORS: APPROXIMATION PROPERTIES IN ORLICZ SPACES

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Abstract. In this paper modular convergence theorems in Orlicz spaces for multivariate extensions of the one-dimensional moment operator are given and the order of modular convergence in modular Lipschitz classes is studied.

1. Introduction

In several papers (see [18], [9], [2], [7], [3]) various approximation properties of the one-dimensional moment operator

$$(M_nf)(s) = \int_0^1 (n+1)t^n f(ts)dt$$

and some of its generalized versions were studied. In particular, these operators have nice pointwise approximation properties. Indeed they reduce the essential jump of the function f at a point $s \in [0,1]$ and they converge to $f(s)$ when s is a Lebesgue point of f. Moreover the use of these operators in problems of calculus of variation was very wide. For example it is possible to show that the sequence M_nf converges in variation or in lenght to f. Again the one-dimensional moment operator has interesting applications to the fractional calculus and it was used in the study of the fractional dimension of measurable sets (see [18], [12]).

In [9] and [11] some bivariate versions of the above operator were studied in connection with the pointwise convergence and convergence with respect to some functional of calculus of variation as, for example, the surface area and perimeter of sets. Some of these extended operators have a kernel given by radial functions which characterizes them as "metric type operators". It is important to remark that in the earlier paper [17], some of the above properties were obtained in a general form by considering families of Urysohn type operators.

Here we give some multidimensional versions of the moment type or metric type operators and we study their convergence properties in the general frame of the Orlicz spaces. This enables us to obtain corresponding results in L^p-spaces. In particular we

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take into consideration two kinds of operators: a multidimensional metric type operator with a suitable weight and a box version of the one-dimensional moment operator.

In Sections 3 and 4, we obtain modular convergence theorems in the Orlicz spaces for the two operators. A key tool in order to obtain these convergence theorems is a multivariate version in modular spaces of the well known Korovkin theorem, proved recently in [5] (for the classical Korovkin approximation theory see e.g. [10] and [1]). This result enables us to check the modular convergence only on a finite set of test functions determined by the projections on the axis. Moreover we study the behaviour of the convergence on certain Lipschitz subclasses of the Orlicz space and we give an order of modular convergence on these subclasses.

Note that the results given here hold also in abstract modular function spaces [15], [6]. In this instance, the generating modular functional ϱ, has to satisfy monotonicity, absolute finiteness, absolute continuity assumptions, some generalized Jensen convexity in integral form and a notion of subboundedness (see e.g. [6]). In particular we have to assume an inequality of the form

$$\varrho[f(t\cdot)] \leqslant F(t)\varrho[f(\cdot)]$$

where F is a measurable function such that

$$\int_A K_n(t)F(t)dt \leqslant D$$

for every $n \in \boldsymbol{N}$ and an absolute constant $D > 0$. These assumptions are automatically satisfied in Orlicz spaces and are fundamental in order to obtain the modular continuity for the family of operators.

2. Notations and Definitions

Let us consider $A = [0,1]^N$ provided with the Lebesgue measure dt. For any two vectors $t = (t_1, \dots, t_N)$, $s = (s_1, \dots, s_N) \in A$, we put $ts = (t_1s_1, \dots, t_Ns_N)$, $\langle t \rangle = t_1 \cdots t_N$ and, as usual, $|t| = (t_1^2 + \dots + t_N^2)^{1/2}$. By θ we denote the vector $\theta = (1, \cdots, 1)$.

We will denote by $X(A)$ the space of all real-valued measurable functions defined on A provided with equality a.e. and by $C(A)$ the space of all continuous and bounded functions.

Let Φ be the class of all functions $\varphi : \boldsymbol{R}_0^+ \to \boldsymbol{R}_0^+$ such that

(i) φ is a convex function,

(ii) $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\lim_{u\to+\infty} \varphi(u) = +\infty$.

For $\varphi \in \Phi$, we define the functional

$$\varrho^\varphi[f] = \int_A \varphi(|f(s)|)ds$$

for every $f \in X(A)$.

As it is well known, ϱ^φ is a convex modular on $X(A)$ and the subspace

$$L^\varphi(A) = \{f \in X(A) : \varrho^\varphi[\lambda f] < +\infty \text{ for some } \lambda > 0\}$$

is the Orlicz space generated by φ, (see [15]). The subspace of $L^\varphi(A)$, defined by

$$E^\varphi(A) = \{f \in X(A) : \varrho^\varphi[\lambda f] < +\infty \text{ for every } \lambda > 0\},$$

is called the space of finite elements of $L^\varphi(A)$. For example every bounded function belongs to $E^\varphi(A)$, (see [15] and [6]).

