Journal of
Mathematical
Inequalities
Volume 2, Number 4 (2008), 555–570

BOUNDEDNESS AND COMPACTNESS OF A CLASS OF MATRIX OPERATORS IN WEIGHTED SEQUENCE SPACES

R. OINAROV AND A. TEMIRKHANOVA

(*communicated by A. Kufner*)

Abstract. We study the problem of boundedness and compactness of operators of multiple summation with weights in weighted sequence spaces.

1. Introduction and preliminaries

Let $1 < p,q < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let $f = \{f_i\}_{i=1}^{\infty}$ be an arbitrary sequence of real numbers. Moreover, suppose that $\{\omega_{i,k}\}_{k=1}^{\infty}$, $i = 1,\dots,n-1$, $\{u_i\}_{i=1}^{\infty}$, and $\{v_i\}_{i=1}^{\infty}$ are weight sequences, i.e., non-negative sequences.

We consider the following weighted estimate:

$$\|S_n f\|_{l_{q,u}} \leqslant C\|f\|_{l_{p,v}}, \; 1 < p,q < \infty, \tag{1.1}$$

where n - tuple summation operator $(S_n f)$ has the form:

$$(S_n f)_i = \sum_{k_1=1}^{i} \omega_{1,k_1} \sum_{k_2=1}^{k_1} \omega_{2,k_2} \sum_{k_3=1}^{k_2} \omega_{3,k_3} \cdots \sum_{k_{n-1}=1}^{k_{n-2}} \omega_{n-1,k_{n-1}} \sum_{j=1}^{k_{n-1}} f_j, \tag{1.2}$$

and the space $l_{p,\upsilon}$ consists of all sequences $f = \{f_i\}_{i=1}^{\infty}$ such that

$$\|f\|_{l_{p,\upsilon}} = \left(\sum_{i=1}^{\infty} |f_i \upsilon_i|^p \right)^{\frac{1}{p}} < \infty, \quad 1 < p < \infty.$$

If in (1.2) we change the order of summation, then we can present it as

$$(S_n f)_i = \sum_{j=1}^{i} f_j A_{n-1,1}(i,j), \quad i \geqslant 1, \tag{1.3}$$

where $A_{n-1,1}(i,j)$ is an element of the expression:

$$A_{l,m}(i,j) = \sum_{k_l=j}^{i} \omega_{l,k_l} \sum_{k_{l-1}=k_l}^{i} \omega_{l-1,k_{l-1}} \cdots \sum_{k_m=k_{m+1}}^{i} \omega_{m,k_m}$$

Mathematics subject classification (2000): 39B82, 44B20, 46C05.
Keywords and phrases: Weighted sequences, Matrix operators.

Paper JMI-02-50

for $m \leqslant l \leqslant n-1$. When $0 \leqslant l < m$, we have that $A_{l,m}(i,j) = 1$.

When $n = 1$, the operator (1.3) has the form $(S_1 f)_i = \sum\limits_{j=1}^{i} f_j$ that coincides with the discrete Hardy operator. Its boundedness was proved by K. F. Andersen and H. P. Heinig in 1983 ([1], Theorem 4.1) for the case $1 \leqslant p \leqslant q < \infty$.

In 1987 – 1991 G. Bennett [2, 3, 4] investigated weighted Hardy type inequalities and presented their full characterizations for all relations between p and q except for the case $0 < q < 1 < p < \infty$. The remaining case $0 < q < 1 < p < \infty$ was characterized by M. Sh. Braverman and V. D. Stepanov in [5].

When $n = 2$, the operator (1.3) is a matrix operator of the following form:

$$(Af)_i = \sum_{j=1}^{i} a_{i,j} f_j. \tag{1.4}$$

The matrix operator (1.4) was studied in many papers in different sequence spaces. The almost complete collection of these results is presented in the work by M. Stieglitz and H. Tietz [10]. There the mappings of the operator (1.4) are considered in 11 sequence spaces except its mapping from $l_{p,\upsilon}$ to $l_{q,u}$. The remaining case is still an open problem.

However, there is the series of works ([8]-[10]) devoted to the operator (1.4) acting from $l_{p,\upsilon}$ to $l_{q,u}$ but with some additional conditions on the matrix elements $(a_{i,j})$, $a_{i,j} \geqslant 0$. For example, in [8], when $1 < p \leqslant q < \infty$, the validity of (1.1) for (1.4) is found under the condition $a_{i,j} \approx a_{i,k} + a_{k,j}$, $i \geqslant k \geqslant j \geqslant 1$. In the paper by R. Oinarov, C. A. Okpoti and L.-E. Persson ([9], Theorem 2.1), when $1 < q < p < \infty$, the correctness of (1.1) for (1.4) is given under the condition $a_{i,j} \approx \frac{a_{i,k}}{c_k} c_j + \frac{a_{k,j}}{b_k} b_i$, $i \geqslant k \geqslant j \geqslant 1$, where $c = \{c_i\}_{i=1}^{\infty}$ and $b = \{b_i\}_{i=1}^{\infty}$ are sequences of positive numbers.

Let us notice that when $n \geqslant 3$ for the operator (1.3), then the conditions on the matrix elements from [8] and [9] do not fulfil.

In 1998 A.O. Baiarystanov [6] considered the continuous analogue of the operator (1.3). Namely, he investigated the problem of the operator boundedness from L_p into L_q. However, the presented method was based on absolute continuity of integral. This method is impossible in the discrete case. Thus, here we establish the validity of (1.1) by other method. Moreover, we study the problem of compactness of the operator (1.3).

To prove our main results we use the criteria on precompactness of sets in l_p ([11], p. 32) and the result for a standard weighted Hardy inequality, when $1 \leqslant p \leqslant q < \infty$ ([1], Theorem 4.1). For better presentation let us state them here.

THEOREM A. *Let T be a set from l_p, $1 \leqslant p < \infty$. The set T is compact if and only if T is bounded and for all $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for all $x = \{x_i\}_{i=1}^{\infty} \in T$ the following inequality*

$$\sum_{i=N}^{\infty} |x_i|^p < \varepsilon$$

holds.

THEOREM B. *Let* $1 \leqslant p \leqslant q < \infty$. *The inequality*

$$\left(\sum_{i=1}^{\infty}\left(\sum_{\tau=1}^{i} f_\tau\right)^q u_i^q\right)^{\frac{1}{q}} \leqslant C\left(\sum_{i=1}^{\infty}|v_i f_i|^p\right)^{\frac{1}{p}} \tag{1.5}$$

holds for all sequences $\{f_i\}_{i=1}^{\infty}$, $f_i \geqslant 0$, $i \geqslant 1$, *with the best constant* $C > 0$ *if and only if*

$$B^1 = \sup_{1<j<\infty}\left(\sum_{i=j}^{\infty} u_i^q\right)^{\frac{1}{q}}\left(\sum_{\tau=1}^{j} v_i^{-p'}\right)^{\frac{1}{p'}} < \infty.$$

Moreover, $B^1 \leqslant C \leqslant p'q^{\frac{1}{q}}B^1$.

