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MEDA INEQUALITY FOR REARRANGEMENTS OF THE B–CONVOLUTIONS AND SOME APPLICATIONS

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Abstract. In this paper we prove the Meda inequality for rearrangements of the convolution operator (B–convolution) associated with the Laplace-Bessel differential operator. By using the Meda inequality for rearrangements we obtain an O'Neil type inequality for the B–convolution. As applications of these results, we obtain necessary and sufficient conditions on the parameters for the boundedness of the fractional B-maximal operator and B–fractional integral operator with rough kernels, from the spaces $L_{p,\gamma}$ to $L_{q,\gamma}$ and from the spaces $L_{1,\gamma}$ to the weak spaces $WL_{q,\gamma}$.

1. Introduction

Let $K_\alpha \in WL_{n/(n-\alpha)}(\mathbb{R}^n)$, $0 < \alpha < n$, and $f \in L_p(\mathbb{R}^n)$, $1 < p < n/\alpha$. Then for the convolution $K_\alpha * f$, S. Meda [10] proved the following pointwise rearrangement estimate

$$(K_\alpha * f)^{**}(t) \leqslant C\left(\delta^\alpha f^{**}(t) + \delta^{\alpha-n/p}\|f\|_p\right), \quad \delta > 0, \tag{1}$$

and gave a new proof of the Hardy-Littlewood-Sobolev theorem for $K_\alpha * f$ by using this inequality.

The potential type integral operators associated with the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=k+1}^{n} \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$$

(see [1]–[6], [8, 9]), are playing an important role in harmonic analysis, theory of functions and partial differential equations.

In this paper we study the convolution (B-convolution), the fractional maximal function (fractional B-maximal function) and fractional integral (B-frac-tional integral) with rough kernels, associated with the Laplace-Bessel differential operator. We get the Meda inequality given in (1) for rearrangements of the B–convolution. By using the Meda inequality for rearrangements we obtain an O'Neil type inequality for the

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B–convolution. As applications, we obtain necessary and sufficient conditions on the parameters for the boundedness of the fractional B-maximal operator and B–fractional integral operator with rough kernels from the spaces $L_{p,\gamma}$ to $L_{q,\gamma}$ and from the spaces $L_{1,\gamma}$ to the weak spaces $WL_{q,\gamma}$.

Let $\mathbb{R}^n_{k,+} = \{x = (x_1,...,x_n) \in \mathbb{R}^n : x_1 > 0,\dots,x_k > 0,\ 1 \leqslant k \leqslant n\}$, and define

$$L_{p,\gamma} \equiv L_{p,\gamma}(\mathbb{R}^n_{k,+}) = \left\{ f : \|f\|_{p,\gamma} \equiv \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx \right)^{1/p} < \infty \right\},\ \ 1 \leqslant p < \infty$$

where $(x')^\gamma = x_1^{\gamma_1} \cdot \ldots \cdot x_k^{\gamma_k}$, $\gamma = (\gamma_1,...,\gamma_k)$ is a multi-index consisting of fixed positive numbers such that $|\gamma| = \gamma_1 + ... + \gamma_k$.

If $p = \infty$, we assume

$$L_{\infty,\gamma}(\mathbb{R}^n_{k,+}) = L_\infty(\mathbb{R}^n_{k,+}) = \{f : \|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n_{k,+}} |f(x)| < \infty\}.$$

Denote by T^y the shift operator (B–shift) acting according to the law

$$T^y f(x) = C_{k,\gamma} \int_0^\pi \dots \int_0^\pi f\left((x',y')_\alpha, x''-y''\right)\, d\nu(\alpha),$$

where

$$C_{k,\gamma} = \pi^{-k/2} \prod_{i=1}^k \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)},\ (x_i,y_i)_{\alpha_i} = \sqrt{x_i^2 - 2x_iy_i\cos\alpha_i + y_i^2},\ \ 1 \leqslant i \leqslant k,$$

$$(x',y')_\alpha = \left((x_1,y_1)_{\alpha_1},\dots,(x_k,y_k)_{\alpha_k}\right) \text{ and } d\nu(\alpha) = \prod_{i=1}^k \sin^{\gamma_i-1}\alpha_i\, d\alpha_i,\ \ 1 \leqslant k \leqslant n.$$

