Journal of Mathematical Inequalities
Volume 2, Number 3 (2008), 377–382

SHAFER–FINK TYPE INEQUALITIES FOR THE ELLIPTIC FUNCTION $\mathbf{sn(u|k)}$

A. McD. Mercer

(*communicated by J. Pečarić*)

Abstract. The inequalities of Shafer and Fink, namely,

$$\frac{3x}{2+\sqrt{1-x^2}} \leqslant \sin^{-1}(x) \leqslant \frac{\pi x}{2+\sqrt{1-x^2}}, \qquad x \in [0,1)$$

are generalized to similar inequalities for the elliptic function $\mathrm{sn}(u|k)$.

1. Introduction

We start by presenting two very simple proofs of the Shafer and Fink inequalities. Consider the functions

$$f(\theta) = \pi \sin\theta - 2\theta - \theta\cos\theta, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

and

$$g(\theta) = 3\sin\theta - 2\theta - \theta\cos\theta, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

Each of these is concave over its interval and the following boundary conditions are satisfied $f(0) = f(\frac{\pi}{2}) = 0$ and $g(0) = g'(0) = 0$. Hence

$$g(\theta) \leqslant 0 \text{ and } f(\theta) \geqslant 0, \qquad \theta \in [0, \frac{\pi}{2}]$$

That is

$$\frac{3\sin\theta}{2+\cos\theta} \leqslant \theta \leqslant \frac{\pi\sin\theta}{2+\cos\theta}, \qquad \theta \in [0, \frac{\pi}{2}]$$

or, on putting $x = \sin\theta$

$$\frac{3x}{2+\sqrt{1-x^2}} \leqslant \sin^{-1} x \leqslant \frac{\pi x}{2+\sqrt{1-x^2}}, \qquad x \in [0,1] \tag{1}$$

These are, respectively, the Shafer and Fink inequalities.

The origins of these are to be found in [1] and [2] and a large bibliography concerning them and their extensions appears in [3].

Mathematics subject classification (2000): 26D05.
Key words and phrases: Shafer-Fink inequalities, Jacobi elliptic function.

Paper JMI-02-33

Our purpose in this note is to generalize these results to the case in which $\sin\theta$ is replaced by the Jacobi elliptic function $\mathrm{sn}(u|k)$. In short, it is our purpose to prove the following inequalities:

THEOREM. *Let* $0 < k < 1$. *Then if*

$$K(k) = \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$$

we have

$$\frac{3x}{2+\sqrt{1-x^2}\sqrt{1-k^2x^2}} \leqslant \mathrm{sn}^{-1}(x|k) \leqslant \frac{2K(k)x}{2+\sqrt{1-x^2}\sqrt{1-k^2x^2}}, \quad x \in [0,1] \tag{2}$$

2. The Jacobi elliptic functions

A very succinct introduction to these functions, when the independent variable is real, can be found in [4]. And in [5] there is a comprehensive list of their properties. In this section we remind the reader of some of these facts.

(a) Definitions. With $0 < k < 1$ the three Jacobi elliptic functions $\mathrm{sn}(u|k)$, $\mathrm{cn}(u|k)$ and $\mathrm{dn}(u|k)$ are usually defined by integrals such as, for example,

$$u = \int_0^{\mathrm{sn}(u|k)} \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$$

Let us follow [4] and write

$$x = \mathrm{sn}(u|k), \quad y = \mathrm{cn}(u|k) \text{ and } z = \mathrm{dn}(u|k)$$