In the sequel the symbol $M \ll K$ means that $M \leqslant cK$, where $c > 0$ is a constant depending only on unessential parameters. If $M \ll K \ll M$, then we write $M \approx K$.

2. Main results

Denote

$$(B_m^n)_j = \left(\sum_{i=j}^{\infty} A_{m-1,1}^q(i;j)u_i^q\right)^{\frac{1}{q}}\left(\sum_{\tau=1}^{j} v_\tau^{-p'} A_{n-1,m}^{p'}(j;\tau)\right)^{\frac{1}{p'}},$$

$$B_m^n = \sup_{1\leqslant j\leqslant\infty}(B_m^n)_j, \qquad m = 1,\dots,n.$$

THEOREM 1. *Let* $1 < p \leqslant q < \infty$. *The inequality (1.1) holds if and only if*

$$B^n = \max_{1\leqslant m\leqslant n} B_m^n < \infty. \tag{2.1}$$

Moreover, $B^n \approx C$, *where* C *is the best constant in (1.1).*

To prove Theorem 1 we need the following helpful property of $A_{n,m}(i,\tau)$:

LEMMA 1. *For all* $i,j,\tau : 1 \leqslant \tau \leqslant j \leqslant i < \infty$ *we have that*

$$\max_{m\leqslant r\leqslant k+1}(A_{r-1,m}(i,j)A_{k,r}(j,\tau)) \leqslant A_{k,m}(i,\tau) \leqslant \sum_{r=m}^{k+1} A_{r-1,m}(i,j)A_{k,r}(j,\tau), \tag{2.2}$$

when $n \geqslant k \geqslant m \geqslant 1$.

Proof. The estimate (2.2) is correct, when $k = m$, $m = 1,2,\dots n$. Indeed, for $1 \leqslant \tau \leqslant j \leqslant i < \infty$, taking into account that

$$A_{m-1,m}(i,j) = A_{m,m+1}(j,\tau) = 1,$$

we have the following upper estimate:

$$\begin{aligned}A_{m,m}(i,\tau) &= \sum_{k_m=\tau}^{i} \omega_{m,k_m} = \sum_{k_m=\tau}^{j} \omega_{m,k_m} + \sum_{k_m=j+1}^{i} \omega_{m,k_m} \\ &\leqslant \sum_{k_m=\tau}^{j} \omega_{m,k_m} + \sum_{k_m=j}^{i} \omega_{m,k_m} = A_{m,m}(j,\tau) + A_{m,m}(i,j) \\ &= A_{m,m}(i,j)A_{m,m+1}(j,\tau) + A_{m-1,m}(i,j)A_{m,m}(j,\tau) \\ &= \sum_{r=m}^{m+1} A_{r-1,m}(i,j)A_{m,r}(j,\tau),\end{aligned}$$

and the following lower estimates:

$$A_{m,m}(i,\tau) \geqslant \sum_{k_m=\tau}^{j} \omega_{m,k_m} = A_{m,m}(j,\tau) \quad \text{and} \quad A_{m,m}(i,\tau) \geqslant \sum_{k_m=j}^{i} \omega_{m,k_m} = A_{m,m}(i,j),$$

or

$$A_{m,m}(i,\tau) \geqslant A_{m-1,m}(i,j)A_{m,m}(j,\tau) \quad \text{and} \quad A_{m,m}(i,\tau) \geqslant A_{m,m}(i,j)A_{m,m+1}(j,\tau).$$

Then

$$A_{m,m}(i,\tau) \geqslant \max_{m \leqslant r \leqslant m+1} (A_{r-1,m}(i,j)A_{m,r}(j,\tau)).$$

Suppose that (2.2) holds for $k = s \geqslant m \geqslant 1$. Let us show that it is correct also for $k = s+1$. Assume that $1 \leqslant \tau \leqslant j \leqslant i < \infty$. From the definition of $A_{n,m}(i,\tau)$ we have

$$\begin{aligned}A_{s+1,m}(i,\tau) &= \sum_{k_{s+1}=\tau}^{i} \omega_{s+1,k_{s+1}} \sum_{k_s=k_{s+1}}^{i} \omega_{s,k_s} \cdots \sum_{k_m=k_{m+1}}^{i} \omega_{m,k_m} \\ &= \sum_{k_{s+1}=\tau}^{i} \omega_{s+1,k_{s+1}} A_{s,m}(i,k_{s+1}) \\ &\leqslant \sum_{k_{s+1}=\tau}^{j} \omega_{s+1,k_{s+1}} A_{s,m}(i,k_{s+1}) + \sum_{k_{s+1}=j}^{i} \omega_{s+1,k_{s+1}} A_{s,m}(i,k_{s+1}) \\ &= \sum_{k_{s+1}=\tau}^{j} \omega_{s+1,k_{s+1}} A_{s,m}(i,k_{s+1}) + A_{s+1,m}(i,j)\end{aligned}$$

[we use the fact that the estimate (2.2) holds for $A_{s,m}(i,k_{s+1})$]

$$\begin{aligned}&\leqslant A_{s+1,m}(i,j) + \sum_{k_{s+1}=\tau}^{j} \omega_{s+1,k_{s+1}} \sum_{r=m}^{s+1} A_{r-1,m}(i,j)A_{s,r}(j,k_{s+1}) \\ &= A_{s+1,m}(i,j)A_{s+1,s+2}(j,\tau) + \sum_{r=m}^{s+1} A_{r-1,m}(i,j) \sum_{k_{s+1}=\tau}^{j} \omega_{s+1,k_{s+1}} A_{s,r}(j,k_{s+1})\end{aligned}$$

$$= A_{s+1,m}(i,j)A_{s+1,s+2}(j,\tau) + \sum_{r=m}^{s+1} A_{r-1,m}(i,j)A_{s+1,r}(j,\tau)$$
$$= \sum_{r=m}^{s+2} A_{r-1,m}(i,j)A_{s+1,r}(j,\tau).$$

Now, we prove the lower estimate of (2.2). From the definition of $A_{s+1,m}(i,\tau)$ we have

$$A_{s+1,m}(i,\tau) \geqslant \sum_{k_{s+1}=j}^{i} \omega_{s+1,k_{s+1}} A_{s,m}(i,k_{s+1})$$
$$= A_{s+1,m}(i,j) = A_{s+1,m}(i,j)A_{s+1,s+2}(i,\tau),$$

and

$$A_{s+1,m}(i,\tau) \geqslant \sum_{k_{s+1}=\tau}^{j} \omega_{s+1,k_{s+1}} A_{s,m}(i,k_{s+1})$$