We remark that the B-shift is closely related to the Laplace-Bessel differential operator Δ_B. The shift operator T^y generates the corresponding convolution (B-convolution)

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y) T^y g(x) (y')^\gamma dy.$$

Let $\Omega \in L_{s,\gamma}(S^{n-1}_{k,+})$, $s \geqslant 1$, $S^{n-1}_{k,+} = \{x \in \mathbb{R}^n_{k,+} : |x| = 1\}$, and Ω be homogeneous of degree zero on $\mathbb{R}^n_{k,+}$, i.e., $\Omega(tx) = \Omega(x)$ for all $t > 0$, $x \in \mathbb{R}^n_{k,+}$, and let $0 < \alpha < Q$, $Q = n + |\gamma|$. We define the fractional B-maximal function with a rough kernel by

$$M_{\Omega,\alpha,\gamma} f(x) = \sup_{r>0} \frac{1}{r^{Q-\alpha}} \int_{B(0,r)} |\Omega(y)|\, T^y |f(x)|\, (y')^\gamma\, dy,$$

and the B–fractional integral with a rough kernel by

$$I_{\Omega,\alpha,\gamma} f(x) = \int_{\mathbb{R}^n_{k,+}} \frac{\Omega(y)}{|y|^{Q-\alpha}}\, T^y f(x)\, (y')^\gamma\, dy,$$

where $B(0,r) = \{x \in \mathbb{R}^n_{k,+} : |x| < r\}$. It is clear that, when $\Omega \equiv 1$, $M_{\Omega,\alpha,\gamma}$ and $I_{\Omega,\alpha,\gamma}$ are the usual fractional B–maximal operator $M_{\alpha,\gamma}$ ([4]) and the B–Riesz potential $I_{\alpha,\gamma}$ ([1, 3, 9]), respectively.

The paper is organized as follows. In Section 2, we give some lemmas needed to facilitate the proofs of our theorems. In Section 3, we show that the Meda inequality for rearrangements of the B-convolution holds. In Section 4, we prove an O'Neil type inequality for B-convolutions. In Section 5, we give some applications of the results above. We show that the conditions on the parameters ensuring the boundedness cannot be weakened for the fractional B-maximal operator and B–fractional integral operator with rough kernels from the spaces $L_{p,\gamma}$ to $L_{q,\gamma}$, and from the spaces $L_{1,\gamma}$ to the weak spaces $WL_{q,\gamma}$.

2. Some auxiliary lemmas

In this section we formulate some lemmas that will be needed later. We establish a relation between shift operator $T^y f$ and γ-rearrangement of f.

For the B-shift operator the following two lemmas hold.

LEMMA 1. *1. Let* $1 \leqslant p \leqslant \infty$, $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$, *then for all* $y \in \mathbb{R}^n_{k,+}$

$$\|T^y f(\cdot)\|_{p,\gamma} \leqslant \|f\|_{p,\gamma}. \tag{2}$$

2. Let $1 \leqslant p,\ r \leqslant q \leqslant \infty$, $1/p - 1/q = 1/r'$, $(r' = r/(r-1))$, $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$, $g \in L_{r,\gamma}(\mathbb{R}^n_{k,+})$. *Then* $f \otimes g \in L_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|f \otimes g\|_{q,\gamma} \leqslant \|f\|_{p,\gamma}\, \|g\|_{r,\gamma}.$$

LEMMA 2. *For any measurable set* $A = (A', A'') \subset \mathbb{R}^n_{k,+}$, $A' = A_1 \times \ldots \times A_k \subset (0,\infty)^k$, $A'' \subset \mathbb{R}^{n-k}$ *and for any* $y \in \mathbb{R}^n_{k,+}$ *the following equality holds*

$$\int_A T^y g(x)(x')^\gamma dx = C_{k,\gamma} \int_{(y,0)+\overline{A}} g\left(\sqrt{z_1^2 + \overline{z}_1^2}, \cdots, \sqrt{z_k^2 + \overline{z}_k^2}, z''\right) d\mu(z, \overline{z'}), \tag{3}$$