Then, whenever the independent variable u is restricted to the real field an equivalent definition of these functions is

$$\frac{dx}{du} = yz, \quad \frac{dy}{du} = -zx, \quad \frac{dz}{du} = -k^2xy$$

with

$$x(0) = 0, \quad y(0) = z(0) = 1$$

The parameter k is called the *elliptic modulus* and k', defined by

$$k' = \sqrt{1-k^2}$$

is the *complementary elliptic modulus.*

Another constant involved in these matters is the following ;

$$K(k) = \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$$

K is called *the complete elliptic integral of the first kind.*

(b) Properties. We have, using the notations of [4] :

$$x(0) = 0, \quad x(K) = 1$$
$$y(0) = 1, \quad y(K) = 0$$
$$z(0) = 1, \quad z(K) = k' = \sqrt{1-k^2}$$

$$x^2 + y^2 = 1,$$
$$k^2x^2 + z^2 = 1,$$
$$k^2y^2 + k'^2 = z^2,$$
$$y^2 + k'^2x^2 = z^2$$

Visualization of these functions may be helped by mentioning that $x(u)$ increases from 0 to 1 and $y(u)$ decreases from 1 to 0 in $[0,K]$, their graphs generally resembling those of $\sin(u)$ and $\cos(u)$. The function z decreases from 1 to k' in $[0,K]$ with a minimum at K when

$$z = \sqrt{1-k^2} > 0$$

These properties have been proved in [2], for example. The constant k will be fixed in $(0,1)$ throughout but we note that as $k \to 0$

$$x \to \sin, \; y \to \cos, \; z \to 1 \text{ and } K \to \frac{\pi}{2}$$

and, as $k \to 1$

$$x \to \tanh, \quad y \to \text{sech}, \; z \to \text{sech and } K \to \infty$$

We shall not be concerned with behaviour outside $[0,K]$ but it may be mentioned here that $4K$ is a period of x and y while $2K$ is a period of z.

3. The proofs

In the proofs of the Lemma and of the Theorem which follow it is convenient to write

$$Q \equiv [z^2 + k^2y^2] \; - 4k^2x^2$$

We have:

LEMMA.

$$\textit{If } Q > 0, \textit{ then } \frac{x}{yz} \geqslant u, \quad u \in [0,K)$$

Proof. Consider

$$w(u) = x - uyz$$

Then

$$\begin{aligned} w'(u) &= yz - yz + u[xz^2 + k^2xy^2] \\ &= ux[z^2 + k^2y^2] \\ w''(u) &= x[z^2 + k^2y^2] + uyz[z^2 + k^2y^2] + ux[-k^2 2zxy - k^2 2yxz] \\ &= x[z^2 + k^2y^2] + uyz\{[z^2 + k^2y^2] - 4k^2x^2]\} \\ &= x[z^2 + k^2y^2] + uyzQ \end{aligned}$$

Since $Q > 0$ then $w''(u) > 0$. So $w(u)$ is convex and, since

$$w(0) = w'(0) = 0$$

we have

$$w(u) \geqslant 0, \quad u \in [0, K]$$

and so

$$\frac{x}{yz} \geqslant u, \quad u \in [0, K)$$

as was to be proved.

Proof of the Theorem (left side). Consider

$$f(u) = 3\,\mathrm{sn}(u|k) - 2u - u\,\mathrm{cn}(u|k)\,\mathrm{dn}(u|k)$$

or equivalently,

$$f(u) = 3x - 2u - uyz$$

Differentiating with respect to u, we get

$$f'(u) = 2yz - 2 + ux[z^2 + k^2y^2]$$

Then

$$\begin{aligned} f''(u) &= -2xz^2 - 2k^2xy^2 + x[z^2 + k^2y^2] + uyz[z^2 + k^2y^2] \\ &\quad + ux[-k^2 2zxy - k^2 2yxz] \\ &= -x[z^2 + k^2y^2] + uyz\{[z^2 + k^2y^2] - 4k^2x^2\} \\ &= -x[z^2 + k^2y^2] + uyzQ \end{aligned}$$

where, again,

$$Q \equiv [z^2 + k^2y^2] - 4k^2x^2$$

When $u = 0$, $Q = 1 + k^2 > 0$ and when $u = K$, $Q = 1 - 5k^2$, so that, depending on k, Q may take both signs.