[we again use the fact that the estimate (2.2) is correct for $A_{s,m}(i,k_{s+1})$]

$$\geqslant \sum_{k_{s+1}=\tau}^{j} \omega_{s+1,k_{s+1}} \max_{m\leqslant r\leqslant s+1} A_{r-1,m}(i,j)A_{s,r}(j,k_{s+1})$$
$$= \max_{m\leqslant r\leqslant s+1} A_{r-1,m}(i,j) \sum_{k_{s+1}=\tau}^{j} \omega_{s+1,k_{s+1}} A_{s,r}(j,k_{s+1})$$
$$= \max_{m\leqslant r\leqslant s+1} A_{r-1,m}(i,j)A_{s+1,r}(j,\tau).$$

Then

$$A_{s+1,m}(i,\tau) \geqslant \max\{A_{s+1,m}(i,j)A_{s+1,s+2}(j,\tau), \max_{m\leqslant r\leqslant s+1} A_{r-1,m}(i,j)A_{s+1,r}(j,\tau)\}.$$

Hence,

$$A_{s+1,m}(i,\tau) \geqslant \max_{m\leqslant r\leqslant s+2} A_{r-1,m}(i,j)A_{s+1,r}(j,\tau).$$

Consequently, (2.2) holds for all $k,m : n \geqslant k \geqslant m \geqslant 1$. The proof of Lemma 1 is complete.

Proof. [Proof of Theorem 1] *Necessity.* Suppose that the inequality (1.1) holds with the best constant $C > 0$. Let us show that $B^n < \infty$.

We take a test sequence $f = \{f_s\}_{s=1}^{\infty}$ such that

$$f_s = \begin{cases} A_{n-1,m}^{p'-1}(j,s)v_s^{-p'}, & 1 \leqslant s \leqslant j, \\ 0, & s > j, \end{cases} \tag{2.3}$$

for the fixed $m = 1,\dots,n$, and $j : 1 \leqslant j < \infty$.

Substituting (2.3) in the right side of the inequality (1.1), we have

$$\left(\sum_{s=1}^{\infty} v_s^p f_s^p\right)^{\frac{1}{p}} = \left(\sum_{s=1}^{j} v_s^p A_{n-1,m}^{(p'-1)p}(j,s) v_s^{-p'p}\right)^{\frac{1}{p}}$$
$$= \left(\sum_{s=1}^{j} A_{n-1,m}^{p'}(j,s) v_s^{-p'}\right)^{\frac{1}{p}}. \tag{2.4}$$

Substituting (2.3) in the left side of the inequality (1.1), we get

$$\sum_{i=1}^{\infty}\left(\sum_{s=1}^{i} f_s A_{n-1,1}(i,s)\right)^q u_i^q \geqslant \sum_{i=j}^{\infty}\left(\sum_{s=1}^{j} f_s A_{n-1,1}(i,s)\right)^q u_i^q$$

[we use the inequality $A_{n-1,1}(i,s) \geqslant A_{n-1,m}(j,s) A_{m-1,1}(i,j)$ for $i \geqslant j \geqslant s \geqslant 1$ that follows from (2.2)]

$$\geqslant \sum_{i=j}^{\infty}\left(\sum_{s=1}^{j} f_s A_{n-1,m}(j,s) A_{m-1,1}(i,j)\right)^q u_i^q$$
$$= \left(\sum_{s=1}^{j} f_s A_{n-1,m}(j,s)\right)^q \sum_{i=j}^{\infty} A_{m-1,1}^q(i,j) u_i^q$$
$$= \left(\sum_{s=1}^{j} A_{n-1,m}^{p'-1}(j;s) v_s^{-p'} A_{n-1,m}(j;s)\right)^q \sum_{i=j}^{\infty} A_{m-1,1}^q(i;j) u_i^q$$
$$= \left(\sum_{s=1}^{j} A_{n-1,m}^{p'}(j,s) v_s^{-p'}\right)^q \sum_{i=j}^{\infty} A_{m-1,1}^q(i,j) u_i^q. \tag{2.5}$$

Consequently, substituting (2.4) and (2.5) in (1.1), we have

$$\left(\sum_{s=1}^{j} A_{n-1,m}^{p'}(j,s) v_s^{-p'}\right)\left(\sum_{i=j}^{\infty} A_{m-1,n}^q(i,j) u_i^q\right)^{\frac{1}{q}} \leqslant C\left(\sum_{s=1}^{j} A_{n-1,m}^{p'}(j,s) v_s^{-p'}\right)^{\frac{1}{p}}.$$

Hence,

$$(B_m^n)_j = \left(\sum_{s=1}^{j} A_{n-1,m}^{p'}(j,s) v_s^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{i=j}^{\infty} u_i^q A_{m-1,1}^q(i,j)\right)^{\frac{1}{q}} \leqslant C. \tag{2.6}$$

Since, the best constant $C > 0$ of (1.1) does not depend on $j,m : 1 \leqslant m \leqslant n$, then

$$B^n = \max_{1\leqslant m\leqslant n} \sup_{1\leqslant j<\infty} (B_m^n)_j \leqslant C < \infty. \tag{2.7}$$

Therefore, $B^n < \infty$. The proof of necessity is complete.

Sufficiency. Let $B^n < \infty$. Now, we prove the inequality:

$$\left(\sum_{i=1}^{\infty}\left(\sum_{\tau=1}^{i} f_\tau A_{n-1,1}(i,\tau)\right)^q u_i^q\right)^{\frac{1}{q}} \ll B^n\left(\sum_{j=1}^{\infty}|v_j f_j|^p\right)^{\frac{1}{p}}. \tag{2.8}$$

When $n=1$, we have that $m=1$, $A_{m-1,1}(i,\tau)\equiv A_{n-1,1}(i,\tau)\equiv A_{0,1}(i,\tau)\equiv 1$, and $B^n = B^1 = \sup\limits_{1\leqslant j<\infty}\left(\sum\limits_{i=j}^{\infty} u_i^q\right)^{\frac{1}{q}}\left(\sum\limits_{\tau=1}^{j} v_\tau^{-p'}\right)^{\frac{1}{p'}}$. Moreover, in this case the inequality (2.8) coincides with the inequality (1.5). Consequently, on the bases of Theorem B the inequality (2.8) is valid.

Suppose that the inequality (2.8) is valid for $1\leqslant n\leqslant l$. Let us prove that it is valid for $n=l+1$. Since $A_{l,1}(i,\tau)\geqslant 0$, then for $f=\{f_\tau\geqslant 0, \tau\in N\}$ the sequence $(S_{l+1}f)_i$ is increasing.