We say that a sequence of functions $(f_n)_{n\in \boldsymbol{N}} \subset L^\varphi(A)$ is modularly convergent to a function $f \in L^\varphi(A)$, if there exists $\lambda > 0$ such that

$$\lim_{n\to+\infty} \varrho^\varphi[\lambda(f_n - f)] = 0.$$

This notion extends the norm-convergence in L^p-spaces.

Moreover a sequence of functions $(f_n)_{n\in \boldsymbol{N}} \subset L^\varphi(A)$ is norm-convergent (or strongly convergent) to $f \in L^\varphi(A)$ if

$$\lim_{n\to+\infty} \varrho^\varphi[\lambda(f_n - f)] = 0$$

for every $\lambda > 0$. The two notions of convergence are equivalent if and only if the function φ satisfies a Δ_2-condition, i.e. there exists a constant $M > 0$ such that $\varphi(2u) \leqslant M\varphi(u)$, for every $u \geqslant 0$. For example, this happens for every L^p-spaces generated by the φ-functions defined as $\varphi(u) = u^p$, $u \geqslant 0$ (see [15], [6]).

Let $\mathbf{T} = (T_n)_{n\in \boldsymbol{N}}$ be a sequence of modularly continuous positive linear operators $T_n : L^\varphi(A) \to L^\varphi(A)$, i.e. there exists a positive constant $W > 0$ such that

$$\varrho^\varphi[T_n f] \leqslant W\varrho^\varphi[f]$$

for every $f \in L^\varphi(A)$ and for every $n \in \boldsymbol{N}$.

Let us consider the functions $e_i \in L^\varphi(A)$, $i = 0, \dots, N+1$ defined by

$$e_0(t) = 1,\ e_i(t) = t_i,\ i = 1, \dots, N \text{ and } e_{N+1}(t) = |t|^2.$$

Note that the above system of functions satisfies the following property: there exist continuous functions a_i, $i = 0, \dots, N+1$ such that the function

$$P_s(t) = \sum_{i=0}^{N+1} a_i(s)e_i(t),\ s, t \in A$$

is positive and equal to zero if and only if $s = t$. Indeed we can take $a_0(s) = |s|^2$, $a_i(s) = -2s_i$, $i = 1, \dots, N$ and $a_{N+1}(s) = 1$, $s \in A$.

Our main convergence theorems make use of the following result which is a consequence of a general Korovkin type theorem in abstract modular spaces proved in [5]. The proof of this theorem is mainly based on a modular density theorem of the subspace $C(A)$ (see [14]).

For convenience of the reader we reformulate the statement of the theorem in this special instance

THEOREM 1. *Assume that*

$$\lim_{n\to+\infty} T_n e_i = e_i,\ i = 0,\ldots,N+1$$

strongly in $L^\varphi(A)$ *then*

$$\lim_{n\to+\infty} T_n f = f$$

modularly in $L^\varphi(A)$.

3. Weighted metric type kernels

Let $N \geqslant 2$ and let, for every $n \in \boldsymbol{N}$, $K_n : A \to \boldsymbol{R}_0^+$ be the function defined by

$$K_n(t) = c_n \langle\, t\,\rangle\, |t|^{2n} \chi_{A_n}, \quad t \in A,$$

where for every $n \in \boldsymbol{N}$, $A_n = [1/n, 1]^N$, χ_{A_n} is the characteristic function of A_n and

$$\frac{1}{c_n} = \int_A \langle\, t\,\rangle\, |t|^{2n} \chi_{A_n} dt.$$

Let us consider the family of operators defined by

$$(\mathscr{M}_n f)(s) = \int_A K_n(t) f(ts)\, dt,$$

for every function f belonging to the domain $\mathscr{D} = \operatorname{Dom}\mathscr{M} = \bigcap_{n\in\boldsymbol{N}} \operatorname{Dom}\mathscr{M}_n$, where $\operatorname{Dom}\mathscr{M}_n$ is the subset of $X(A)$ on which $\mathscr{M}_n f$ is well defined as a measurable function of $s \in A$.