When $Q < 0$ we see that $f''(u)$ is negative. And when $Q > 0$ we find that $f''(u)$ is negative again, by virtue of the Lemma and the fact that

$$z^2 + k^2y^2 > Q = [z^2 + k^2y^2] - 4k^2x^2$$

So $f(u)$ is concave. And since $f(0) = f'(0) = 0$ then

$$f(u) < 0, \quad u \in [0, K]$$

and so

$$u > \frac{3\,\mathrm{sn}(u|k)}{2 + \mathrm{cn}(u|k)\,\mathrm{dn}(u|k)}, \quad u \in [0, K].$$

Putting $x = \mathrm{sn}(u)$ this reads

$$\mathrm{sn}^{-1}(x|k) > \frac{3x}{2 + \sqrt{1 - x^2}\sqrt{1 - k^2x^2}}, \quad x \in [0, 1]$$

which is the left inequality of (2).

Proof of the Theorem (right side). The proof of this is very similar. We consider

$$g(u) = 2K\,\mathrm{sn}(u|k) - 2u - u\,\mathrm{cn}(u|k)\,\mathrm{dn}(u|k)$$

or equivalently,

$$g(u) = 2Kx - 2u - uyz$$

Differentiating with respect to u, we get

$$g'(u) = (2K-1)yz - 2 + ux[z^2 + k^2y^2]$$

and

$$g''(u) = -2(K-1)x[z^2 + k^2y^2] + uyzQ$$

Just as previously we see that if $Q < 0$ then $g''(u) < 0$. And if $Q > 0$ then $g''(u) < 0$ by virtue the fact that

$$z^2 + k^2y^2 > Q = [z^2 + k^2y^2] - 4k^2x^2$$

and the Lemma, which, in this case, gives

$$\frac{x}{yz} > \frac{u}{2K-2}$$

since $2K - 2 \geqslant \pi - 2 > 1$.

(Note that since $K(0) = \pi/2$ and $K = K(k)$ increases with k (see[2]), this inequality persists for $k \in (0,1)$). So $g(u)$ is concave. And since $g(0) = g(K) = 0$ we have

$$g(u) \geqslant 0, \quad u \in [0,K]$$

Hence

$$u < \frac{2K\,\mathrm{sn}(u|k)}{2 + \mathrm{cn}(u|k)\,\mathrm{dn}(u|k)}, \quad u \in [0,K].$$

Putting $x = \mathrm{sn}\,(u|k)$ this reads

$$\mathrm{sn}^{-1}(x|k) < \frac{2Kx}{2 + \sqrt{1-x^2}\sqrt{1-k^2x^2}}, \qquad x \in [0,1]$$

and this is the right side of (2).

So the proof of the Theorem is complete.

A final note. If we let $k \to 0$ in (2) we recover (1) and if we let $k \to 1$ (2) becomes

$$\tanh^{-1} x > \frac{3x}{3 - x^2}, \qquad x \in [0,1].$$

REFERENCES

[1] R. E. Shafer, *Problem E1867*, Amer. Math. Monthly, **74** (6) (1967), 726–727.
[2] A. M. Fink, *Two Inequalities*, Univ. Beograd. Pub. Elekt. Fak., **6** (1995), 48–49.
[3] Ling Zhu, *On Shafer-Fink-Type Inequalities*, J. Ineq. Appl. Vol. (2007), Article ID 67430.
[4] Meyer, K. R., *Jacobi elliptic functions from a dynamical systems point of view*, Amer. Math. Monthly, **108** (8) (0ctober 2001), 729–737.
[5] Wolfram MathWorld, *URL:* http://mathworld.wolfram.com/JacobiEllpticFunctions.html
[6] G. T. Whittaker, G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, Cambridge (1952).

(Received November 16, 2007)

A. McD. Mercer
The Dept. of Math. and Stats.
University of Guelph Ontario
Box 12, RR7, Belleville
Ontario K8N4Z8
Canada
e-mail: rex@mail2pisces.com