For all $i\geqslant 1$ we define the following positive number set:

$$T_i = \{k\in Z : 2^k \leqslant (S_{l+1}f)_i\}, \quad \max T_i = k_i.$$

From the definition of k_i and the property of $(S_{l+1}f)_i$ it follows that

$$2^{k_i}\leqslant (S_{l+1}f)_i < 2^{k_i+1}, \quad i\geqslant 1.$$

Let $m_1=1$. Moreover, suppose that m_2 is such that $\sup M_1+1=m_2$ and $m_2> m_1$, where $M_1=\{i\in N : k_i=k_1=k_{m_1}\}$. Obviously, if the set M_1 is upper bounded, then $m_2<\infty$ and $m_2-1=\max M_1=\sup M_1$. However, otherwise we have that $m_2=\infty$. Let us define numbers $m_1<m_2<\ldots<m_s<\infty$, $s\geqslant 1$. Then to define m_{s+1} we assume that $\sup M_s+1=m_{s+1}$, where $M_s=\{i\in N : k_i=k_{m_s}\}$.

Let $N_0=\{s\in N : m_s<\infty\}$. Further, we assume that $k_{m_s}=n_s$, $s\in N_0$. Then from the definitions of k_i and m_s we find

$$2^{n_s}\leqslant (S_{l+1}f)_j < 2^{n_s+1}, \quad m_s\leqslant j<m_{s+1}, \forall\, s\in N_0, \tag{2.9}$$

and

$$N = \bigcup_{s\in N_0}[m_s, m_{s+1}). \tag{2.10}$$

Applying (2.10) to the left side of (1.1), we have

$$\|uS_{l+1}f\|_{l_q}^q = \sum_{s\in N_0}\sum_{j=m_s}^{m_{s+1}-1}(S_{l+1}f)_j^q u_j^q. \tag{2.11}$$

Next, we assume that $\sum\limits_{j=m_s}^{m_{s+1}-1}=0$, if $m_s=\infty$. Then the expression (2.11) we rewrite in the following form:

$$\|uS_{l+1}f\|_{l_q}^q = \sum_{j=m_1}^{m_2-1}(S_{l+1}f)_j^q u_j^q + \sum_{j=m_2}^{m_3-1}(S_{l+1}f)_j^q u_j^q + \sum_{s\geqslant 3}\sum_{j=m_s}^{m_{s+1}-1}(S_{l+1}f)_j^q u_j^q. \tag{2.12}$$

Since $m_1 = 1 \in N_0$, then using (2.9), we have

$$\sum_{j=m_1}^{m_2-1} (S_{l+1}f)_j^q u_j^q \leqslant \sum_{j=1}^{\infty} u_j^q 2^{(n_1+1)q} \leqslant 2^q \sum_{j=1}^{\infty} u_j^q (S_{l+1}f)_1^q$$
$$\leqslant 2^q \sum_{j=1}^{\infty} u_j^q \left(A_{l,1}^{p'}(1,1)v_1^{-p'}\right)^{\frac{q}{p'}} \left(\sum_{i=1}^{\infty} |v_i f_i|^p\right)^{\frac{q}{p}}$$
$$\leqslant 2^q \left(B_1^{l+1}\right)_1^q \|vf\|_{l_p}^q \leqslant \left(2B^{l+1}\right)^q \|vf\|_{l_p}^q. \tag{2.13}$$

If $m_2 < \infty$, i.e., $2 \in N_0$, then arguing as before for (2.13), we obtain

$$\sum_{j=m_2}^{m_3-1} (S_{l+1}f)_j^q u_j^q \leqslant \sum_{j=m_2}^{\infty} u_j^q 2^{(n_2+1)q} \leqslant 2^q \sum_{j=m_2}^{\infty} u_j^q (S_{l+1}f)_{m_2}^q$$
$$\leqslant 2^q \sum_{j=m_2}^{\infty} u_j^q \left(\sum_{i=1}^{m_2} A_{l,1}^{p'}(m_2,i)v_i^{-p'}\right)^{\frac{q}{p'}} \left(\sum_{i=1}^{\infty} |v_i f_i|^p\right)^{\frac{q}{p}}$$
$$\leqslant 2^q \left(B_1^{l+1}\right)_{m_2}^q \|vf\|_{l_p}^q \leqslant \left(2B^{l+1}\right)^q \|vf\|_{l_p}^q. \tag{2.14}$$

For $s \geqslant 3$ and $s \in N_0$ at first we estimate the value 2^{n_s-1}:

$$2^{n_s-1} = 2^{n_s} - 2^{n_s-1} \leqslant 2^{n_s} - 2^{n_{s-2}+1} \leqslant (S_{l+1}f)_{m_s} - (S_{l+1}f)_{m_{s-1}-1}$$
$$= \sum_{\tau=1}^{m_s} f_\tau A_{l,1}(m_s,\tau) - \sum_{\tau=1}^{m_{s-1}-1} f_\tau A_{l,1}(m_{s-1}-1,\tau)$$
$$= \sum_{\tau=m_{s-1}}^{m_s} f_\tau A_{l,1}(m_s,\tau) + \sum_{\tau=1}^{m_{s-1}-1} f_\tau A_{l,1}(m_s,\tau) - \sum_{\tau=1}^{m_{s-1}-1} f_\tau A_{l,1}(m_{s-1}-1,\tau)$$
$$= \sum_{\tau=m_{s-1}}^{m_s} f_\tau A_{l,1}(m_s,\tau) + \sum_{\tau=1}^{m_{s-1}-1} f_\tau \left[A_{l,1}(m_s,\tau) - A_{l,1}(m_{s-1}-1,\tau)\right]. \tag{2.15}$$

Using Lemma 1 for $m_{s-1} - 1 \geqslant \tau$, we have

$$A_{l,1}(m_s,\tau) - A_{l,1}(m_{s-1}-1,\tau) \leqslant \sum_{r=1}^{l+1} A_{l,r}(m_{s-1}-1,\tau)A_{r-1,1}(m_s,m_{s-1}-1)$$
$$- A_{l,1}(m_{s-1}-1,\tau) = A_{l,1}(m_{s-1}-1,\tau)A_{0,1}(m_s,m_{s-1}-1)$$
$$+ \sum_{r=2}^{l+1} A_{l,r}(m_{s-1}-1,\tau)A_{r-1,1}(m_s,m_{s-1}-1) - A_{l,1}(m_{s-1}-1,\tau)$$
$$= \sum_{r=2}^{l+1} A_{l,r}(m_{s-1}-1,\tau)A_{r-1,1}(m_s,m_{s-1}-1)$$
$$= \sum_{r=1}^{l} A_{l,r+1}(m_{s-1}-1,\tau)A_{r,1}(m_s,m_{s-1}-1).$$

