

SOME SUFFICIENT CONDITIONS FOR CERTAIN INTEGRAL OPERATORS

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Abstract. In this paper, we consider some sufficient conditions for two integral operators to be starlike, close-to-convex and uniformly convex functions defined in the open unit disk.

1. Introduction and definitions

Let \mathcal{A} denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z: |z| < 1\}$. Also let $\mathcal{S}^*(\alpha)$, $\mathcal{H}(\alpha)$, $\mathcal{C}(\alpha)$, and \mathcal{UCV} denote the subclasses of \mathcal{A} consisting of functions which are, respectively, starlike, convex, close -to-convex of order $\alpha(0 \leq \alpha < 1)$ in \mathcal{U} and uniformly convex function . Thus, we have (see, for details, [4, 6]; see also [10, 11])

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathcal{U}; 0 \leq \alpha < 1) \right\} \tag{1.2}$$

$$\mathcal{H}(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathcal{U}; 0 \leq \alpha < 1) \right\} \tag{1.3}$$

$$\mathcal{C}(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \operatorname{Re} \left(\frac{f'(z)}{g'(z)} \right) > \alpha \quad (z \in \mathcal{U}; 0 \leq \alpha < 1; g \in \mathcal{H}) \right\} \tag{1.4}$$

and

$$\mathcal{UCV} := \left\{ f : f \in \mathcal{A} \text{ and } \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathcal{U}). \right\} \tag{1.5}$$

Recently, Breaz and Breaz in [1] introduced and studied the integral operator

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$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \dots \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{1.6}$$

and the integral operator

$$F_{\alpha_1, \dots, \alpha_n}(z) = \int_0^z (f'_1(t))^{\alpha_1} \dots (f'_n(t))^{\alpha_n} dt \tag{1.7}$$

for $\alpha_i > 0$, was introduced by Breaz et al. [3].

In [2], Breaz and Güney considered the above integral operators and they obtained their properties on the classes $\mathcal{S}^*_\alpha(b)$, $\mathcal{C}_\alpha(b)$ of starlike and convex functions of complex order b and type α introduced and studied by the author [5].

In the present paper, we obtain some sufficient conditions for the above integral operators $F_n(z)$ and $F_{\alpha_1, \dots, \alpha_n}(z)$ to be in the classes $\mathcal{S}^*(0) = \mathcal{S}^*$, $\mathcal{C}(\alpha)$ and \mathcal{UCV} .

In order to derive our main results, we have to recall here the following results:

LEMMA 1.1. ([12]) *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \frac{3}{2} \quad (z \in \mathcal{U})$$

then $f \in \mathcal{S}^*$.

LEMMA 1.2. ([7]) *If $f \in \mathcal{A}$ satisfies*

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| < 2 \quad (z \in \mathcal{U})$$

then $f \in \mathcal{S}^*$.

LEMMA 1.3. ([9]) *If $f \in \mathcal{A}$ satisfies*

$$\left| \frac{zf''(z)}{f'(z)} \right| < \frac{1}{2} \quad (z \in \mathcal{U})$$

then $f \in \mathcal{UCV}$.

LEMMA 1.4. ([8]) *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \frac{1+3\alpha}{2(1+\alpha)}, \quad (z \in \mathcal{U}; 0 \leq \alpha < 1).$$

Then

$$\operatorname{Re}\{f'(z)\} > \frac{1+\alpha}{2}, \quad (z \in \mathcal{U}; 0 \leq \alpha < 1),$$

or equivalently,

$$f \in \mathcal{C} \left(\frac{1+\alpha}{2} \right), \quad (0 \leq \alpha < 1).$$

LEMMA 1.5. ([8]) If $f \in \mathcal{A}$ satisfies

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) < \frac{3 + 2\alpha}{2 + \alpha}, \quad (z \in \mathcal{U}; 0 \leq \alpha < 1).$$

Then

$$\operatorname{Re}\{f'(z)\} > -\alpha, \quad (z \in \mathcal{U}; 0 \leq \alpha < 1).$$

2. Sufficient conditions for the integral operator F_n

Applying Lemma 1.1, we derive

THEOREM 2.1. Let $\alpha_i, i \in \{1, \dots, n\}$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, \dots, n\}$. If $f_i \in \mathcal{A}$ for $i \in \{1, \dots, n\}$ satisfies

$$\operatorname{Re} \left(\frac{zf'_i(z)}{f_i(z)} \right) < 1 + \frac{1}{2 \sum_{i=1}^n \alpha_i} \tag{2.1}$$

then $F_n \in \mathcal{S}^*$.