where $(x,0) = (x, \underbrace{0,\ldots,0}_{k-times})$, $\overline{z'} = (\overline{z}_1, \ldots, \overline{z}_k)$, $d\mu(z,\overline{z'}) = \overline{z'}^{\gamma-1} dz d\overline{z'}$, $d\overline{z'} = d\overline{z}_1 \cdots d\overline{z}_k$, $\overline{z'}^{\gamma-1} = \overline{z}_1^{\gamma_1 - 1} \cdots \overline{z}_k^{\gamma_k - 1}$, $(z, \overline{z'}) \in \mathbb{R}^n_{k,+} \times (0,\infty)^k$, $m_i = \sup A_i$, $i = 1, \ldots, k$, $\overline{A} = ((-m_1, m_1) \times [0, m_1) \times \ldots. \times (-m_k, m_k) \times [0, m_k)) \times A''$.

The proof of Lemma 2 is straightforward after applying the following substitutions

$$z'' = x'', z_i = x_i \cos\alpha_i,\ \ \overline{z_i} = x_i \sin\alpha_i,\ \ 0 \leqslant \alpha_i < \pi,$$

$$i = 1, \ldots, k,\ \overline{z'} = (\overline{z}_1, \ldots, \overline{z}_k),\ (z, \overline{z'}) \in \mathbb{R}^n_{k,+} \times (0,\infty)^k.$$

Let $f : \mathbb{R}^n_{k,+} \to \mathbb{R}$ be a measurable function and for any measurable set E, $|E|_\gamma = \int_E (x')^\gamma dx$. We define γ-rearrangement of f in decreasing order by

$$f^*(t) = \inf\{s > 0 : f_*(s) \leqslant t\},\ \ \forall t \in [0, \infty),$$

where $f_*(t)$ denotes the γ-distribution function of f given by

$$f_*(t) = \left|\{x \in \mathbb{R}^n_{k,+} : |f(x)| > t\}\right|_\gamma .$$

We note the following properties of γ-rearrangement of functions (see [5, 7, 12]):

1) if $0 < p < \infty$, then

$$\int_{\mathbb{R}^n_+} |f(x)|^p (x')^\gamma dx = \int_0^\infty f^*(t)^p \, dt;$$

2) for any $t > 0$,

$$\sup_{|E|_\gamma = t} \int_E |f(x)|(x')^\gamma dx = \int_0^t f^*(s)\, ds;$$

3)

$$\int_{\mathbb{R}^n_+} |f(x)g(x)|(x')^\gamma dx \leqslant \int_0^\infty f^*(t) g^*(t)\, dt.$$

The function f^{**} on $(0,\infty)$ is defined by $f^{**}(t) = \frac{1}{t}\int\limits_0^t f^*(s)\, ds, \quad t > 0.$

If $1 < p < \infty$ the following inequality is valid

$$\|f^{**}\|_{L_p(0,\infty)} \leqslant p' \, \|f^*\|_{L_p(0,\infty)},$$

where $p' = p/(p-1)$.

We denote by $WL_{p,\gamma}(\mathbb{R}^n_{k,+})$ the weak $L_{p,\gamma}$ space of all measurable functions f with finite norm

$$\|f\|_{WL_{p,\gamma}} = \sup_{t>0} t f_*(t)^{1/p}, \quad 1 \leqslant p < \infty.$$

LEMMA 3. *For any measurable set $A \subset \mathbb{R}^n_{k,+}$ and for any $y \in \mathbb{R}^n_{k,+}$, the following equality holds*

$$\sup_{|A|_\gamma = t} \int_A T^y |f(x)|(x')^\gamma dx = C_{k,\gamma} \int_0^t f^*(s) ds.$$

Proof. From Lemma 2 we have

$$\int_A T^y |f(x)|(x')^\gamma dx = C_{k,\gamma} \int_{(y,0)+\overline{A}} \left|\overline{f}(z,\overline{z'})\right| d\mu(z,\overline{z'}), \tag{4}$$

where $\overline{f}(z,\overline{z'}) = f\left(\sqrt{z_1^2 + \overline{z}_1^2}, \cdots, \sqrt{z_k^2 + \overline{z}_k^2}, z''\right)$. For the function $\overline{f}(z,\overline{z'})$ the analogous equality (2) is also valid (see, for example [7])

$$\sup_{\mu(\overline{A}) = t} \int_{\overline{A}} |\overline{f}(z,\overline{z'}) d\mu(z,\overline{z'}) = \int_0^t \left(\overline{f}\right)^*_\mu (s) ds, \tag{5}$$

where $\left(\overline{f}\right)^*_\mu(s) = \inf\left\{t > 0 \;:\; \mu\left(\left\{(z,\overline{z'}) \;:\; \left|\overline{f}(z,\overline{z'})\right| > t\right\}\right) \leqslant s\right\}$.