Hence, from (2.15) we obtain

$$
\begin{aligned}
2^{n_s-1} &\leqslant \sum_{\tau=m_{s-1}}^{m_s} f_\tau A_{l,1}(m_s,\tau)+\sum_{\tau=1}^{m_{s-1}-1} f_\tau \sum_{r=1}^{l} A_{l,r+1}(m_{s-1}-1,\tau)A_{r,1}(m_s,m_{s-1}-1)\\
&= \sum_{\tau=m_{s-1}}^{m_s} f_\tau A_{l,1}(m_s,\tau)+\sum_{r=1}^{l} A_{r,1}(m_s,m_{s-1}-1)\\
&\qquad \times \sum_{\tau=1}^{m_{s-1}-1} f_\tau A_{l,r+1}(m_{s-1}-1,\tau).
\end{aligned} \tag{2.16}
$$

Applying (2.9) and (2.10), we estimate the last summand in (2.12).

$$
\begin{aligned}
&\sum_{s\geqslant 3}\sum_{i=m_s}^{m_{s+1}-1}(S_{l+1}f)_i^q u_i^q < \sum_{s\geqslant 3}\sum_{i=m_s}^{m_{s+1}-1} 2^{(n_s+1)q}u_i^q\\
&\quad = \sum_{s\geqslant 3} 2^{(n_s+1)q}\sum_{i=m_s}^{m_{s+1}-1} u_i^q = 4^q\sum_{s\geqslant 3} 2^{(n_s-1)q}\sum_{i=m_s}^{m_{s+1}-1} u_i^q\\
&\quad \leqslant 4^q\sum_{s\geqslant 3}\left(\sum_{\tau=m_{s-1}}^{m_s} f_\tau A_{l,1}(m_s,\tau)+\sum_{r=1}^{l} A_{r,1}(m_s,m_{s-1}-1)\times\right.\\
&\qquad\qquad \left.\times\sum_{\tau=1}^{m_{s-1}-1} f_\tau A_{l,r+1}(m_{s-1}-1,\tau)\right)^q\sum_{i=m_s}^{m_{s+1}-1} u_i^q\\
&\quad \leqslant 4^q\sum_{s}(l+1)^{q-1}\left[\left(\sum_{\tau=m_{s-1}}^{m_s} f_\tau A_{l,1}(m_s,\tau)\right)^q\right.\\
&\qquad\quad \left.+\sum_{r=1}^{l} A_{r,1}^q(m_s,m_{s-1}-1)\left(\sum_{\tau=1}^{m_{s-1}-1} f_\tau A_{l,r+1}(m_{s-1}-1,\tau)\right)^q\right]\times\\
&\qquad\qquad \times\sum_{i=m_s}^{m_{s+1}-1} u_i^q = 4^q(l+1)^{q-1}\sum_{s\geqslant 3}\left(\sum_{i=m_s}^{m_{s+1}-1} u_i^q\right)\left(\sum_{\tau=m_{s-1}}^{m_s} f_\tau A_{l,1}(m_s,\tau)\right)^q\\
&\qquad\quad +\sum_{s\geqslant 3}\sum_{r=1}^{l}\left(\sum_{i=m_s}^{m_{s+1}-1} u_i^q\right)A_{r,1}^q(m_s,m_{s-1}-1)\left(\sum_{\tau=1}^{m_{s-1}-1} f_\tau A_{l,r+1}(m_{s-1}-1,\tau)\right)^q\\
&\quad \ll \sum_{s\geqslant 3}\left(\sum_{i=m_s}^{m_{s+1}-1} u_i^q\right)\left(\sum_{\tau=m_{s-1}}^{m_s} f_\tau A_{l,1}(m_s,\tau)\right)^q\\
&\qquad\quad +\sum_{r=1}^{l}\sum_{s\geqslant 3}\left(\sum_{i=m_s}^{m_{s+1}-1} u_i^q\right)^q A_{r,1}^q(m_s,m_{s-1}-1)\times\\
&\qquad\qquad \times\left(\sum_{\tau=1}^{m_{s-1}-1} f_\tau A_{l,r+1}(m_{s-1}-1,\tau)\right)^q = I_0+\sum_{r=1}^{l} I_r,
\end{aligned} \tag{2.17}
$$

where

$$I_0=\sum_{s\geqslant 3}\left(\sum_{i=m_s}^{m_{s+1}-1}u_i^q\right)\left(\sum_{\tau=m_{s-1}}^{m_s}f_\tau A_{l,1}(m_s,\tau)\right)^q$$

and

$$I_r=\sum_{s\geqslant 3}\left(\sum_{i=m_s}^{m_{s+1}-1}u_i^q\right)A_{r,1}^q(m_s,m_{s-1}-1)\left(\sum_{\tau=1}^{m_{s-1}-1}f_\tau A_{l,r+1}(m_{s-1}-1,\tau)\right)^q, \quad r=1,\dots l.$$

Now, we consider I_0. Since $A_{l,l+1}(i,m_s)\equiv 1$, when $n-1\geqslant l\geqslant 0$, and

$$\sum_{i=m_s}^{\infty}A_{0,1}^q(i,m_s)u_i^q=\sum_{i=m_s}^{\infty}u_i^q\geqslant\sum_{i=m_s}^{m_{s+1}-1}u_i^q,$$

then

$$\begin{aligned}
I_0&=\sum_{s\geqslant 3}\left(\sum_{i=m_s}^{m_{s+1}-1}u_i^q\right)\left(\sum_{\tau=m_{s-1}}^{m_s}f_\tau A_{l,1}(m_s,\tau)\right)^q\\
&\leqslant\sum_{s\geqslant 3}\left(\sum_{i=m_s}^{\infty}A_{0,1}^q(i,m_s)u_i^q\right)\left(\sum_{\tau=m_{s-1}}^{m_s}f_\tau A_{l,1}(m_s,\tau)\right)^q\\
&=\sum_{s\geqslant 3}\left(\sum_{i=m_s}^{\infty}A_{0,1}^q(i,m_s)u_i^q\right)\left(\sum_{\tau=m_{s-1}}^{m_s}A_{l,1}(m_s,\tau)v_\tau^{-1}v_\tau f_\tau\right)^q
\end{aligned}$$

