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INTERPOLATION INEQUALITIES IN WEIGHTED SOBOLEV SPACES

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Abstract. In this paper we prove some interpolation inequalities between functions and their derivatives in the class of weighted Sobolev spaces defined on unbounded open subset $\Omega \subset \mathbb{R}^n$.

1. Introduction

We consider the problem of determining some upper bounds for L^q-norms ($q > 1$) of derivatives $\partial^k u$, $k < r$ $(k, r \in \mathbb{N})$, of functions u belonging to appropriate Sobolev spaces defined on an open subset $\Omega \subset \mathbb{R}^n$, in terms of the L^p-norms ($p \geqslant 1$) of u and its derivative $\partial^r u$. Such interpolation inequalities have been obtained by many writers with different regularity properties on Ω. Such regularity is normally expressed in terms of geometrical conditions that may or may not be satisfied by a given domain.

For example, if Ω is a bounded domain with the cone property, several authors (see, for instance, [8], [5], [7]) proved some interpolation inequalities of the type

$$\|\partial^k u\|_{L^q(\Omega)} \leqslant c\left(\|\partial^r u\|^a_{L^p(\Omega)} \cdot \|u\|^{1-a}_{L^{p_o}(\Omega)} + \|u\|_{L^{p_o}(\Omega)}\right), \tag{1.1}$$

for functions u in a suitable Sobolev space $X(\Omega)$, with $p_o \geqslant 1, p > 1, q$ in an appropriate interval depending on $p,\ k,\ r,\ n$, and a determined by q.

Several inequalities of kind (1.1) have been obtained when the functions u are required to be in a suitable weighted Sobolev space (see, for instance, [9], [3], [2], [6]).

In particular, in [9] the author proved some interpolation inequalities between functions and their derivatives in the within of weighted Sobolev spaces defined on bounded open subset $\Omega \subset \mathbb{R}^n$. Here the weights are suitable powers of a measurable function $\rho : \Omega \to \mathbb{R}_+$ such that

$$\sup_{\substack{x,y\in\Omega \\ |x-y|<\rho(y)}} \left|\log\frac{\rho(x)}{\rho(y)}\right| < +\infty\,. \tag{1.2}$$

Moreover, the required regularity property on Ω is:

H) For any $x \in \Omega$, $\Omega \cap B(x,\rho(x))$ is a set with the cone property with characteristic cone of opening $\theta \in]0, \frac{\pi}{2}[$ and height $\lambda\,\rho(x)$, where $\theta,\ \lambda$ are independent of x ($B(x,\rho(x))$ denote the open ball of radius $\rho(x)$ centered at x).

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The aim of this paper is to generalize the above quoted results of [9] to the case of an unbounded open set Ω and with regularity property weaker then $H)$ (see condition $h_o)$ in section 4).

In section 5 we will give some examples of open sets Ω for which condition $H)$ is not verified, but our condition $h_o)$ is verified.

As an application of the main obtained result, we consider the case of constant weight functions and this allows us to obtain an inequality like (1.1) in unbounded domains. Moreover, we prove an interpolation inequality for intermediate derivatives, without condition $h_o)$, in the case in which Ω has the cone property and the weight function ρ verifies the conditions

$$\inf_{\Omega} \rho > 0, \qquad \sup_{\Omega} \rho = +\infty\,. \tag{1.3}$$

Finally, we consider the case of an other class of weight functions, the class $\mathscr{G}(\Omega)$ (see section 4 for the definition). Here we prove an interpolation inequality in which all derivatives of the function u are in the same weighted Sobolev space.

We note that these inequalities play an important role in the study of elliptic equations with discontinuous (or also singular) coefficients in weighted Sobolev spaces.

2. Weight functions and weighted spaces

Let E be a generic Lebesgue measurable subset of $\mathbb{R}^n$. We denote by $|E|$ the Lebesgue measure of E. Moreover, we denote by $L^p_{\text{loc}}(E)$, $p \in [1,+\infty[$, the class of functions f defined on E such that $\zeta f \in L^p(E)$ for all $\zeta \in \mathscr{D}(E)$, where $\mathscr{D}(E)$ is the class of restrictions to E of functions $\zeta \in C^\infty_o(\mathbb{R}^n)$ with $\overline{E} \cap \operatorname{supp}\zeta \subset E$. If $\varphi : E \to \mathbb{R}_+$ is a measurable function and $s \in \mathbb{R}$, we denote by $L^p_s(E)$ the class of functions f defined on E such that $\varphi^s f \in L^p(E)$ equipped with the norm $\|f\|_{L^p_s(E)} = \|\varphi^s f\|_{L^p(E)}$.

Let Ω be an open subset of $\mathbb{R}^n$. We denote by $\mathscr{A}(\Omega)$ the class of measurable functions $\rho : \Omega \to \mathbb{R}_+$ such that

$$\sup_{\substack{x,y\in\Omega \\ |x-y|<\rho(y)}} \left|\log\frac{\rho(x)}{\rho(y)}\right| < +\infty\,. \tag{2.1}$$

It is easy to show that $\rho \in \mathscr{A}(\Omega)$ if and only if there exist $c_1, c_2 \in \mathbb{R}_+$ independent of x and y such that

$$c_1\,\rho(y) \leqslant \rho(x) \leqslant c_2\,\rho(y) \qquad \forall\, y \in \Omega\,, \quad \forall\, x \in \Omega \cap B(y,\rho(y))\,, \tag{2.2}$$

where $B(y,\rho(y))$ is the open ball of radius $\rho(y)$ centered at y.

REMARK 2.1. We note that if $\rho \in \mathscr{A}(\Omega)$ and $a \in]0,1[$, then the function $\omega(x) = a\,\rho(x)\ (x \in \Omega)$ belongs to $\mathscr{A}(\Omega)$.