Note that $\mu\left((y,0)+\overline{A})\right) = |A|_\gamma$ and $\left(\overline{f}\right)^*_\mu(s) = f^*(s)$.

From (4) and (5) we get

$$\begin{aligned}\sup_{|A|_\gamma=t}\int_A T^y|f(x)|(x')^\gamma dx &= C_{k,\gamma}\sup_{\mu(\overline{A})=t}\int_{(y,0)+\overline{A}}|\overline{f}(z,\overline{z'})|d\mu(z,\overline{z'})\\ &= C_{k,\gamma}\int_0^t\left(\overline{f}\right)^*_\mu(s)ds\\ &= \int_0^t f^*(s)ds.\end{aligned}$$

Thus Lemma 3 is proved. □

3. Meda inequality for rearrangements of B-convolutions

In this section we prove the Meda inequality for rearrangements of the B-convolution.

THEOREM 1. *Let $g \in WL_{r,\gamma}(\mathbb{R}^n_{k,+})$, $1 < r < \infty$, $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$, $1 \leqslant p < r'$. Then for any $\delta > 0$*

$$(g\otimes f)^{**}(t) \leqslant C_1\delta^{Q/r'}f^{**}(t) + C_2\,\delta^{Q/r'-Q/p}\|f\|_{p,\gamma}, \tag{6}$$

where $C_1 = B_1\|g\|^r_{WL_{r,\gamma}}$, $B_1 = 2^r(2^{r-1}-1)^{-1}$ and $C_2 = B_2\|g\|^{r/p'}_{WL_{r,\gamma}}$, $B_2 = 2\left(2^{p'-r}-1\right)^{-1/p'}$ for $1 < p < r'$, and $B_2 = 1$ for $p = 1$.

Proof. Suppose $F_\delta = \{y \in \mathbb{R}^n_{k,+} : |g(y)| \geqslant \delta^{-Q/r}\}$, then

$$|(g\otimes f)(x)| \leqslant \left(\int_{F_\delta} + \int_{\mathbb{R}^n_{k,+}\setminus F_\delta}\right)T^y|f(x)|\;|g(y)|\;(y')^\gamma dy = D_1(x) + D_2(x).$$

Suppose that $F_\delta = \bigcup_{j=1}^\infty F'_{\delta,j}$, where

$$F'_{\delta,j} = \{y \in \mathbb{R}^n_{k,+} \;:\; 2^{j-1}\delta^{-Q/r} \leqslant |g(y)| < 2^j\delta^{-Q/r}\}.$$

Then taking into account Lemma 3, we get

$$\begin{aligned}\frac{1}{|E|_\gamma}\int_E D_1(x)(x')^\gamma dx &= \frac{1}{|E|_\gamma}\int_E\left(\int_{F_\delta}|g(y)|T^y|f(x)|(y')^\gamma dy\right)(x')^\gamma dx\\ &\leqslant \sum_{j=1}^\infty 2^j\delta^{-Q/r}\int_{F'_{\delta,j}}\left(\frac{1}{|E|_\gamma}\int_E T^y|f(x)|(x')^\gamma dx\right)(y')^\gamma dy\\ &\leqslant \delta^{-Q/r}f^{**}(t)\sum_{j=1}^\infty 2^j\int_{F'_{\delta,j}}(y')^\gamma dy\end{aligned}$$

$$\leqslant 2^r \delta^{-Q/r} f^{**}(t)\ \|g\|^r_{WL_{r,\gamma}}\ \sum_{j=1}^{\infty} 2^j\ (2^j \delta^{-Q/r})^{-r}$$
$$= C_1 \delta^{Q/r'} f^{**}(t),$$

where $C_1 = 2^r(2^{r-1}-1)^{-1}\ \|g\|^r_{WL_{r,\gamma}}$.