A similar bivariate version of the above operator (without the weight $\langle\, t\,\rangle$), was considered in [11] (see also [9]).

3.1. Convergence theorem

We begin with some lemmas.

LEMMA 1. *There holds, for every* $n \in \boldsymbol{N}$,

$$c_n \leqslant \frac{2^{2N}(n+1)^N}{N^n}.$$

Proof. We estimate

$$\frac{1}{c_n} = \int_A \langle\, t\,\rangle\, |t|^{2n} \chi_{A_n} dt.$$

Using $N-1$ times the Newton formula we can write, for $n \geqslant 2$,

$$\begin{aligned}
\frac{1}{c_n} &= \sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{N-1=0}}^{k_{N-2}} \binom{n}{k_1}\binom{k_1}{k_2}\cdots\binom{k_{N-2}}{k_{N-1}} \\
&\quad \times \int_{A_n} t_1^{2(n-k_1)+1} t_2^{2(k_1-k_2)+1}\cdots t_N^{2k_{N-1}+1}dt \\
&= \sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{N-1=0}}^{k_{N-2}} \binom{n}{k_1}\binom{k_1}{k_2}\cdots\binom{k_{N-2}}{k_{N-1}} \frac{1}{2(n-k_1)+2}\cdots\frac{1}{2k_{N-1}+2} \\
&\quad \times \left(1-\frac{1}{n^{2(n-k_1)+2}}\right)\cdots\left(1-\frac{1}{n^{2k_{N-1}+2}}\right) \\
&\geqslant \frac{1}{2^{2N}}\frac{1}{(n+1)^N}\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{N-1=0}}^{k_{N-2}} \binom{n}{k_1}\binom{k_1}{k_2}\cdots\binom{k_{N-2}}{k_{N-1}} \\
&= \frac{1}{2^{2N}}\frac{N^n}{(n+1)^N}
\end{aligned}$$

and so the assertion follows. □

LEMMA 2. *For every ball $B_\delta(\theta)$ centered at the point θ with radius $\delta < 1$ there holds*

$$\lim_{n\to+\infty}\int_{A\setminus B_\delta(\theta)} K_n(t)dt = 0.$$

Proof. Let $\widetilde{B_\delta}$ be the ball centered at the origin with radius $\sqrt{N-1+(1-\delta)^2}$. Since $A_n \setminus B_\delta(\theta) \subset A\setminus B_\delta(\theta) \subset \widetilde{B_\delta}$ we have

$$\begin{aligned}
\int_{A_n\setminus B_\delta(\theta)} K_n(t)dt &\leqslant c_n\int_{A\setminus B_\delta(\theta)} |t|^{2n}dt \leqslant c_n\int_{\widetilde{B_\delta}}|t|^{2n}dt \\
&= c_n\frac{2\pi^{N/2}}{\Gamma(N/2)}\int_0^{\sqrt{N-1+(1-\delta)^2}} v^{2n}v^{N-1}dv \\
&= c_n\frac{2\pi^{N/2}}{\Gamma(N/2)}\frac{(N-1+(1-\delta)^2)^{n+N/2}}{2n+N} \\
&\leqslant \frac{2^{2N}(n+1)^N}{N^n}\frac{2\pi^{N/2}}{\Gamma(N/2)}\frac{(N-1+(1-\delta)^2)^{n+N/2}}{2n+N} \\
&= \frac{2^{2N+1}\pi^{N/2}(N-1+(1-\delta)^2)^{N/2}}{\Gamma(N/2)}\frac{(n+1)^N}{2n+N}\left(\frac{N-1+(1-\delta)^2}{N}\right)^n.
\end{aligned}$$

Since $\delta < 1$ we have the assertion. □

LEMMA 3. *There exists a constant* $W > 0$ *such that*

$$\int_A K_n(t)\langle t\rangle^{-1}dt \leqslant W$$

for every $n \in \mathbf{N}$.