For any weight function $\rho \in \mathscr{A}(\Omega)$, we put

$$S_\rho = \{z \in \partial\Omega\ :\ \lim_{x\to z}\rho(x) = 0\}\,. \tag{2.3}$$

It is know that S_ρ is a closed subset of $\partial\Omega$ and that

$$\rho \in L^\infty_{\text{loc}}(\overline{\Omega}), \qquad \rho^{-1} \in L^\infty_{\text{loc}}(\overline{\Omega} \setminus S_\rho), \tag{2.4}$$

(see [10], [4]).

If $r \in \mathbb{N}$, $1 \leqslant p \leqslant +\infty$, $s \in \mathbb{R}$ and $\rho \in \mathscr{A}(\Omega)$, we consider the space $W^{r,p}_s(\Omega)$ of distributions u on Ω such that $\rho^{s+|\alpha|-r}\,\partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \leqslant r$, equipped with the norm

$$\|u\|_{W^{r,p}_s(\Omega)} = \sum_{|\alpha|\leqslant r} \left\|\rho^{s+|\alpha|-r}\,\partial^\alpha u\right\|_{L^p(\Omega)}. \tag{2.5}$$

A detailed account of properties of the above defined function spaces can be found in [11].

3. Some preliminary results

We observe that $\mathscr{A}(\Omega)$ contains the class of all functions $\rho : \Omega \to \mathbb{R}_+$ which are Lipschitz continuous in Ω with Lipschitz constant less than 1. On the other hand the only continuity of a positive function ρ may not be imply that ρ belongs to $\mathscr{A}(\Omega)$. Nevertheless, fixed a weight function $\rho \in \mathscr{A}(\Omega)$, under suitable conditions on the open set Ω, it is possible to find a continuous weight function in $\mathscr{A}(\Omega)$ which is equivalent to ρ (see section 4).

In this section we prove some results for continuous weight functions belonging to $\mathscr{A}(\Omega)$.

Let $\tau \in C^0(\Omega)$ be a positive function. Consider the following condition on Ω:

$i_o)$ There exists $\theta \in]0, \frac{\pi}{2}[$ such that

$$\forall x \in \Omega \quad \exists\, C_\theta(x) \ : \quad \overline{C_\theta(x, \tau(x))} \subset \Omega, \tag{3.1}$$

where $C_\theta(x)$ is an indefinite cone with vertex at x and opening θ, and $C_\theta(x, \tau(x)) = C_\theta(x) \cap B(x, \tau(x))$.

For any fixed $x \in \Omega$ and $b \in [1, +\infty[$, we denote by $\Omega_b(x)$ the union of all open cones C with opening θ and height $b^{-1}\,\tau(x)$, such that $C \subset\subset \Omega$ and $x \in C$. For each $x \in \Omega$, put

$$F_b(x) = \{y \in \Omega \ : \ x \in \Omega_b(y)\}, \tag{3.2}$$

$$\Lambda = \{(x, y) \in \Omega \times \Omega \ : \ x \in \Omega \quad y \in \Omega_b(x)\}. \tag{3.3}$$

Obviously we have

$$y \in F_b(x) \Longleftrightarrow x \in \Omega_b(y). \tag{3.4}$$

LEMMA 3.1. *Assume that condition $i_o)$ holds. Then the sets $F_b(y)$ ($y \in \Omega$) and Λ are open sets in Ω and $\Omega \times \Omega$ respectively.*

Proof. Let $y_o \in \Omega$ and $x_o \in F_b(y_o)$. By (3.4), we have that there exists $C_\theta(z, b^{-1}\,\tau(x_o))$ $(z \in \Omega)$ such that

$$x_o,\, y_o \in C_\theta(z, b^{-1}\,\tau(x_o)), \qquad \overline{C_\theta(z, b^{-1}\,\tau(x_o))} \subset \Omega. \tag{3.5}$$

From (3.5) we have that there exists $\epsilon > 0$ such that

$$x_o,\ y_o \in C_\theta(z, b^{-1}\,\tau(x_o) - \epsilon), \qquad \overline{C_\theta(z, b^{-1}\,\tau(x_o) + \epsilon)} \subset \Omega\,. \tag{3.6}$$

Since $\tau \in C^0(\Omega)$, we deduce that there exists a neighborhood $A_1(x_o)$ of x_o such that

$$\begin{gathered} A_1(x_o) \subset C_\theta(z, b^{-1}\,\tau(x_o) - \epsilon) \\ b^{-1}\,\tau(x_o) - \epsilon < b^{-1}\,\tau(x) < b^{-1}\,\tau(x_o) + \epsilon \qquad \forall\, x \in A_1(x_o)\,. \end{gathered} \tag{3.7}$$

So we have for any $x \in A_1(x_o)$

$$\begin{gathered} C_\theta(z, b^{-1}\,\tau(x_o) - \epsilon) \subset C_\theta(z, b^{-1}\,\tau(x)) \\ \overline{C_\theta(z, b^{-1}\,\tau(x))} \subset \overline{C_\theta(z, b^{-1}\,\tau(x_o) + \epsilon)} \subset \Omega\,. \end{gathered} \tag{3.8}$$

Therefore for each $x \in A_1(x_o)$ we have

$$x,\ y_o \in C_\theta(z, b^{-1}\,\tau(x)), \qquad \overline{C_\theta(z, b^{-1}\,\tau(x))} \subset \Omega, \tag{3.9}$$

and so $y_o \in \Omega_b(x)$. Therefore $F_b(y_o)$ is an open set.