Thus

$$\frac{1}{|E|_\gamma}\int_E D_1(x)(x')^\gamma dx \leqslant C_1 \delta^{Q/r'} f^{**}(t).$$

Let $p = 1$. Then from the inequality (2) we have

$$|D_2(x)| \leqslant \|T^x f(\cdot)\|_{1,\gamma} \sup_{\mathbb{R}^n_{k,+}\setminus F_\delta} |g(y)| \leqslant \|f\|_{1,\gamma}\delta^{-Q/r}.$$

Now let $1 < p < r'$. By using Hölder inequality and the inequality (2) we get

$$|D_2(x)| \leqslant \|T^x f(\cdot)\|_{p,\gamma}\left(\int_{\mathbb{R}^n_{k,+}\setminus F_\delta} |g(y)|^{p'} (y')^\gamma dy\right)^{1/p'}$$
$$\leqslant \|f\|_{p,\gamma}\left(\int_{\mathbb{R}^n_{k,+}\setminus F_\delta} |g(y)|^{p'} (y')^\gamma dy\right)^{\frac{1}{p'}}.$$

Since $g \in WL_{r,\gamma}(\mathbb{R}^n_{k,+})$ and $\mathbb{R}^n_{k,+}\setminus F_\delta = \bigcup_{j=1}^{\infty} B_{\delta,\gamma}$, where

$$B_{\delta,j} = \{y \in \mathbb{R}^n_{k,+} : 2^{-j}\delta^{-Q/r} \leqslant |g(x)| < 2^{-j+1}\delta^{-Q/r}\},$$

then

$$\begin{aligned}
\int_{\mathbb{R}^n_{k,+}\setminus F_\delta} |g(y)|^{p'} (y')^\gamma dy &= \sum_{j=1}^{\infty}\int_{B_{\delta,j}} |g(y)|^{p'} (y')^\gamma dy \\
&\leqslant \sum_{j=1}^{\infty}(2^{-j+1}\delta^{-Q/r})^{p'} \int_{\{y\in\mathbb{R}^n_{k,+}:\ |g(y)|\geqslant 2^{-j}\delta^{-Q/r}\}} (y')^\gamma dy \\
&\leqslant 2^{p'}\delta^{-Qp'/r}\|g\|^r_{WL_{r,\gamma}}\sum_{j=1}^{\infty} 2^{-jp'}(2^{-j}\delta^{-Q/r})^{-r} \\
&= 2^{p'}\ \delta^{Q-Qp'/r}\ \|g\|^r_{WL_{r,\gamma}}\ \sum_{j=1}^{\infty} 2^{-j(p'-r)} \\
&= C_2^{p'}\delta^{Q-Q/rp'},
\end{aligned}$$

where $C_2 = 2\left(2^{p'-r}-1\right)^{-1/p'}\ \|g\|^{r/p'}_{WL_{r,\gamma}}$.

Hence

$$|D_2(x)| \leqslant C_2\|f\|_{p,\gamma}\ \delta^{Q/r'-Q/p}.$$

Thus

$$\frac{1}{|E|_\gamma}\int_E |(g\otimes f)(x)|(x')^\gamma dx \leqslant C_1\delta^{Q/r'}f^{**}(t)+C_2\delta^{Q/r'-Q/p}\|f\|_{p,\gamma}.$$

Therefore we get (6) and Theorem 1 is proved. □

COROLLARY 1. *Let Ω be homogeneous of degree zero on $\mathbb{R}^n_{k,+}$ and $0<\alpha<Q$, $\Omega\in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$. Then for any $\delta>0$ the following inequality holds*

$$\left(I_{\Omega,\alpha,\gamma}f\right)^{**}(t)\leqslant C_3\delta^{\alpha}f^{**}(t)+C_4\delta^{\alpha-Q/p}\|f\|_{p,\gamma},$$

where $C_3=B_3(A/Q)$, $C_4=B_4(A/Q)^{1/p'}$, $B_3=2^{Q/(Q-\alpha)}(2^{\alpha/(Q-\alpha)}-1)^{-1}$, $B_4=2\left(2^{p'-Q/(Q-\alpha)}-1\right)^{-1/p'}$ for $1<p<Q/\alpha$ and $B_4=1$ for $p=1$.