Proof. Putting $A^k = [0,1]^k$, $[t]_k = t_1 + t_2 + \ldots + t_k$ for $k = 1, \ldots, N$ we have successively

$$\begin{aligned}
\int_A K_n(t)\langle t\rangle^{-1}dt &= c_n \int_{A_n} |t|^{2n}dt \leqslant c_n \int_A [t]^n dt \\
&\leqslant \frac{c_n}{n+1}\int_{A^{N-1}} \left[([t]_{N-1}+1)^{n+1} - ([t]_{N-1})^{n+1}\right] dt_1 \ldots dt_{N-1} \\
&\leqslant \frac{c_n}{n+1}\int_{A^{N-1}} ([t]_{N-1}+1)^{n+1} dt_1 \ldots dt_{N-1} \\
&= \frac{c_n}{n+1}\int_{A^{N-2}} dt_1 \ldots dt_{N-2} \int_0^1 ([t]_{N-1}+1)^{n+1} dt_{N-1} \\
&\leqslant \frac{c_n}{(n+1)(n+2)}\int_{A^{N-2}} ([t]_{N-2}+2)^{n+2} dt_1 \ldots dt_{N-2} \\
&\leqslant \frac{c_n}{(n+1)(n+2)\cdots(n+N-1)}\int_0^1 (t_1+N-1)^{n+N-1} dt_1 \\
&\leqslant c_n \frac{N^{n+N}}{(n+1)\cdots(n+N)} \leqslant c_n \frac{N^{n+N}}{(n+1)^N} \\
&\leqslant \frac{2^{2N}(n+1)^N}{N^n}\frac{N^{n+N}}{(n+1)^N} = 2^{2N}N^N.
\end{aligned}$$

So we can take $W = 2^{2N}N^N$ and the assertion follows. □

Let $\varphi \in \Phi$ be fixed and let $\varrho^\varphi[f]$ be the modular generated by φ. As a consequence of Lemma 3 we have the following

PROPOSITION 1. *For every* $n \in \mathbf{N}$ *and* $f \in L^\varphi(A)$ *there holds*

$$\varrho^\varphi[\mathscr{M}_n f] \leqslant W\varrho^\varphi[f].$$

In particular we have $\mathscr{M}_n f \in L^\varphi(A)$ *whenever* $f \in L^\varphi(A)$.

Proof. By Jensen inequality and Fubini-Tonelli theorem, we have

$$\begin{aligned}
\varrho^\varphi[\mathscr{M}_n f] &\leqslant \int_A K_n(t)\left[\int_A \varphi(|f(ts)|)ds\right]dt \leqslant \int_A K_n(t)\langle t\rangle^{-1}\varrho^\varphi[f]dt \\
&\leqslant W\varrho^\varphi[f]
\end{aligned}$$

i.e. the assertion. □

Hence we obtain that $L^\varphi(A) \subset \mathscr{D}$.

Using Lemma 2, we have the following

THEOREM 2. *Putting* $e_i(t) = t_i,\ i = 1,\dots,N$ *and* $e_{N+1}(t) = |t|^2$, *there holds*

$$\lim_{n\to+\infty} \mathscr{M}_n e_i = e_i,\ i = 1,\cdots,N+1$$

strongly in $L^{\varphi}(A)$.

Proof. Note that

$$|(\mathscr{M}_n e_i)(s) - e_i(s)| \leqslant s_i \int_A K_n(t)(1-t_i)dt \leqslant \int_A K_n(t)(1-t_i)dt.$$

Let $0 < \varepsilon < 1$ be fixed and let $B_\varepsilon(\theta)$ be the ball centered at the point θ with radius ε. We have

$$|\mathscr{M}_n e_i(s) - e_i(s)| \leqslant \left\{ \int_{A\setminus B_\varepsilon(\theta)} + \int_{B_\varepsilon(\theta)\cap A} \right\} K_n(t)(1-t_i)dt = J_1 + J_2.$$

For J_1 from Lemma 2, there exists an integer $\overline{n}$ such that

$$J_1 \leqslant 2\int_{A\setminus B_\varepsilon(\theta)} K_n(t)dt < 2\varepsilon,$$

for every $n \geqslant \overline{n}$. Moreover

$$J_2 \leqslant \varepsilon \int_{B_\varepsilon(\theta)\cap A} K_n(t)dt \leqslant \varepsilon.$$

Thus

$$|\mathscr{M}_n e_i(s) - e_i(s)| \leqslant 3\varepsilon.$$

Let now $\lambda > 0$ be fixed and let us assume that $3\varepsilon < 1$. We have, by the convexity of the function φ,

$$\varrho^{\varphi}[\lambda(\mathscr{M}_n e_i - e_i)] < 3\varepsilon\varrho^{\varphi}[\lambda]$$

and so for the arbitrariness of ε we obtain the assertion for the functions $e_i,\ i = 1,\dots,N$. For the function e_{N+1} we can repeat the same arguments as before in order to prove that

$$\lim_{n\to+\infty} \mathscr{M}_n e_i^2 = e_i^2$$

strongly in $L^{\varphi}(A)$ and so we obtain easily the assertion. □

COROLLARY 1. *There holds*

$$\lim_{n\to+\infty} \mathscr{M}_n f = f$$

modularly in $L^{\varphi}(A)$ *for every* $f \in L^{\varphi}(A)$.