Fix now $(x_o, y_o) \in \Lambda$. We have already proved that there exists a neighborhood $A_1(x_o)$ of x_o such that $y_o \in \Omega_b(x)$ for any $x \in A_1(x_o)$. Fix $x \in A_1(x_o)$, let $A_2(y_o)$ be a neighborhood of y_o such that

$$A_2(y_o) \subset C_\theta(z, b^{-1}\,\tau(x))\,. \tag{3.10}$$

By (3.9) it follows that $A_2(y_o) \subset \Omega_b(x)$ for all $x \in A_1(x_o)$.

Then we deduce that

$$A_1(x_o) \times A_2(y_o) \subset \Lambda\,. \tag{3.11}$$

From (3.11) we deduce that Λ is an open set. □

Consider the function $\Phi_b : \Omega \times \Omega \to \mathbb{R}$ defined by

$$\Phi_b(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \Omega \times \Omega_b(x) \\ 0 & \text{if } (x,y) \in \Omega \times \big(\Omega \setminus \Omega_b(x)\big)\,. \end{cases} \tag{3.12}$$

We observe that

$$\Phi_b(x,y) = \begin{cases} 1 & \text{if } (x,y) \in F_b(y) \times \Omega \\ 0 & \text{if } (x,y) \in \big(\Omega \setminus F_b(y)\big) \times \Omega\,. \end{cases} \tag{3.13}$$

LEMMA 3.2. *Assume that condition $i_o)$ holds. Then the function Φ_b defined by (3.12) is a measurable function.*

Proof. Let $t \in \mathbb{R}$. Consider the set

$$\mathscr{F}(t) = \{(x,y) \in \mathbb{R}^2 \ : \ \Phi_b(x,y) > t\}\,. \tag{3.14}$$

For $t < 0$ we have $\mathscr{F}(t) = \Omega \times \Omega$, so $\mathscr{F}(t)$ is a measurable set. If $t \geqslant 1$, then $\mathscr{F}(t) = \emptyset$ and hence the set is measurable too. Suppose now $0 \leqslant t < 1$, then $\mathscr{F}(t) = \Lambda$, where Λ is defined by (3.3). In this case the result follows from Lemma 3.1. □

Suppose now that $\tau \in C^o(\Omega) \cap \mathscr{A}(\Omega)$. So there exists $\delta \in [1,+\infty[$, independent of x and y, such that

$$\delta^{-1}\,\tau(y) \leqslant \tau(x) \leqslant \delta\,\tau(y) \qquad \forall\, y \in \Omega\,, \quad \forall\, x \in \Omega \cap B(y,\tau(y))\,. \tag{3.15}$$

In the sequel, we denote by $\Omega(x)$, $F(x)$ and $\Phi(x,y)$ the sets $\Omega_\delta(x)$, $F_\delta(x)$ and the function $\Phi_\delta(x,y)$ respectively.

LEMMA 3.3. *Assume that condition i_o) holds. Then there exist c_1, $c_2 \in \mathbb{R}_+$ such that*

$$c_1\,\tau^n(x) \leqslant |F(x)| \leqslant c_2\,\tau^n(x) \qquad \forall\, x \in \Omega\,, \tag{3.16}$$

where c_1 depends on θ, n, δ, and c_2 depends only on n.

Proof. Fix $x \in \Omega$. To prove (3.16), we observe that if $y \in F(x)$ then $x \in \Omega \cap B(y, \delta^{-1}\,\tau(y))$. Therefore from (3.15) we deduce that $y \in \Omega \cap B(x,\tau(x))$ and so

$$|F(x)| \leqslant |B(x,\tau(x))| \leqslant c_3\,\tau^n(x)\,, \qquad \forall\, x \in \Omega\,, \tag{3.17}$$

where $c_3 \in \mathbb{R}_+$ depends only on n.

Now let $y \in C_\theta(x, \delta^{-2}\,\tau(x))$. From (3.15) it follows that

$$C_\theta(x,\delta^{-2}\,\tau(x)) \subset C_\theta(x,\delta^{-1}\,\tau(y)) \subset C_\theta(x,\tau(x))\,. \tag{3.18}$$

Therefore from condition i_o) we have

$$\overline{C_\theta(x,\delta^{-1}\,\tau(y))} \subset \Omega\,. \tag{3.19}$$

Since $\overline{C_\theta(x,\delta^{-1}\,\tau(y))}$ is a compact set in Ω, we deduce that there exists a cone C with opening θ and height $\delta^{-1}\,\tau(y)$, such that $x,y \in C$ and $C \subset\subset \Omega$. So $x \in \Omega(y)$ and then $y \in F(x)$. It follows that

$$|F(x)| \geqslant |C_\theta(x,\delta^{-2}\,\tau(x))| \geqslant c_4\,\tau^n(x)\,, \tag{3.20}$$

where $c_4 \in \mathbb{R}_+$ depends on θ, n and δ. □

The above lemmas can be used to prove the following result, which will be essential in the proof of Theorem 4.4.

Let S_τ the set defined by (2.3) in correspondence of the weight function τ.

Let ψ a function defined on Ω which verifies the following conditions:

$$\begin{aligned} &\psi(x) > 0 \qquad \forall\, x \in \Omega \\ &\exists\, \mu \in \mathbb{R}_+ \ : \ \mu^{-1}\,\psi(y) \leqslant \psi(x) \leqslant \mu\,\psi(y) \qquad \forall\, x \in \Omega \quad \forall\, y \in \Omega(x)\,. \end{aligned} \tag{3.21}$$

LEMMA 3.4. *Suppose that condition $i_o)$ holds and fix a function ψ which verifies (3.21). Then for any $p,\, q \in [1,+\infty[$, with $q \geqslant p$, there exist $c_1, c_2 \in \mathbb{R}_+$ such that*

$$\int_\Omega \psi(x)\,\tau^{-n}(x)\,\|u\|^p_{L^p(\Omega(x))}\,dx \geqslant c_1 \int_\Omega \psi(x)\,|u(x)|^p\,dx \tag{3.22}$$

$$\int_\Omega \psi^{\frac{q}{p}}(x)\,\tau^{-n}(x)\,\|u\|^q_{L^p(\Omega(x))}\,dx \leqslant c_2 \Big(\int_\Omega \psi(x)\,|u(x)|^p\,dx\Big)^{\frac{q}{p}}, \tag{3.23}$$

for any $u \in L^p_{loc}(\overline{\Omega}\setminus S_\tau)$, where c_1 depends on $\theta,\, n,\, \delta,\, \mu$, and c_2 depends on $n,\, \delta,\, \mu,\, p$ and q.