[we apply Hölder's inequality to the second bracket]

$$\begin{aligned}
&\leqslant\sum_{s\geqslant 3}\left(\sum_{i=m_s}^{\infty}A_{0,1}^q(i,m_s)u_i^q\right)\left(\sum_{\tau=m_{s-1}}^{m_s}A_{l,1}^{p'}(m_s,\tau)v_\tau^{-p'}\right)^{\frac{q}{p'}}\left(\sum_{\tau=m_{s-1}}^{m_s}|f_\tau v_\tau|^p\right)^{\frac{q}{p}}\\
&\leqslant\sum_{s\geqslant 3}\left(\sum_{i=m_s}^{\infty}A_{0,1}^q(i,m_s)u_i^q\right)\left(\sum_{\tau=1}^{m_s}A_{l,1}^{p'}(m_s,\tau)v_\tau^{-p'}\right)^{\frac{q}{p'}}\left(\sum_{\tau=m_{s-1}}^{m_s}|f_\tau v_\tau|^p\right)^{\frac{q}{p}}\\
&=\sum_{s\geqslant 3}\left[\left(\sum_{i=m_s}^{\infty}A_{0,1}^q(i,m_s)u_i^q\right)^{\frac{1}{q}}\left(\sum_{\tau=1}^{m_s}A_{l,1}^{p'}(m_s,\tau)v_\tau^{-p'}\right)^{\frac{1}{p'}}\right]^q\left(\sum_{\tau=m_{s-1}}^{m_s}|f_\tau v_\tau|^p\right)^{\frac{q}{p}}\\
&\leqslant\left[\sup_{s\in N_0}\left(B_1^{l+1}\right)_{m_s}\right]^q\sum_{s\geqslant 3}\left(\sum_{\tau=m_{s-1}}^{m_s}|f_\tau v_\tau|^p\right)^{\frac{q}{p}}\\
&\leqslant\left[\sup_j\left(B_1^{l+1}\right)_j\right]^q\sum_{s\geqslant 3}\left(\sum_{\tau=m_{s-1}}^{m_s}|f_\tau v_\tau|^p\right)^{\frac{q}{p}}
\end{aligned}$$

[we use Iensen's inequality for $q\geqslant p$]

$$=\left(B_1^{l+1}\right)^q\left(\sum_{s\geqslant 3}\sum_{\tau=m_{s-1}}^{m_s}|f_\tau v_\tau|^p\right)^{\frac{q}{p}}\leqslant\left(B_1^{l+1}\right)^q\left(\sum_{s\geqslant 3}\sum_{\tau=m_{s-1}}^{m_{s+1}-1}|f_\tau v_\tau|^p\right)^{\frac{q}{p}}$$

$$= \left(B_1^{l+1}\right)^q 2^{\frac{q}{p}} \left(\sum_{\tau=1}^{\infty} |f_\tau v_\tau|^p\right)^{\frac{q}{p}} \ll \left(B^{l+1}\right)^q \left(\sum_{\tau=1}^{\infty} |f_\tau v_\tau|^p\right)^{\frac{q}{p}}. \tag{2.18}$$

To estimate I_r, $r = 1,2,...,l$, at first we transform it.

$$\begin{aligned} I_r &= \sum_{s\geqslant 3} \left(\sum_{i=m_s}^{m_{s+1}-1} u_i^q\right) A_{r,1}^q(m_s, m_{s-1}-1) \left(\sum_{\tau=1}^{m_{s-1}-1} f_\tau A_{l,r+1}(m_{s-1}-1,\tau)\right)^q \\ &= \sum_{s\geqslant 3} \left(\sum_{i=m_s}^{m_{s+1}-1} u_i^q\right) A_{r,1}^q(m_s, m_{s-1}-1) \sum_{j=1}^{\infty} \left(\sum_{\tau=1}^{j} f_\tau A_{l,r+1}(j,\tau)\right)^q \delta_j(m_{s-1}-1) \\ &= \sum_{j=1}^{\infty} \left(\sum_{\tau=1}^{j} f_\tau A_{l,r+1}(j,\tau)\right)^q \sum_{s\geqslant 3} \left(\sum_{i=m_s}^{m_{s+1}-1} u_i^q\right) A_{r,1}^q(m_s, m_{s-1}-1)\delta_j(m_{s-1}-1), \end{aligned}$$

where

$$\delta_j(m_{s-1}-1) = \begin{cases} 1, & j = m_{s-1}-1, \\ 0, & j \neq m_{s-1}-1. \end{cases}$$

Hence,

$$I_r = \sum_{j=1}^{\infty} \left(\sum_{\tau=1}^{j} f_\tau A_{l,r+1}(j,\tau)\right)^q \Delta_j(r), \tag{2.19}$$

where

$$\Delta_j(r) = \sum_{s\geqslant 3} \left(\sum_{i=m_s}^{m_{s+1}-1} u_i^q\right) A_{r,1}^q(m_s, m_{s-1}-1)\delta_j(m_{s-1}-1). \tag{2.20}$$

Next, we introduce a new notation

$$\overline{\omega}_{\lambda,k_\lambda} = \omega_{\lambda+r,k_{\lambda+r}}, \quad 1 \leqslant \lambda \leqslant l-r.$$

Then for $\lambda \geqslant t$ we have

$$\overline{A}_{\lambda,t}(i,\tau) = \sum_{k_\lambda=\tau}^{i} \overline{\omega}_{\lambda,k_\lambda} \sum_{k_{\lambda-1}=k_\lambda}^{i} \overline{\omega}_{\lambda-1,k_{\lambda-1}} \cdots \sum_{k_t=k_{t+1}}^{i} \overline{\omega}_{t,k_t}$$

$$= \sum_{k_{\lambda+r}=\tau}^{i} \omega_{\lambda+r,k_{\lambda+r}} \sum_{k_{\lambda+r-1}=k_{\lambda+r}}^{i} \omega_{\lambda+r-1,k_{\lambda+r-1}} \cdots \sum_{k_{t+r}=k_{t+r+1}}^{i} \omega_{t+r,k_{t+r}} = A_{\lambda+r,t+r}(i;\tau).$$

In view of this the notation (2.19) has the following form:

$$I_r = \sum_{j=1}^{\infty} \left(\sum_{\tau=1}^{j} f_\tau \overline{A}_{l-r,1}(j,\tau)\right)^q \Delta_j(r). \tag{2.21}$$

By the assumption the inequality (2.8) holds for the operator

$$\left(\overline{S}_m f\right)_i = \sum_{j=1}^{i} f_j \overline{A}_{m-1,1}(i,j) \ \text{ for } \ 0 \leqslant m \leqslant l.$$

Since in (2.21) we have that $l-r \leqslant l$, when $1 \leqslant r \leqslant l$, then

$$I_r = \sum_{j=1}^{\infty}\left(\sum_{\tau=1}^{j} f_\tau \overline{A}_{l-r,1}(j,\tau)\right)^q \Delta_j(r) \ll \left(\overline{B}^{l-r+1}(\Delta(r))\right)^q \left(\sum_{i=1}^{\infty}|f_i v_i|^p\right)^{\frac{q}{p}}, \tag{2.22}$$

where

$$\overline{B}^{l-r+1}(\Delta(r)) = \max_{1\leqslant m\leqslant l-r+1}\sup_{1\leqslant j<\infty}\left(\overline{B}_m^{l-r+1}(\Delta(r))\right)_j$$

and

$$\left(\overline{B}_m^{l-r+1}(\Delta(r))\right)_j = \left(\sum_{i=j}^{\infty}\overline{A}^q_{m-1,1}(i,j)\Delta_i(r)\right)^{\frac{1}{q}}\left(\sum_{\tau=1}^{j} v_\tau^{-p'}\overline{A}^{p'}_{l-r,m}(j,\tau)\right)^{\frac{1}{p'}}. \tag{2.23}$$