Proof. The proof follows from Theorem 1. □

3.2. Rate of modular convergence

Let $\mathscr{T}$ be the class of all functions $\tau : A \to \boldsymbol{R}_0^+$ such that $\tau(\theta) = 0$ and $\tau(t) \neq 0$ for $t \neq \theta$. For a fixed $\tau \in \mathscr{T}$ we define the class:

$$Lip_\tau(\varphi) = \{f \in L^\varphi(A) : \exists\, \lambda > 0 \text{ with } \varrho^\varphi[\lambda|f(t\cdot) - f(\cdot)|] = \mathscr{O}(\tau(t)),\ t \to \theta\}$$

where, for any two functions $f, g \in X(A)$, $f(t) = \mathscr{O}(g(t)),\ t \to \theta$ means that there is a constant $C > 0$ and $\delta > 0$ such that $|f(t)| \leqslant C|g(t)|$, for every $t \in B_\delta(\theta)$.

THEOREM 3. *Let $\alpha \in]0,1]$ and $\tau \in \mathscr{T}$ be fixed. Assume that there is $\delta > 0$ such that*

$$\int_{B_\delta(\theta)\cap A} K_n(t)\tau(t)dt = \mathscr{O}(n^{-\alpha}),\ n \to +\infty. \tag{1}$$

If $f \in Lip_\tau(\varphi)$ then for sufficiently small $\lambda > 0$ we have

$$\varrho^\varphi[\lambda(\mathscr{M}_n f - f)] = \mathscr{O}(n^{-\alpha}),\ n \to +\infty.$$

Proof. Let $\lambda > 0$ be fixed. We have

$$\begin{aligned}
\varrho^\varphi[\lambda(\mathscr{M}_n f - f)] &\leqslant \int_A \varphi\left(\lambda \int_A K_n(t)|f(ts) - f(s)|dt\right) ds \\
&\leqslant \int_A K_n(t)\left(\int_A \varphi(\lambda|f(ts) - f(s)|ds\right) dt \\
&= \left(\int_{B_\delta(\theta)\cap A} + \int_{A\setminus B_\delta(\theta)}\right) K_n(t)\left(\int_A \varphi(\lambda|f(ts) - f(s)|)ds\right) dt \\
&= I_1 + I_2.
\end{aligned}$$

For I_1 we can choose λ and δ such that

$$\varrho^\varphi[\lambda|f(t\cdot) - f(\cdot)|] \leqslant C\tau(t)$$

for every $t \in B_\delta(\theta) \cap A$ and (1) holds. So by (1), for $\alpha \in]0,1]$, we obtain

$$I_1 \leqslant C\int_{B_\delta(\theta)\cap A} K_n(t)\tau(t)dt = \mathscr{O}(n^{-\alpha}).$$

For I_2 we have

$$\begin{aligned}
I_2 &\leqslant \int_{A\setminus B_\delta(\theta)} K_n(t)\left(\int_A \varphi(2\lambda|f(ts)|)ds\right) dt + \int_{A\setminus B_\delta(\theta)} K_n(t)\left(\int_A \varphi(2\lambda|f(s)|)ds\right) dt \\
&\leqslant c_n \int_{A\setminus B_\delta(\theta)} |t|^{2n}\varrho^\varphi[2\lambda f]dt + \int_{A\setminus B_\delta(\theta)} K_n(t)\varrho^\varphi[2\lambda f]dt.
\end{aligned}$$