Proof. From (1.6), we have

$$F'_n(z) = \left(\frac{f_1(z)}{z} \right)^{\alpha_1} \dots \left(\frac{f_n(z)}{z} \right)^{\alpha_n} \tag{2.2}$$

and

$$F''_n(z) = \alpha_1 \left(\frac{f'_1(z)}{f_1(z)} - \frac{1}{z} \right) + \dots + \alpha_n \left(\frac{f'_n(z)}{f_n(z)} - \frac{1}{z} \right) F'_n(z) \tag{2.3}$$

Then from (2.2) and (2.3), we obtain

$$\frac{zF''_n(z)}{F'_n(z)} = \sum_{i=1}^n \alpha_i \left(\frac{zf'_i(z)}{f_i(z)} - 1 \right)$$

or, equivalently,

$$1 + \frac{zF''_n(z)}{F'_n(z)} = \sum_{i=1}^n \alpha_i \left(\frac{zf'_i(z)}{f_i(z)} \right) + 1 - \sum_{i=1}^n \alpha_i. \tag{2.4}$$

Taking the real part of both terms of (2.4), we have

$$\operatorname{Re} \left(1 + \frac{zF''_n(z)}{F'_n(z)} \right) = \sum_{i=1}^n \alpha_i \operatorname{Re} \left(\frac{zf'_i(z)}{f_i(z)} \right) + 1 - \sum_{i=1}^n \alpha_i. \tag{2.5}$$

Using (2.5) and (2.1), we obtain

$$\operatorname{Re} \left(1 + \frac{zF''_n(z)}{F'_n(z)} \right) < \frac{3}{2}.$$

Hence by Lemma 1.1, we get $F_n \in \mathcal{S}^*$. This completes the proof. Letting $n = 1$ in Theorem 2.1, we have

COROLLARY 2.2. Let $\alpha_1 > 0$. If $f_1 \in \mathcal{A}$ and

$$\operatorname{Re} \left(\frac{zf_1'(z)}{f_1(z)} \right) < 1 + \frac{1}{2\alpha_1} \quad (2.6)$$

then $F_1 \in \mathcal{S}^*$.

THEOREM 2.3. Let $\alpha_i, i \in \{1, \dots, n\}$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, \dots, n\}$. If $f_i \in \mathcal{A}$ for $i \in \{1, \dots, n\}$ satisfies

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| < \frac{1}{\sum_{i=1}^n \alpha_i} \quad (2.7)$$

then $F_n \in \mathcal{S}^*$.

Proof. From (2.4) we have

$$1 + \frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^n \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) + 1 \quad (2.8)$$

and hence

$$\left| 1 + \frac{zF_n''(z)}{F_n'(z)} \right| \leq \sum_{i=1}^n |\alpha_i| \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + 1 \quad (2.9)$$

Using (2.9), (2.7) and applying Lemma 1.2, we get $F_n \in \mathcal{S}^*$. Letting $n = 1$ in Theorem 2.3, we have

COROLLARY 2.4. Let $\alpha_1 > 0$. If $f_1 \in \mathcal{A}$ and

$$\left| \frac{zf_1'(z)}{f_1(z)} - 1 \right| < \frac{1}{\alpha_1} \quad (2.10)$$

then $F_1 \in \mathcal{S}^*$.

Applying Lemma 1.3 and using (2.8), we easily get

THEOREM 2.5. Let $\alpha_i, i \in \{1, \dots, n\}$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, \dots, n\}$. If $f_i \in \mathcal{A}$ for $i \in \{1, \dots, n\}$ satisfies

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| < \frac{1}{2 \sum_{i=1}^n \alpha_i} \quad (2.11)$$

then $F_n \in \mathcal{UCV}$.

Letting $n = 1$ in Theorem 2.5, we have

COROLLARY 2.6. *Let $\alpha_1 > 0$. If $f_1 \in \mathcal{A}$ and*

$$\left| \frac{zf_1'(z)}{f_1(z)} - 1 \right| < \frac{1}{2\alpha_1} \tag{2.12}$$

then $F_1 \in \mathcal{UCV}$.

THEOREM 2.7. *Let $\alpha_i, i \in \{1, \dots, n\}$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, \dots, n\}$. If $f_i \in \mathcal{A}$ for $i \in \{1, \dots, n\}$ satisfies*

$$\operatorname{Re} \left(\frac{zf_i'(z)}{f_i(z)} \right) > \frac{-1 + \alpha + 2(1 + \alpha) \sum_{i=1}^n \alpha_i}{2(1 + \alpha) \sum_{i=1}^n \alpha_i} \tag{2.13}$$

then $F_n \in \mathcal{C} \left(\frac{1+\alpha}{2} \right)$, where $0 \leq \alpha < 1$.