Now, we estimate

$$\begin{aligned} A(j,\Delta(r)) &\equiv \sum_{i=j}^{\infty}\overline{A}^q_{m-1,1}(i,j)\Delta_i(r) \\ &= \sum_{i=j}^{\infty}\overline{A}^q_{m-1,1}(i,j)\sum_{s\geqslant 3}\left(\sum_{k=m_s}^{m_{s+1}-1}u_i^q\right)A^q_{r,1}(m_s,m_{s-1}-1)\delta_i(m_{s-1}-1) \\ &= \sum_{s\geqslant 3}\left(\sum_{i=m_s}^{m_{s+1}-1}u_i^q\right)A^q_{r,1}(m_s,m_{s-1}-1)\sum_{i=j}^{\infty}\overline{A}^q_{m-1,1}(i,j)\delta_i(m_{s-1}-1). \end{aligned}$$

Hence,

$$A(j,\Delta(r)) = 0 \ \text{ for } \ j > m_{s-1}-1. \tag{2.24}$$

If $m_{s-1}-1 \geqslant j$, then

$$\begin{aligned} A(j,\Delta(r)) &= \sum_{s\geqslant 3}\left(\sum_{i=m_s}^{m_{s+1}-1}u_i^q\right)A^q_{r,1}(m_s,m_{s-1}-1)\overline{A}^q_{m-1,1}(m_{s-1}-1,j) \\ &= \sum_{s\geqslant 3}\left(\sum_{i=m_s}^{m_{s+1}-1}u_i^q\right)A^q_{r,1}(m_s,m_{s-1}-1)A^q_{r+m-1,r+1}(m_{s-1}-1,j) \end{aligned}$$

[we use the inequality $A_{r+m-1,1}(m_s,j) \geqslant A_{r,1}(m_s,m_{s-1}-1)A_{r+m-1,r+1}(m_{s-1}-1,j)$ for $m_{s-1}-1 \geqslant j$ that follows from (2.2)]

$$\leqslant \sum_{m_{s-1}-1\geqslant j}\left(\sum_{i=m_s}^{m_{s+1}-1}u_i^q\right)A^q_{r+m-1,1}(m_s,j)$$

$$= \sum_{m_{s-1}-1 \geqslant j} \left(\sum_{i=m_s}^{m_{s+1}-1} u_i^q A_{r+m-1,1}^q (m_s, j) \right)$$
$$\leqslant \sum_{m_{s-1}-1 \geqslant j} \left(\sum_{i=m_s}^{m_{s+1}-1} u_i^q A_{r+m-1,1}^q (i, j) \right) \leqslant \sum_{i=j}^{\infty} u_i^q A_{r+m-1,1}^q (i, j). \tag{2.25}$$

From (2.24) and (2.25) it follows that $A(j,\Delta(r)) \leqslant \sum_{i=j}^{\infty} u_i^q A_{r+m-1,1}^q(i,j)$ for all $j \geqslant 1$. Then substituting this estimate in (2.23), we obtain

$$\overline{B}_m^{l-r+1}(\Delta(r))_j \leqslant \left(\sum_{i=j}^{\infty} u_i^q A_{r+m-1,1}^q (i,j) \right)^{\frac{1}{q}} \left(\sum_{\tau=1}^{j} v_\tau^{-p'} \overline{A}_{l-r,m}^{p'}(j,\tau) \right)^{\frac{1}{p'}}$$
$$= \left(\sum_{i=j}^{\infty} u_i^q A_{r+m-1,1}^q (i,j) \right)^{\frac{1}{q}} \left(\sum_{\tau=1}^{j} v_\tau^{-p'} A_{l,m+r}^{p'}(j,\tau) \right)^{\frac{1}{p'}}$$
$$= \left(B_{m+r}^{l+1} \right)_j, \quad \forall\, j \geqslant 1.$$

Consequently,

$$\overline{B}^{l-r+1}(\Delta(r)) \leqslant \max_{1 \leqslant m \leqslant l-r+1} \sup_{1 \leqslant j < \infty} \left(B_{m+r}^{l+1} \right)_j \leqslant \max_{1 \leqslant k \leqslant l+1} \sup_{1 \leqslant j < \infty} \left(B_k^{l+1} \right)_j = B^{l+1}.$$

Then (2.22) yields that

$$I_r \ll \left(B^{l+1} \right)^q \left(\sum_{i=1}^{\infty} |f_i v_i|^p \right)^{\frac{q}{p}}. \tag{2.26}$$

According to (2.17), (2.18), and (2.26), we have that (2.8) holds, when $n = l+1$. Therefore, (2.8) is valid for all $1 < n < \infty$. Moreover, for the best constant C in (1.1) we obtain the estimate $C \ll B^n$, which together with (2.7) gives $C \approx B^n$.

The proof of Theorem 1 is complete.

Since the finiteness of B_m^n is equivalent to the condition $\overline{\lim\limits_{j\to\infty}} (B_m^n)_j < \infty$, then from Theorem 1 we have

COROLLARY 1. *Let $1 < p \leqslant q < \infty$. The operator S_n is bounded from $l_{p,v}$ into $l_{q,u}$ if and only if*

$$\overline{\lim_{j\to\infty}} \max_{1 \leqslant m \leqslant n} (B_m^n)_j < \infty.$$

THEOREM 2. *Let $1 < p \leqslant q < \infty$. The operator S_n is compact from $l_{p,v}$ into $l_{q,u}$ if and only if*

$$\lim_{j\to\infty} \max_{1 \leqslant m \leqslant n} (B_m^n)_j = 0. \tag{2.27}$$

Proof. Necessity. Let the operator S_n be compact from $l_{p,v}$ into $l_{q,u}$. For all $l \geqslant 1$ and $m : 1 \leqslant m \leqslant n$ we introduce the following sequence:

$$g_{l,m} = \{(g_{l,m})_s\}_{s=1}^{\infty} : \quad (g_{l,m})_s = \frac{(f_{l,m})_s}{\|vf_{l,m}\|_{l_p}},$$

where $f_{l,m} = \{(f_{l,m})_s\}_{s=1}^{\infty} : \ (f_{l,m})_s = \begin{cases} A_{n-1,m}^{p'-1}(l,s)v_s^{-p'}, & 1 \leqslant s \leqslant l, \\ 0, & s \geqslant l+1. \end{cases}$