Following the proof of Lemma 2 we easily obtain for every $\alpha > 0$

$$I_2 = \mathscr{O}(n^{-\alpha})$$

and so the assertion follows. □

EXAMPLE 1. Here we give a bidimensional example of Theorem 3. We take for $\alpha\in]0,1]$, $\tau(x,y)=|\log x|^{\alpha}|\log y|^{\alpha}$ which is defined in a neighbourhood of $\theta=(1,1)$. We will show that our bidimensional kernel $K_n(x,y)$ satisfies assumption (1) of Theorem 3. Indeed, by the concavity of the function $g(t)=t^{\alpha},\ t\geqslant 0$ we have

$$\int_{B_\delta(\theta)\cap A} K_n(x,y)\tau(x,y)dxdy \leqslant \left(c_n\int_A (x^2+y^2)^n|\log x||\log y|dxdy\right)^{\alpha}.$$

Let us consider now the integral

$$I_n=c_n\int_A (x^2+y^2)^n|\log x||\log y|dxdy.$$

We have, by elementary calculation,

$$\begin{aligned}
&c_n\sum_{k=0}^{n}\binom{n}{k}\left(\int_0^1 x^{2k}|\log x|dx\right)\left(\int_0^1 y^{2(n-k)}|\log y|dy\right)\\
&\quad=c_n\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(2k+1)^2}\frac{1}{(2(n-k)+1)^2}\\
&\quad\leqslant c_n\sum_{k=0}^{n}\binom{n}{k}\frac{1}{[(k+1)((n-k)+1)]^2}\\
&\quad=\frac{c_n}{(n+2)^2}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{(k+1)}+\frac{1}{n-k+1}\right)^2\\
&\quad=\frac{c_n}{(n+2)^2}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(n-k+1)^2}+\frac{c_n}{(n+2)^2}\sum_{k=0}^{n}\binom{n}{k}\frac{2}{(n-k+1)(k+1)}\\
&\qquad+\frac{c_n}{(n+2)^2}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+1)^2}\\
&\quad=I_1+I_2+I_3.
\end{aligned}$$

For I_1 we have

$$I_1\leqslant\frac{c_n}{(n+2)^2}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(n-k+1)}=\frac{c_n}{(n+2)^2}\frac{2^{n+1}-1}{(n+1)}.$$

In the same way

$$I_3\leqslant\frac{c_n}{(n+2)^2}\frac{2^{n+1}-1}{(n+1)}.$$

Finally for I_2 we get

$$I_2=\frac{2c_n}{(n+2)^3}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{k+1}+\frac{1}{n-k+1}\right)=\frac{4c_n}{(n+2)^3}\frac{2^{n+1}-1}{(n+1)}.$$

Thus we have the estimate

$$\begin{aligned}\int_{B_\delta(\theta)\cap A} & K_n(x,y)\tau(x,y)dxdy \\ & \leqslant \left[2c_n(2^{n+1}-1)\left(\frac{1}{(n+2)^2(n+1)}+\frac{2}{(n+2)^3(n+1)}\right)\right]^\alpha \\ & \leqslant \left[2^5\frac{(n+1)^2}{2^n}(2^{n+1}-1)\left(\frac{1}{(n+2)^2(n+1)}+\frac{2}{(n+2)^3(n+1)}\right)\right]^\alpha \\ & = \mathscr{O}(n^{-\alpha})\end{aligned}$$

REMARKS.

1. We can take other examples of functions $\tau(x,y)$ for example

$$\tau(x,y) = (1-x)^\alpha(1-y)^\alpha,$$

obtaining the same error of approximation.

2. In (1) it is possible also to consider other comparison sequences $\xi(n)$ such that $\lim_{n\to+\infty}\xi(n)=0$, in place of $n^{-\alpha}$. In this case we obtain an order of approximation $\mathscr{O}(\xi(n))$.

4. Another extension of one-dimensional moment kernels

Here we introduce a direct extension to the multivariate case of the classical one-dimensional moment kernel (see [18], [2], [9],[7], [3]). Let $A=[0,1]^N$ and let, for every $n\in\boldsymbol{N}$, $K_n : A\to\boldsymbol{R}_0^+$ be the function defined by

$$K_n(t) = (n+1)^N\langle t\rangle^n, \quad t\in A.$$

Let us consider the family of operators defined by

$$(M_nf)(s) = \int_A K_n(t)f(ts)dt,$$

for every function f belonging to the domain $\mathscr{D}$ of the operator M_n.