Proof. If we take $g(x)=\Omega(x)/|x|^{Q-\alpha}$, $0<\alpha<Q$, and $r=Q/(Q-\alpha)$ in Theorem 1, then the proof of Corollary 1 is straightforward, where Ω is homogeneous of degree zero on $\mathbb{R}^n_{k,+}$ and $\Omega\in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$. In this case

$$g_*(t)=(A/Q)\,t^{-Q/(Q-\alpha)},\quad g^*(t)=(A/Q)t^{1-\alpha/Q},\ \textit{and}\ A=\|\Omega\|^{Q/(Q-\alpha)}_{L_{Q/(Q-\alpha),\gamma}},$$

where

$$\|\Omega\|_{L_{Q/(Q-\alpha),\gamma}}=\left(\int_{S^{n-1}_{k,+}}|\Omega(x')|^{Q/(Q-\alpha)}dx'\right)^{(Q-\alpha)/Q}.$$

Therefore $g\in WL_{Q/(Q-\alpha),\gamma}(\mathbb{R}^n_{k,+})$ and $\|g\|_{WL_{Q/(Q-\alpha),\gamma}}=(A/Q)^{1-\alpha/Q}$. □

COROLLARY 2. *For the B-Riesz potential*

$$I_{\alpha,\gamma}f(x)=\int_{\mathbb{R}^n_{k,+}}T^y|x|^{\alpha-Q}f(y)(y')^\gamma dy,\quad 0<\alpha<Q,$$

for all $0<t<\infty$

$$\left(I_{\alpha,\gamma}f\right)^{**}(t)\leqslant C_5\delta^{\alpha}f^{**}(t)+C_6\delta^{\alpha-Q/p}\|f\|_{p,\gamma},$$

where $C_5=\omega(n,k,\gamma)B_3$ and $C_6=\omega(n,k,\gamma)^{1/p'}B_4$ for $1\leqslant p<Q/\alpha$.

Proof. By the same argument in Corollary 1 if we take

$$g(x)=|x|^{\alpha-Q}\in WL_{Q/(Q-\alpha),\gamma}(\mathbb{R}^n_{k,+}),\quad 0<\alpha<Q,$$

in Theorem 1, we easily get the proof of the Corollary. In this case

$$g_*(t)=\omega(n,k,\gamma)\,t^{-Q/(Q-\alpha)},\ \ g^*(t)=\left(\omega(n,k,\gamma)\,t^{-1}\right)^{1-\alpha/Q},$$

and

$$\|g\|_{WL_{Q/(Q-\alpha),\gamma}}=\omega(n,k,\gamma)^{1-\alpha/Q},$$

where $\omega(n,k,\gamma)=|B(0,1)|_\gamma$. □

Note that, the following estimate

$$M_{\Omega,\alpha,\gamma} f(x) \leqslant I_{|\Omega|,\alpha,\gamma}(|f|)(x) \tag{7}$$

is valid. Indeed, for all $r > 0$ we have

$$\begin{aligned} I_{|\Omega|,\alpha,\gamma}(|f|)(x) &\geqslant \int_{B(0,r)} \frac{|\Omega(y)|}{|y|^{Q-\alpha}} T^y |f(x)| \, (y')^\gamma \, dy \\ &\geqslant \frac{1}{r^{Q-\alpha}} \int_{B(0,r)} |\Omega(y)| \, T^y |f(x)| \, (y')^\gamma \, dy. \end{aligned}$$

Taking supremum over all $r > 0$, we get (7).

From Corollary 1 and inequality (7) we get the following.

COROLLARY 3. *Suppose that* Ω *is homogeneous of degree zero on* $\mathbb{R}^n_{k,+}$ *and* $\Omega \in L_{(Q)/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$, $0 < \alpha < Q$, *then for all* $\delta > 0$

$$\left(M_{\Omega,\alpha,\gamma} f\right)^{**}(t) \leqslant C_3 \delta^\alpha f^{**}(t) + C_4 \|f\|_{p,\gamma} \delta^{\alpha - Q/p}.$$

4. O'Neil type inequality for the B-convolution

In this section we prove an O'Neil type inequality for the B-convolution (see [11]).