Proof. Fix $p \in [1,+\infty[$ and $u \in L^p_{\mathrm{loc}}(\overline{\Omega}\setminus S_\tau)$. From (3.15), (3.21) and (3.4) we obtain

$$\begin{aligned}\int_\Omega \psi(x)\,\tau^{-n}(x)\,\|u\|^p_{L^p(\Omega(x))}\,dx &= \int_\Omega \Big(\int_{\Omega(x)} \psi(x)\,\tau^{-n}(x)\,|u(y)|^p\,dy\Big)\,dx\\ &\geqslant c_3 \int_\Omega \psi(y)\,\tau^{-n}(y)\,|u(y)|^p \Big(\int_\Omega \Phi(x,y)\,dx\Big)\,dy\\ &= c_3 \int_\Omega \psi(y)\,\tau^{-n}(y)\,|u(y)|^p \Big(\int_{F(y)} dx\Big)\,dy,\end{aligned} \tag{3.24}$$

where $c_3 \in \mathbb{R}_+$ depends on $\delta,\, \mu$ and n.

From Lemma 3.3 and (3.24) it follows (3.22).

Fix now $q \geqslant p$ and put

$$h(x) = \psi^{\frac{q}{p}-1}(x)\Big(\int_{\Omega(x)} |u(y)|^p\,dy\Big)^{\frac{q}{p}-1} \quad x \in \Omega. \tag{3.25}$$

From (3.15), (3.21) and Hölder inequality, it follows that

$$\begin{aligned}&\int_\Omega \psi^{\frac{q}{p}}(x)\,\tau^{-n}(x)\,\|u\|^q_{L^p(\Omega(x))}\,dx\\ &= \int_\Omega \psi(x)\,\tau^{-n}(x)\,h(x)\Big(\int_{\Omega(x)} |u(y)|^p\,dy\Big)\,dx\\ &\leqslant c_4 \int_\Omega \psi(y)\,\tau^{-n\frac{p}{q}}(y)\,|u(y)|^p \Big(\int_\Omega \tau^{-n\frac{(q-p)}{q}}(x)\,h(x)\,\Phi(x,y)\,dx\Big)\,dy\\ &\leqslant c_4 \int_\Omega \Big[\psi(y)\,\tau^{-n\frac{p}{q}}(y)\,|u(y)|^p \Big(\int_\Omega \tau^{-n}(x)\,\big(h(x)\big)^{\frac{q}{q-p}}\,dx\Big)^{1-\frac{p}{q}} \Big(\int_\Omega \big(\Phi(x,y)\big)^{\frac{q}{p}}\,dx\Big)^{\frac{p}{q}}\Big]\,dy\\ &= c_4 \int_\Omega \psi(y)\,\tau^{-n\frac{p}{q}}(y)\,|u(y)|^p\,|F(y)|^{\frac{p}{q}}\,dy \cdot \Big(\int_\Omega \psi^{\frac{q}{p}}(x)\,\tau^{-n}(x)\,\|u\|^q_{L^p(\Omega(x))}\,dx\Big)^{1-\frac{p}{q}},\end{aligned} \tag{3.26}$$

where $c_4 \in \mathbb{R}_+$ depends on $n,\, \delta,\, \mu,\, p$ and q. So (3.23) follows from (3.26) and Lemma 3.3. □

4. Main results

Fix $\rho \in \mathscr{A}(\Omega)$. So there exists $\gamma \in [1,+\infty[$, independent of $x, y \in \Omega$, such that

$$\gamma^{-1}\rho(y) \leqslant \rho(x) \leqslant \gamma\,\rho(y) \qquad \forall y \in \Omega, \quad \forall x \in \Omega \cap B(y,\rho(y)). \tag{4.1}$$

The results of previous section can be used only if we consider a continuous weight function. Preliminarily, in this section, we want to find some conditions on Ω that guarantee the existence of a continuous weight function which is equivalent to ρ. To this aim, we establish some introductory results.

For any $x \in \Omega$ we put

$$I(x) = \Omega \cap B(x,\rho(x)), \tag{4.2}$$

$$J(x) = \{y \in \Omega \,:\, x \in I(y)\}. \tag{4.3}$$

LEMMA 4.1. *For any $x \in \Omega$ the set $J(x)$ is measurable.*

Proof. Fix $x \in \Omega$. We observe that

$$J(x) = \{y \in \Omega \,:\, \rho(y) - |x-y| > 0\}. \tag{4.4}$$

Then the result follows from the measurability of the function $y \mapsto \rho(y) - |x-y|$. □

Consider the following condition on Ω:

$h_o)$ There exists $\theta \in]0,\frac{\pi}{2}[$ such that

$$\forall x \in \Omega \quad \exists\, C_\theta(x) \;:\quad \overline{C_\theta(x,\rho(x))} \subset \Omega. \tag{4.5}$$

LEMMA 4.2. *Assume that condition $h_o)$ holds. Then there exist $c_1, c_2 \in \mathbb{R}_+$ such that*

$$c_1\,\rho^n(x) \leqslant |J(x)| \leqslant c_2\,\rho^n(x) \qquad \forall x \in \Omega, \tag{4.6}$$

where c_1 depends on θ, n, γ, and c_2 depends on n and γ.