At first, note that

$$\int_A K_n(t)dt = \prod_{j=1}^N (n+1)\int_0^1 t_j^n dt_j = 1$$

and

$$\int_A K_n(t)\langle t\rangle^{-1}dt = (n+1)^N\int_A\langle t\rangle^{n-1}dt = \left(\frac{n+1}{n}\right)^N \leqslant 2^N.$$

Moreover, putting $R_\delta(\theta) = [1-\delta, 1]^N$, we have $A \setminus R_\delta(\theta) \subset V_\delta = \bigcup_{k=1}^N U_k$ where $U_k = [0,1] \times \ldots \overbrace{[0,1-\delta]}^{k-place} \times \ldots [0,1]$ and

$$\int_{A\setminus R_\delta(\theta)} K_n(t)dt = \mathscr{O}(n^{-\alpha})$$

for every $\alpha > 0$. Indeed we have

$$\begin{aligned} \int_{A\setminus R_\delta(\theta)} K_n(t)dt &\leqslant \int_{V_\delta} K_n(t)dt = \sum_{k=1}^N \prod_{j=1, j\neq k}^N (n+1)\int_0^1 t_j^n dt_j \cdot (n+1)\int_0^{1-\delta} t_k^n dt_k \\ &= \sum_{k=1}^N (n+1)\int_0^{1-\delta} t_k^n dt_k = N(1-\delta)^{n+1} \end{aligned}$$

and so the assertion follows.

As before, we can prove that $\varrho^\varphi[M_n f] \leqslant 2^N \varrho^\varphi[f]$ and so $M_n f \in L^\varphi(A)$ whenever $f \in L^\varphi(A)$. Hence we obtain that $L^\varphi(A) \subset \mathscr{D}$.

THEOREM 4. *Putting* $e_i(t) = t_i,\ i = 1,\ldots,N$ *and* $e_{N+1} = |t|^2$ *there holds*

$$\lim_{n\to+\infty} M_n e_i = e_i,\ i = 1,\ldots,N+1,$$

strongly in $L^\varphi(A)$.

Proof. We have easily

$$\begin{aligned} |(M_n e_i)(s) - e_i(s)| &\leqslant \int_A K_n(t)(1-t_i)dt = \prod_{j=1}^N (n+1)\int_0^1 t_j^n(1-t_i)dt_j \\ &= (n+1)\int_0^1 t_i^n(1-t_i)dt_i = \frac{1}{n+2}. \end{aligned}$$

Taking into account that any constant function defined on A belongs to the space $E^\varphi(A)$, passing to the modular, we have the assertion letting $n \to +\infty$. For the function e_{N+1} we can argue as in Theorem 2. □

COROLLARY 2. *For the operator* M_n *we have*

$$\lim_{n\to+\infty} M_n f = f$$

modularly in $L^\varphi(A)$ *for every* $f \in L^\varphi(A)$.

Regarding the rate of modular convergence in Lipschitz classes, using the notations of previous section, we have the following theorem

THEOREM 5. *Let* $\alpha \in]0,1]$ *and* $\tau \in \mathscr{T}$ *be fixed. Assume that there is* $\delta > 0$ *such that*

$$\int_{R_\delta(\theta)} K_n(t)\tau(t)dt = \mathscr{O}(n^{-\alpha}),\ n \to +\infty. \tag{2}$$

If $f \in Lip_\tau(\varphi)$ *then for sufficiently small* $\lambda > 0$ *we have*

$$\varrho^\varphi[\lambda(M_n f - f)] = \mathscr{O}(n^{-\alpha}),\ n \to +\infty.$$

EXAMPLE 2. Let $\tau(t) = \prod_{j=1}^N (1-t_j)^{\alpha/N}$ with $\alpha \in]0,1]$, then we have

$$\begin{aligned} n^\alpha \int_A K_n(t)\tau(t)dt &= n^\alpha \prod_{j=1}^N (n+1) \int_0^1 t_j^n (1-t_j)^{\alpha/N} dt_j \\ &= n^\alpha (n+1)^N B^N(n+1, \frac{\alpha}{N}+1). \end{aligned}$$

Since $\lim_{n\to+\infty}(n+1)^{\frac{\alpha}{N}+1} B(n+1, \frac{\alpha}{N}+1) = \Gamma(\frac{\alpha}{N}+1)$ (see [16]) we obtain (2).

Note that we can obtain similar order of approximation by different choises of the function τ. For example we can take

$$\tau(t) = \prod_{j=1}^N |\log t_j|^{\alpha/N}$$

for $\alpha \in]0,1]$.

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