THEOREM 2. *1. Let* $g \in WL_{r,\gamma}(\mathbb{R}^n_{k,+})$, $1 < r < \infty$, $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < r'$ *and* $1/p - 1/q = 1/r'$. *Then* $f \otimes g \in L_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|f \otimes g\|_{q,\gamma} \leqslant 2 B_1^{1-p/r'} B_2^{p/r'} \, (p')^{p/q} \, \|g\|_{WL_{r,\gamma}} \, \|f\|_{p,\gamma}.$$

2. Let $f \in L_{1,\gamma}(\mathbb{R}^n_{k,+})$, $g \in WL_{q,\gamma}(\mathbb{R}^n_{k,+})$, $1 < q < \infty$. *Then* $f \otimes g \in WL_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|f \otimes g\|_{WL_{q,\gamma}} \leqslant 2 B_1^{1/q} \, \|g\|_{WL_{q,\gamma}} \, \|f\|_{1,\gamma}.$$

Proof. Step 1. $g \in WL_{r,\gamma}(\mathbb{R}^n_{k,+})$, $1 < r < \infty$, $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < r'$ and $1/p - 1/q = 1/r'$. If we take

$$\delta = \left(\frac{C_1 f^{**}(t)}{C_2 \|f\|_{p,\gamma}} \right)^{-p/Q}$$

in (6), then we get

$$\begin{aligned} (f \otimes g)^{**}(t) &\leqslant 2 (C_1 f^{**}(t))^{1-p/r'} (C_2 \|f\|_{p,\gamma})^{p/r'} \\ &= 2 C_1^{1-p/r'} C_2^{p/r'} f^{**}(t)^{p/q} \|f\|_{p,\gamma}^{1-p/q} \\ &= 2 B_1^{1-p/r'} B_2^{p/r'} \, \|g\|_{WL_{r,\gamma}} \, \|f\|_{p,\gamma}^{1-p/q} f^{**}(t)^{p/q}. \end{aligned}$$

Thus

$$\begin{aligned}
\|f\otimes g\|_{q,\gamma} &\leqslant \|(f\otimes g)^{**}\|_{L_q(0,\infty)}\\
&\leqslant 2B_1^{1-p/r'}B_2^{p/r'}\ \|g\|_{WL_{r,\gamma}}\|f\|_{p,\gamma}^{1-p/q}\|(f^{**})^{p/q}\|_{L_q(0,\infty)}\\
&= 2B_1^{1-p/r'}B_2^{p/r'}\ \|g\|_{WL_{r,\gamma}}\|f\|_{p,\gamma}^{1-p/q}\|f^{**}\|_{L_p(0,\infty)}^{p/q}\\
&= 2B_1^{1-p/r'}B_2^{p/r'}\ (p')^{p/q}\,\|g\|_{WL_{r,\gamma}}\,\|f\|_{p,\gamma}.
\end{aligned}$$

Step 2. Let $p=1$, $1<q<\infty$, $f\in L_{1,\gamma}(\mathbb{R}^n_{k,+})$ and $g\in WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.
We take

$$\delta=\left(\frac{C_1 f^{**}(t)}{C_2\|f\|_{p,\gamma}}\right)^{-1/Q}$$

in (6), then we get

$$(f\otimes g)^{**}(t)\leqslant 2B_1^{1/q}\,\|g\|_{WL_{q,\gamma}}\,\|f\|_{1,\gamma}^{1/q'}\,f^{**}(t)^{1/q}.$$

Thus

$$\begin{aligned}
\|(f\otimes g)\|_{WL_{q,\gamma}} &= \sup_{t>0} t^{1/q}(f\otimes g)^{*}(t)\\
&\leqslant 2B_1^{1/q}\,\|g\|_{WL_{q,\gamma}}\,\|f\|_{1,\gamma}^{1/q'}\,\sup_{t>0} t^{1/q}f^{**}(t)^{1/q}\\
&= 2B_1^{1/q}\,\|g\|_{WL_{q,\gamma}}\,\|f\|_{1,\gamma}^{1/q'}\,\sup_{t>0}\left(\int_0^t f^{*}(s)ds\right)^{1/q}\\
&\leqslant 2B_1^{1/q}\,\|g\|_{WL_{q,\gamma}}\,\|f\|_{1,\gamma}^{1/q'}\,\|f^{*}\|_{L_1(0,\infty)}^{1/q}\\
&= 2B_1^{1/q}\,\|g\|_{WL_{q,\gamma}}\,\|f\|_{1,\gamma}
\end{aligned}$$

and therefore the proof of Theorem 2 is completed. □

COROLLARY 4. *Let Ω be homogeneous of degree zero on $\mathbb{R}^n_{k,+}$ and $\Omega\in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$, $0<\alpha<Q$.*