Proof. Fix $x \in \Omega$. Using the same argument of the proof of Lemma 3.3, it is possible to prove that

$$C_\theta\big(x,\gamma^{-2}\rho(x)\big) \subset J(x) \subset B(x,\gamma\,\rho(x)). \tag{4.7}$$

So, the result follows from (4.7). □

REMARK 4.3. From Lemma 4.2, we deduce that

$$\inf_{x\in\Omega} \frac{|J(x)|}{\rho^n(x)} > 0. \tag{4.8}$$

From a known result of [10], it follows that there exists a function $\sigma \in \mathscr{A}(\Omega) \cap C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$ and a constant $\mu \in \mathbb{R}_+$, independent of x, such that

$$\mu^{-1}\rho(x) \leqslant \sigma(x) \leqslant \mu\,\rho(x) \qquad \forall x \in \Omega. \tag{4.9}$$

Now we put

$$\tau(x) = \frac{\sigma(x)}{\mu} \qquad \forall x \in \Omega\,, \tag{4.10}$$

where the function σ and the constant μ are defined in Remark 4.3. In view of Remarks 2.1 and 4.3, and condition $h_o)$, the function τ verifies the following conditions:

$$\begin{aligned} &\tau \in \mathscr{A}(\Omega) \cap C^0(\Omega) \\ &\mu^{-2}\rho(x) \leqslant \tau(x) \leqslant \rho(x) \qquad \forall x \in \Omega\,, \\ &S_\tau = S_\rho \\ &\exists\, \theta \in]0, \tfrac{\pi}{2}[\quad : \quad \forall x \in \Omega \quad \exists\, C_\theta(x) \text{ such that } \overline{C_\theta(x, \tau(x))} \subset \Omega\,. \end{aligned} \tag{4.11}$$

For any $x \in \Omega$, we denote by $\Omega(x)$ and $F(x)$ the sets of $\mathbb{R}^n$ defined in section 3 in correspondence of the function τ in (4.10).

Fix $k \in \mathbb{N}_0$, $r \in \mathbb{N}$, $s \in \mathbb{R}$, $p_o \in [1, +\infty[$, p, $q \in]1, +\infty[$ and consider the following condition

$h_1)$

$$k < r, \qquad p_o \leqslant p \leqslant q, \qquad \frac{1}{q} \geqslant \frac{1}{p} - \frac{r-k}{n}\,. \tag{4.12}$$

Put

$$\alpha_o = s - n\Big(\frac{1}{p_o} - \frac{1}{p}\Big), \qquad \alpha = s - n\Big(\frac{1}{q} - \frac{1}{p}\Big)\,, \tag{4.13}$$

$$a = \frac{\frac{1}{p_o} - \frac{1}{q} + \frac{k}{n}}{\frac{1}{p_o} - \frac{1}{p} + \frac{r}{n}}\,. \tag{4.14}$$

We observe that if condition $h_1)$ holds, then

$$\frac{k}{r} \leqslant a \leqslant 1 \qquad \text{and} \quad a = 1 \Longleftrightarrow \frac{1}{q} = \frac{1}{p} - \frac{r-k}{n} \tag{4.15}$$

THEOREM 4.4. *Suppose that conditions $h_o)$ and $h_1)$ hold. Then there exists $c \in \mathbb{R}_+$ such that for any function u for which $\partial^r u \in L^p_s(\Omega)$, $u \in L^{p_o}_{\alpha_o - r}(\Omega)$, we have*

$$\|u\|_{W^{k,q}_{\alpha-(r-k)}(\Omega)} \leqslant c\left(\|\partial^r u\|^a_{L^p_s(\Omega)} \cdot \|u\|^{1-a}_{L^{p_o}_{\alpha_o - r}(\Omega)} + \|u\|_{L^{p_o}_{\alpha_o - r}(\Omega)}\right), \tag{4.16}$$

where c depends on θ, n, ρ, r, k, p_o, p and q.

Proof. Fixed $x \in \Omega$, consider the function

$$\Psi^* \;:\; y \in \Omega \longmapsto x + \frac{y - x}{\tau(x)} \tag{4.17}$$

and put

$$\Omega^*(x) = \Psi^*\big(\Omega(x)\big)\,. \tag{4.18}$$

Obviously by condition $h_o)$ and (4.11) we deduce that $\Omega^*(x)$ is a bounded set with the cone property. Let now $i \in \mathbb{N}_o$ with $i \leqslant k$. Put

$$a_i = \frac{\frac{1}{p_o} - \frac{1}{q} + \frac{i}{n}}{\frac{1}{p_o} - \frac{1}{p} + \frac{r}{n}}\,, \tag{4.19}$$

and observe that $a_i \leqslant a \ \ \forall\, i \leqslant k$, where a is defined by (4.14).