Proof. As in the proof of Theorem 2.1 . From (2.5) and (2.13), we obtain

$$\operatorname{Re} \left(1 + \frac{zF_n''(z)}{F_n'(z)} \right) > \frac{1 + 3\alpha}{2(1 + \alpha)}$$

Hence by Lemma 1.4, we get $F_n \in \mathcal{C} \left(\frac{1+\alpha}{2} \right)$, ($0 \leq \alpha < 1$). Letting $n = 1$ in Theorem 2.7, we have

COROLLARY 2.8. *Let $\alpha_1 > 0$. If $f_1 \in \mathcal{A}$ and*

$$\operatorname{Re} \left(\frac{zf_1'(z)}{f_1(z)} \right) > \frac{-1 + \alpha + 2(1 + \alpha)\alpha_1}{2(1 + \alpha)\alpha_1} \tag{2.14}$$

then $F_1 \in \mathcal{C} \left(\frac{1+\alpha}{2} \right)$, where $0 \leq \alpha < 1$.

Using Lemma 1.5 and applying similar proof as in Theorem 2.7, we obtain

THEOREM 2.9. *Let $\alpha_i, i \in \{1, \dots, n\}$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, \dots, n\}$. If $f_i \in \mathcal{A}$ for $i \in \{1, \dots, n\}$ satisfies*

$$\operatorname{Re} \left(\frac{zf_i'(z)}{f_i(z)} \right) < \frac{1 + \alpha + (2 + \alpha) \sum_{i=1}^n \alpha_i}{(2 + \alpha) \sum_{i=1}^n \alpha_i} \tag{2.15}$$

then

$$\operatorname{Re}\{F_n'(z)\} > -\alpha, \quad (z \in \mathcal{U}; 0 \leq \alpha < 1) \tag{2.16}$$

Letting $n = 1$ in Theorem 2.9, we have

COROLLARY 2.10. *Let $\alpha_1 > 0$. If $f_1 \in \mathcal{A}$ and*

$$\operatorname{Re} \left(\frac{zf_1'(z)}{f_1(z)} \right) < \frac{1 + \alpha + (2 + \alpha)\alpha_1}{(2 + \alpha)\alpha_1} \tag{2.17}$$

$$\operatorname{Re}\{F_1'(z)\} > -\alpha, \quad (z \in \mathcal{U}; 0 \leq \alpha < 1) \tag{2.18}$$

3. Sufficient conditions for the integral operator $F_{\alpha_1, \dots, \alpha_n}$

Applying Lemma 1.1, we derive

THEOREM 3.1. *Let $\alpha_i, i \in \{1, \dots, n\}$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, \dots, n\}$. If $f_i \in \mathcal{A}$ for $i \in \{1, \dots, n\}$ satisfies*

$$\operatorname{Re} \left(1 + \frac{zf_i''(z)}{f_i'(z)} \right) < 1 + \frac{1}{2 \sum_{i=1}^n \alpha_i} \tag{3.1}$$

then $F_{\alpha_1, \dots, \alpha_n} \in \mathcal{S}^*$.

Proof. As in the proof of Theorem 2.1, we have

$$\frac{F''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} = \alpha_1 \frac{f_1''(z)}{f_1'(z)} + \dots + \alpha_n \frac{f_n''(z)}{f_n'(z)} \tag{3.2}$$

or, equivalently,

$$\frac{zF''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} = \sum_{i=1}^n \alpha_i \left(\frac{zf_i''(z)}{f_i'(z)} \right). \tag{3.3}$$

From (3.3) we have,

$$\operatorname{Re} \left(1 + \frac{zF''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} \right) = \sum_{i=1}^n \alpha_i \operatorname{Re} \left(1 + \frac{zf_i''(z)}{f_i'(z)} \right) + 1 - \sum_{i=1}^n \alpha_i \tag{3.4}$$

Using (3.4) and (3.1), we obtain

$$\operatorname{Re} \left(1 + \frac{zF''_{\alpha_1, \dots, \alpha_n}(z)}{F'_{\alpha_1, \dots, \alpha_n}(z)} \right) < \frac{3}{2}.$$

Hence by Lemma 1.1, we get $F_{\alpha_1, \dots, \alpha_n} \in \mathcal{S}^*$.