1) If $1<p<Q/\alpha$, $f\in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and $1/p-1/q=\alpha/Q$, then $I_{\Omega,\alpha,\gamma}f\in L_{q,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|I_{\Omega,\alpha,\gamma}f\|_{q,\gamma}\leqslant 2(A/Q)^{1-\alpha/Q}B_3^{1-\alpha p/Q}B_4^{\alpha p/Q}\ (p')^{p/q}\,\|f\|_{p,\gamma}.$$

2) If $f\in L_{1,\gamma}(\mathbb{R}^n_{k,+})$ and $1-1/q=\alpha/Q$, then $I_{\Omega,\alpha,\gamma}f\in WL_{q,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|I_{\Omega,\alpha,\gamma}f\|_{WL_{q,\gamma}}\leqslant 2(A/Q)^{1-\alpha/Q}B_3^{1/q}\,\|f\|_{1,\gamma}.$$

COROLLARY 5. *Let $0<\alpha<Q$.*

1) If $1<p<Q/\alpha$, $f\in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and $1/p-1/q=\alpha/Q$, then $I_{\alpha,\gamma}f\in L_{q,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|I_{\alpha,\gamma}f\|_{q,\gamma}\leqslant 2\omega(n,k,\gamma)^{1-\alpha/Q}B_3^{1-\alpha p/Q}B_4^{\alpha p/Q}\ (p')^{p/q}\,\|f\|_{p,\gamma},$$

2) If $f \in L_{1,\gamma}(\mathbb{R}^n_{k,+})$ *and* $1-1/q = \alpha/Q$, *then* $I_{\alpha,\gamma}f \in WL_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|I_{\alpha,\gamma}f\|_{WL_{q,\gamma}} \leqslant \omega(n,k,\gamma)^{1-\alpha/Q} B_3^{1/q} \, \|f\|_{1,\gamma} .$$

Note that, Corollary 4 was proved in [5] and Corollary 5 in [1, 3, 9] by using other methods.

By Corollary 4 we obtain necessary and sufficient conditions on the parameters for the fractional B-maximal operator and B-fractional integral operator with rough kernels to be bounded from the spaces $L_{p,\gamma}$ to $L_{q,\gamma}$, and from the spaces $L_{1,\gamma}$ to the weak spaces $WL_{q,\gamma}$.

THEOREM 3. *Let* Ω *be homogeneous of degree zero on* $\mathbb{R}^n_{k,+}$ *and* $\Omega \in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$, $0<\alpha<Q$.

1) If $1<p<Q/\alpha$, *then the condition* $1/p-1/q=\alpha/Q$ *is necessary and sufficient for the boundedness of* $I_{\Omega,\alpha,\gamma}$ *from* $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{q,\gamma}(\mathbb{R}^n_{k,+})$.

2) If $p=1$, *then the condition* $1-1/q=\alpha/Q$ *is necessary and sufficient for the boundedness of* $I_{\Omega,\alpha,\gamma}$ *from* $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ *to* $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.

Proof. The proof of Theorem 3 is similar to that of Theorem 4 in [6]. □

COROLLARY 6. *Let* $0<\alpha<Q$, Ω *be homogeneous of degree zero on* $\mathbb{R}^n_{k,+}$ *and* $\Omega \in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$.

1) If $1<p<Q/\alpha$, *then the condition* $1/p-1/q=\alpha/Q$ *is necessary and sufficient for the boundedness of* $M_{\Omega,\alpha,\gamma}$ *from* $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{q,\gamma}(\mathbb{R}^n_{k,+})$.

2) If $p=1$, *then the condition* $1-1/q=\alpha/Q$ *is necessary and sufficient for the boundedness of* $M_{\Omega,\alpha,\gamma}$ *from* $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ *to* $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.

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