From well known results (see, for instance, [7], [5], [8]), it follows that for any $i \leqslant k$ there exists $c_i \in \mathbb{R}_+$, depending on $\theta,\ \tau,\ r,\ i,\ p_o,\ p$, such that

$$\|\partial^i u\|_{L^q(\Omega^*(x))} \leqslant c_i \left(\|\partial^r u\|^{a_i}_{L^p(\Omega^*(x))} \cdot \|u\|^{1-a_i}_{L^{p_o}(\Omega^*(x))} + \|u\|_{L^{p_o}(\Omega^*(x))} \right). \tag{4.20}$$

It is know that if A and B are positive real numbers and if $0 < \beta_1 \leqslant \beta_2 < 1$, then

$$A^{\beta_1} \cdot B^{1-\beta_1} \leqslant A^{\beta_2} \cdot B^{1-\beta_2} + B\,. \tag{4.21}$$

So, by (4.20) we deduce that there exists $c_1 \in \mathbb{R}_+$, depending on $\theta,\ \tau,\ r,\ k,\ p_o$ and p, such that for each $i \leqslant k$

$$\|\partial^i u\|_{L^q(\Omega^*(x))} \leqslant c_1 \left(\|\partial^r u\|^{a}_{L^p(\Omega^*(x))} \cdot \|u\|^{1-a}_{L^{p_o}(\Omega^*(x))} + \|u\|_{L^{p_o}(\Omega^*(x))} \right). \tag{4.22}$$

From (4.22) and (4.11) we obtain

$$\tau^{i-\frac{n}{q}}(x)\,\|\partial^i u\|_{L^q(\Omega(x))} \leqslant c_1 \Big(\tau^{-\frac{n}{p_o}}(x)\,\|u\|_{L^{p_o}(\Omega(x))} + \tau^{(r-\frac{n}{p})a-\frac{n}{p_o}(1-a)}(x)\,\|\partial^r u\|^{a}_{L^p(\Omega(x))} \cdot \|u\|^{1-a}_{L^{p_o}(\Omega(x))} \Big) \quad \forall\, i \leqslant k\,. \tag{4.23}$$

With easy computations, from (4.23) we obtain for any $i \leqslant k$

$$\tau^{(\alpha-r+i)q-n}(x)\,\|\partial^i u\|^q_{L^q(\Omega(x))} \leqslant c_2 \Big(\tau^{(\alpha_o-r)q-n}(x)\,\|u\|^q_{L^{p_o}(\Omega(x))} + \big(\tau^{sq-n}(x)\,\|\partial^r u\|^q_{L^p(\Omega(x))}\big)^a \cdot \big(\tau^{(\alpha_o-r)q-n}(x)\,\|u\|^q_{L^{p_o}(\Omega(x))}\big)^{1-a} \Big)\,, \tag{4.24}$$

where $c_2 \in \mathbb{R}_+$ depends on the same parameters as c_1 and on q. So by (4.24), (4.11) and Hölder inequality we obtain

$$\int_\Omega \rho^{(\alpha-r+i)q}(x)\,\tau^{-n}(x)\,\|\partial^i u\|^q_{L^q(\Omega(x))} \leqslant c_3 \Bigg(\Big(\int_\Omega \rho^{sq}(x)\,\tau^{-n}(x)\,\|\partial^r u\|^q_{L^p(\Omega(x))}\,dx \Big)^a \cdot \Big(\int_\Omega \rho^{(\alpha_o-r)q}(x)\,\tau^{-n}(x)\,\|u\|^q_{L^{p_o}(\Omega(x))}\,dx \Big)^{1-a} + \int_\Omega \rho^{(\alpha_o-r)q}(x)\tau^{-n}(x)\,\|u\|^q_{L^{p_o}(\Omega(x))}\,dx \Bigg)\,, \tag{4.25}$$

where $c_3 \in \mathbb{R}_+$ depends on $\theta,\ \rho,\ r,\ k,\ p_o,\ p$ and q.

Observing that ρ verifies conditions (3.21), by Lemma 3.4 and (4.25) we have

$$\|\partial^i u\|^q_{L^q_{\alpha-r+i}(\Omega)} \leqslant c_4 \left(\|\partial^r u\|^{aq}_{L^p_s(\Omega)} \cdot \|u\|^{(1-a)q}_{L^{p_o}_{\alpha_o-r}(\Omega)} + \|u\|^q_{L^{p_o}_{\alpha_o-r}(\Omega)} \right), \tag{4.26}$$

where $c_4 \in \mathbb{R}_+$ depends on the same parameters as c_3 and on n. Finally, (4.26) proves the result. □

Now we can obtain some applications of Theorem 4.4. First suppose that the weight $\rho \in \mathscr{A}(\Omega)$ is a constant function. In this case the condition $h_o)$ means that the open set Ω has the cone property.

COROLLARY 4.5. *Let Ω be an open set with the cone property. If condition $h_1)$ holds, then there exists $c \in \mathbb{R}_+$, depending on Ω, n, r, k, p_o, p and q, such that for any function u for which $\partial^r u \in L^p(\Omega)$, $u \in L^{p_o}(\Omega)$, we have*

$$\|u\|_{W^{k,q}(\Omega)} \leqslant c \left(\|\partial^r u\|^a_{L^p(\Omega)} \cdot \|u\|^{1-a}_{L^{p_o}(\Omega)} + \|u\|_{L^{p_o}(\Omega)} \right), \tag{4.27}$$

where a is defined by (4.14).

Proof. The statement easily follows from Remark 2.1 and Theorem 4.4. □

We observe that if the weight $\rho \in \mathscr{A}(\Omega)$ is an unbounded function, the condition $h_o)$ may not be verified even if Ω is a regular open set. However it is possible to prove an interpolation inequality even when the condition $h_o)$ is not verified.

COROLLARY 4.6. *Let Ω be an open set with the cone property and $\rho \in \mathscr{A}(\Omega)$ such that $\inf_\Omega \rho > 0$ and $\sup_\Omega \rho = +\infty$. If condition $h_1)$ holds, then there exists $c \in \mathbb{R}_+$, depending on $\Omega, n, \rho, r, k, p_o, p$ and q, such that for any function u for which $\partial^r u \in L^p_{-s}(\Omega)$, $u \in L^{p_o}_{-\alpha_o+r}(\Omega)$ and for any $i \leqslant k$ $(i \in \mathbb{N}_0)$, we have*

$$\|\partial^i u\|_{L^q_{-\alpha+r-i}(\Omega)} \leqslant c \left(\|\partial^r u\|^a_{L^p_{-s}(\Omega)} \cdot \|u\|^{1-a}_{L^{p_o}_{-\alpha_o+r}(\Omega)} + \|u\|_{L^{p_o}_{-\alpha_o+r}(\Omega)} \right), \tag{4.28}$$

where a is defined by (4.14).