Using (3.3), (3.4) and Lemmas 1.2, 1.3, 1.4 and 1.5, we have, respectively

THEOREM 3.2. Let $\alpha_i, i \in \{1, \dots, n\}$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, \dots, n\}$. If $f_i \in \mathcal{A}$ for $i \in \{1, \dots, n\}$ satisfies

$$\left| 1 + \frac{zf_i''(z)}{f_i'(z)} \right| < \frac{1}{\sum_{i=1}^n \alpha_i} + 1 \tag{3.5}$$

then $F_{\alpha_1, \dots, \alpha_n} \in \mathcal{S}^*$.

THEOREM 3.3. Let $\alpha_i, i \in \{1, \dots, n\}$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, \dots, n\}$. If $f_i \in \mathcal{A}$ for $i \in \{1, \dots, n\}$ satisfies

$$\left| \frac{zf_i''(z)}{f_i'(z)} \right| < \frac{1}{2 \sum_{i=1}^n \alpha_i} \tag{3.6}$$

then $F_{\alpha_1, \dots, \alpha_n} \in \mathcal{UCV}$.

THEOREM 3.4. Let $\alpha_i, i \in \{1, \dots, n\}$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, \dots, n\}$. If $f_i \in \mathcal{A}$ for $i \in \{1, \dots, n\}$ satisfies

$$\operatorname{Re} \left(1 + \frac{zf_i''(z)}{f_i'(z)} \right) > \frac{-1 + \alpha + 2(1 + \alpha) \sum_{i=1}^n \alpha_i}{2(1 + \alpha) \sum_{i=1}^n \alpha_i} \tag{3.7}$$

then $F_{\alpha_1, \dots, \alpha_n} \in \mathcal{C} \left(\frac{1+\alpha}{2} \right)$, where $0 \leq \alpha < 1$.

THEOREM 3.5. Let $\alpha_i, i \in \{1, \dots, n\}$ be real numbers with the properties $\alpha_i > 0$ for $i \in \{1, \dots, n\}$. If $f_i \in \mathcal{A}$ for $i \in \{1, \dots, n\}$ satisfies

$$\operatorname{Re} \left(1 + \frac{zf_i''(z)}{f_i'(z)} \right) < \frac{1 + \alpha + (2 + \alpha) \sum_{i=1}^n \alpha_i}{(2 + \alpha) \sum_{i=1}^n \alpha_i} \tag{3.8}$$

then

$$\operatorname{Re}\{F'_{\alpha_1, \dots, \alpha_n}(z)\} > -\alpha, \quad (z \in \mathcal{U}; 0 \leq \alpha < 1) \tag{3.9}$$

Letting $n = 1$ in Theorems 3.1-3.5, we have, respectively:

COROLLARY 3.6. Let $\alpha_1 > 0$. If $f_1 \in \mathcal{A}$ and

$$\operatorname{Re} \left(1 + \frac{zf_1''(z)}{f_1'(z)} \right) < 1 + \frac{1}{2\alpha_1} \tag{3.10}$$

then $F_{\alpha_1} \in \mathcal{S}^*$.

COROLLARY 3.7. Let $\alpha_1 > 0$. If $f_1 \in \mathcal{A}$ and

$$\left| 1 + \frac{zf_1''(z)}{f_1'(z)} \right| < \frac{1}{\alpha_1} + 1 \quad (3.11)$$

then $F_{\alpha_1} \in \mathcal{S}^*$.

COROLLARY 3.8. Let $\alpha_1 > 0$. If $f_1 \in \mathcal{A}$ and

$$\left| \frac{zf_1''(z)}{f_1'(z)} \right| < \frac{1}{2\alpha_1} \quad (3.12)$$

then $F_{\alpha_1} \in \mathcal{UCV}$.

COROLLARY 3.9. Let $\alpha_1 > 0$. If $f_1 \in \mathcal{A}$ and

$$\operatorname{Re} \left(1 + \frac{zf_1''(z)}{f_1'(z)} \right) > \frac{-1 + \alpha + 2(1 + \alpha)\alpha_1}{2(1 + \alpha)\alpha_1} \quad (3.13)$$

then $F_{\alpha_1} \in \mathcal{C} \left(\frac{1+\alpha}{2} \right)$, where $0 \leq \alpha < 1$.

COROLLARY 3.10. Let $\alpha_1 > 0$. If $f_1 \in \mathcal{A}$ and

$$\operatorname{Re} \left(1 + \frac{zf_1''(z)}{f_1'(z)} \right) < \frac{1 + \alpha + (2 + \alpha)\alpha_1}{(2 + \alpha)\alpha_1} \quad (3.14)$$

then

$$\operatorname{Re}\{F_{\alpha_1}'(z)\} > -\alpha, \quad (z \in \mathcal{U}; 0 \leq \alpha < 1). \quad (3.15)$$

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