Proof. Let h be the height of characteristic cone of Ω. Let $\inf_\Omega \rho = c$ and fix $\lambda \in \mathbb{R}_+$ such that $\lambda \geqslant \max\{\frac{1}{h\,c}, \frac{1}{c^2}\}$. Put $\tilde{\rho}(x) = \frac{1}{\lambda \rho(x)}$ for any $x \in \Omega$. Using (2.2), it is easily to prove that $\tilde{\rho} \in \mathscr{A}(\Omega)$. So, since $\sup_\Omega \tilde{\rho} \leqslant h$, we deduce that condition $h_o)$ is verified with $\tilde{\rho}$ instead of ρ. Therefore from Theorem 4.4 it follows that there exists $c_1 \in \mathbb{R}_+$ such that for any $i \leqslant k$

$$\|\tilde{\rho}^{\,\alpha-r+i}\, \partial^i u\|_{L^q(\Omega)} \leqslant c_1 \left(\|\tilde{\rho}^{\,\alpha_o-r} u\|_{L^{p_o}(\Omega)} + \|\tilde{\rho}^{\,s}\, \partial^r u\|^a_{L^p(\Omega)} \cdot \|\tilde{\rho}^{\,\alpha_o-r} u\|^{1-a}_{L^{p_o}(\Omega)} \right), \tag{4.29}$$

where c_1 depends on $\Omega, n, \rho, r, k, p_o, p$ and q. The statement easily follows from (4.29). □

In section 5 we will give an example of an open set Ω and a weight function ρ for which it is possible to apply such result.

Now we consider further classes of weight functions and weight spaces and we prove an interpolation inequality also in this case.

Fix $d \in \mathbb{R}_+$. We denote by $\mathscr{G}(\Omega)$ the class of measurable functions $\rho : \Omega \to \mathbb{R}_+$ such that

$$\sup_{\substack{x,y\in\Omega \\ |x-y|<d}} \frac{\rho(x)}{\rho(y)} < +\infty\,, \tag{4.30}$$

(see section 5 for examples of functions belonging to this class).

It is easy to show that $\rho \in \mathscr{G}(\Omega)$ if and only if there exists $\nu \in \mathbb{R}_+$ independent of x and y such that

$$\nu^{-1}\,\rho(y) \leqslant \rho(x) \leqslant \nu\,\rho(y) \qquad \forall\, y \in \Omega\,, \quad \forall\, x \in \Omega \cap B(y,d)\,. \tag{4.31}$$

Moreover, if $r \in \mathbb{N}$, $1 \leqslant p \leqslant +\infty$, $s \in \mathbb{R}$ and $\rho \in \mathscr{G}(\Omega)$, we denote by $U^{r,p}_s(\Omega)$ the space of distributions u on Ω such that $\rho^s\,\partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \leqslant r$, equipped with the norm

$$\|u\|_{U^{r,p}_s(\Omega)} = \sum_{|\alpha|\leqslant r} \left\|\partial^\alpha u\right\|_{L^p_s(\Omega)}\,. \tag{4.32}$$

Suppose that Ω has the cone property with characteristic cone having opening θ and height h and fix $\rho \in \mathscr{G}(\Omega)$. Put $d_o = \min\{d,\, h\}$ and consider the function $\tau(x) = d_o,\ \forall\, x \in \Omega$. Obviously the weight function ρ verifies (3.21).

THEOREM 4.7. *Let Ω be an open set with the cone property and fix $\rho \in \mathscr{G}(\Omega)$. If condition $h_1)$ holds, then there exists $c \in \mathbb{R}_+$, depending on $\Omega, \rho, d_o, n, r, k, p_o, p$ and q, such that for any function u for which $\rho^s\,\partial^r u \in L^p(\Omega)$, $\rho^s\,u \in L^{p_o}(\Omega)$, we have*

$$\|u\|_{U^{k,q}_s(\Omega)} \leqslant c\left(\|\partial^r u\|^a_{L^p_s(\Omega)} \cdot \|u\|^{1-a}_{L^{p_o}_s(\Omega)} + \|u\|_{L^{p_o}_s(\Omega)}\right), \tag{4.33}$$

where a is defined by (4.14).

Proof. Fix $x \in \Omega$ and $i \in \mathbb{N}_o$ with $i \leqslant k$. Proceeding as in Theorem 4.4, we deduce that there exists $c_1 \in \mathbb{R}_+$, depending on $\theta, d_o, r, k, p_o, p, q$, such that for each $i \leqslant k$

$$\|\partial^i u\|^q_{L^q(\Omega(x))} \leqslant c_1\left(\|\partial^r u\|^{aq}_{L^p(\Omega(x))} \cdot \|u\|^{(1-a)q}_{L^{p_o}(\Omega(x))} + \|u\|^q_{L^{p_o}(\Omega(x))}\right). \tag{4.34}$$

So by Hölder inequality we obtain

$$\int_\Omega \rho^{sq}(x)\,\|\partial^i u\|^q_{L^q(\Omega(x))} \leqslant c_1\Big(\int_\Omega \rho^{sq}(x)\,\|u\|^q_{L^{p_o}(\Omega(x))}\,dx$$
$$+\Big(\int_\Omega \rho^{sq}(x)\,\|\partial^r u\|^q_{L^p(\Omega(x))}\,dx\Big)^a \cdot \Big(\int_\Omega \rho^{sq}(x)\,\|u\|^q_{L^{p_o}(\Omega(x))}\,dx\Big)^{1-a}\Big). \tag{4.35}$$

The result follows from (4.35) and Lemma 3.4. □

5. Appendix

We have proved the interpolation inequality in Theorem 4.4 under the assumption that the open set Ω verifies the condition $h_o)$ (see section 4). Such condition is, obviously, weaker than condition $H)$ of [9] (see Introduction). Our purpose is to analyze some examples of open sets Ω (also in the case of sufficient regularity of Ω) for which condition $H)$ is not verified but condition $h_o)$ is verified.

In the sequel, for the corresponding definitions of certain regularity properties of Ω we will refer to [1].

Let $\rho \in \mathscr{A}(\Omega)$. First we observe that condition $H)$ is verified in an arbitrary open set Ω if $S_\rho = \partial\Omega$. In fact it is known that if $S_\rho \neq \emptyset$, then $\rho(x) \leqslant \text{dist}(x, S_\rho) \quad \forall x \in \Omega$.

If $S_\rho \neq \partial\Omega$, some hypotheses of regularity are necessary in order to satisfy condition $H)$. In particular the condition $H)$ is verified if:

i) Ω is a bounded set with locally Lipschitz boundary;

ii) Ω is an unbounded set with the strong local Lipschitz property and $\rho \in L^\infty(\Omega)$.

The cone property only for the open set Ω is not sufficient in order to have condition $H)$. For example if

$$\begin{gathered} \Omega = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; x \in]0,1[\cup]1,2[\quad 0 < y < 1 \right\} \\ \rho(x,y) = \tfrac{x}{2}, \end{gathered} \tag{5.1}$$

then condition $H)$ is not verified. We note that if Ω and ρ are defined by (5.1), then condition $h_o)$ is satisfied.

The condition $h_o)$ is verified, for example, if the open set Ω satisfies the following assumption:

(iii) there exists an open set Ω^* in $\mathbb{R}^n$ with the cone property such that

$$\Omega \subset \Omega^* \qquad \partial\Omega \setminus S_\rho \subset \partial\Omega^* \tag{5.2}$$

and $\rho \in L^\infty(\Omega)$

(see [4]).

The required regularity property on Ω to satisfy condition $h_o)$ can be still weaker than iii). If we consider

$$\begin{gathered} \Omega = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; x \in]0,1[\cup]1,2[\quad y > \tfrac{1}{|x-1|} \right\} \\ \rho(x,y) = \tfrac{x}{2}, \end{gathered} \tag{5.3}$$

then we have an example of an open set Ω with the segment property and a weight function $\rho \in \mathscr{A}(\Omega)$ for which condition $h_o)$ is verified but condition $H)$ is not satisfied.

In section 4 we have observed that if $\rho \in \mathscr{A}(\Omega)$ is an unbounded weight function, then the condition $h_o)$ cannot be verified also when Ω is a regular open set. However we have proved an interpolation inequality in the case of open set Ω with the cone property and where the weight function ρ satisfies the condition

$$\inf_\Omega \rho > 0, \qquad \sup_\Omega \rho = +\infty. \tag{5.4}$$

Now we give an example of such result. Consider the set Ω and the weight function ρ defined by

$$\Omega =]2, +\infty[, \qquad \rho(x) = \frac{x}{2} \quad x \in \Omega\,. \tag{5.5}$$

It is easy to prove that ρ verifies (5.4). Let

$$u(x) = \frac{\cos^2 x}{x} \qquad (x \in \Omega)\,. \tag{5.6}$$

Then $u \in L^2(\Omega)$ and $u'' \in L^2_{-2}(\Omega)$. So, from Corollary 4.6, for any $q \in]1, +\infty[$ there exist $c \in \mathbb{R}_+$ and $a \in]0, 1[$ such that for $i \leqslant 1$ ($i \in \mathbb{N}_0$)

$$\|\rho^{-2+n(\frac{1}{q}-\frac{1}{2})-i}\, u^{(i)}\|_{L^q(\Omega)} \leqslant c\left(\|u''\|^a_{L^2_{-2}(\Omega)} \cdot \|u\|^{1-a}_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right). \tag{5.7}$$

As the last application of our results, we have considered also weight functions which do not belong to the class $\mathscr{A}(\Omega)$. In fact we have studied weight functions $\rho \in \mathscr{G}(\Omega)$ (see section 4). It is easy to verify that, for fixed $d \in \mathbb{R}_+$ and $\Omega \subset \mathbb{R}^n$, the functions

$$\rho(x) = e^{t\,|x|}, \qquad \rho(x) = \left(1 + |x|^2\right)^t, \qquad (x \in \Omega) \quad (t \in \mathbb{R})\,, \tag{5.8}$$

belong to the class $\mathscr{G}(\Omega)$.

Observe that, if $\rho \in \mathscr{A}(\Omega)$ and $\inf_\Omega \rho > 0$, then $\rho \in \mathscr{G}(\Omega)$ with $d \leqslant \inf_\Omega \rho$. So for the open set Ω and the weight function ρ defined by (5.5), and for the function u defined by (5.6), for any $q \in]1, +\infty[$ we have also the bound

$$\|u\|_{U^{1,q}_{-2}(\Omega)} \leqslant c\left(\|u''\|^a_{L^2_{-2}(\Omega)} \cdot \|u\|^{1-a}_{L^2_{-2}(\Omega)} + \|u\|_{L^2_{-2}(\Omega)}\right), \tag{5.9}$$

where the Sobolev space $U^{1,q}_{-2}(\Omega)$ is defined in section 4.

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