It is obvious that $\|vg_{l,m}\|_{l_p} = 1$. Since S_n is compact from $l_{p,v}$ into $l_{q,u}$, then the set $\{uS_n g_{l,m},\ l \geqslant 1,\ 1 \leqslant m \leqslant n\}$ is precompact in l_q. Then from the criteria on precompactness of the sets in l_p (see Theorem A) we conclude

$$\lim_{r\to\infty} \sup_{l\geqslant 1} \max_{1\leqslant m\leqslant n} \left(\sum_{i=r}^{\infty} u_i^q \left(S_n g_{l,m}\right)_i^q \right)^{\frac{1}{q}} = 0. \tag{2.28}$$

Thus, we have

$$\sup_{l\geqslant 1} \max_{1\leqslant m\leqslant n} \left(\sum_{i=r}^{\infty} u_i^q \left(S_n g_{l,m}\right)_i^q \right)^{\frac{1}{q}} \geqslant \max_{1\leqslant m\leqslant n} \left(\sum_{i=r}^{\infty} u_i^q \left(S_n g_{r,m}\right)_i^q \right)^{\frac{1}{q}}$$

$$= \max_{1\leqslant m\leqslant n} \left(\sum_{i=r}^{\infty} u_i^q \left(\sum_{j=1}^{i} \frac{(f_{r,m})_j}{\|vf_{r,m}\|_{l_p}} A_{n-1,1}(i,j) \right)^q \right)^{\frac{1}{q}}$$

$$= \max_{1\leqslant m\leqslant n} \left(\sum_{j=1}^{r} v_j^{-p'} A_{n-1,m}^{p'}(r,j) \right)^{-\frac{1}{p}} \left(\sum_{i=r}^{\infty} u_i^q \left(\sum_{j=1}^{r} A_{n-1,m}^{p'-1}(r,j) v_j^{-p'} A_{n-1,1}(i,j) \right)^q \right)^{\frac{1}{q}}$$

[we use the inequality $A_{n-1,1}(i,j) \geqslant A_{n-1,m}(r,j) A_{m-1,1}(i,r)$ that follows from Lemma 1]

$$\geqslant \max_{1\leqslant m\leqslant n} \left(\sum_{j=1}^{r} v_j^{-p'} A_{n-1,m}^{p'}(r,j) \right)^{\frac{1}{p'}} \left(\sum_{i=r}^{\infty} u_i^q A_{m-1,1}^q(i,r) \right)^{\frac{1}{q}}$$

$$= \max_{1\leqslant m\leqslant n} (B_m^n)_r. \tag{2.29}$$

Then from (2.28) and (2.29) it follows (2.27). The proof of necessity is complete.

Sufficiency. Let (2.27) be correct, then by Corollary 1 the operator S_n is bounded from $l_{p,v}$ into $l_{q,u}$. Consequently, the set $\{uS_n f, \|f\|_{l_{p,v}} \leqslant 1\}$ is bounded in l_q. Let us show that this set is precompact in l_q. By the criteria on precompactness of the sets in l_p (see Theorem A), the bounded set $\{uS_n f, \|f\|_{l_{p,v}} \leqslant 1\}$ is compact in l_q if

$$\lim_{r\to\infty} \sup_{\|vf\|_{l_p}\leqslant 1} \left(\sum_{i=r}^{\infty} u_i^q |(S_n f)_i|^q \right)^{\frac{1}{q}} = 0. \tag{2.30}$$

For all $r>1$ we assume that $\widetilde{u}=\{\widetilde{u}_i\}_{i=1}^{\infty}$: $\widetilde{u}_i=\begin{cases}0, & 1\leqslant i\leqslant r-1,\\ u_i, & r\leqslant i\leqslant\infty.\end{cases}$

Then from Theorem 1 we have

$$\sup_{\|vf\|_{l_p}\leqslant 1}\left(\sum_{i=r}^{\infty}u_i^q|(S_nf)_i|^q\right)^{\frac{1}{q}}=\sup_{\|vf\|_{l_p}\leqslant 1}\left(\sum_{i=1}^{\infty}\widetilde{u}_i^q|(S_nf)_i|^q\right)^{\frac{1}{q}}\ll\widetilde{B}^n(r), \tag{2.31}$$

where

$$\begin{aligned}\widetilde{B}^n(r)&=\sup_{j\geqslant 1}\max_{1\leqslant m\leqslant n}\left(\sum_{i=j}^{\infty}A_{m-1,1}^q(i,j)\widetilde{u}_i^q\right)^{\frac{1}{q}}\left(\sum_{\tau=1}^{j}v_\tau^{-p'}A_{n-1,m}^{p'}(j,\tau)\right)^{\frac{1}{p'}}\\&=\sup_{j\geqslant r}\max_{1\leqslant m\leqslant n}\left(\sum_{i=j}^{\infty}A_{m-1,1}^q(i,j)u_i^q\right)^{\frac{1}{q}}\left(\sum_{\tau=1}^{j}v_\tau^{-p'}A_{n-1,m}^{p'}(j,\tau)\right)^{\frac{1}{p'}}\\&=\sup_{j\geqslant r}\max_{1\leqslant m\leqslant n}(B_m^n)_j.\end{aligned}\tag{2.32}$$

From (2.27) and (2.32) it follows that

$$\lim_{r\to\infty}\widetilde{B}^n(r)=\lim_{j\to\infty}\sup_{j\geqslant r}\max_{1\leqslant m\leqslant n}(B_m^n)_j=\overline{\lim_{r\to\infty}}\max_{1\leqslant m\leqslant n}(B_m^n)_r=\lim_{r\to\infty}\max_{1\leqslant m\leqslant n}(B_m^n)_r=0.$$

Then (2.31) yields (2.30).

The proof of Theorem 2 is complete.

Finally, let us consider the operator

$$(S^*g)_j=\sum_{i=j}^{\infty}g_iA_{n-1,1}(i,j),\quad j\geqslant 1, \tag{2.33}$$

which is conjugate to the operator (1.3). It is easy to see that the operator (2.33) is n-tuple summation operator with weights defined as

$$(S^*g)_j=\sum_{k_{n-1}=j}^{\infty}w_{n-1,k_{n-1}}\sum_{k_{n-2}=k_{n-1}}^{\infty}w_{n-2,k_{n-2}}\cdots\sum_{k_1=k_2}^{\infty}w_{1,k_1}\sum_{i=k_1}^{\infty}g_i.$$

According to properties of conjugate operators the operator (1.3) is bounded and compact from $l_{p,v}$ into $l_{q,u}$ if and only if the operator (2.33) is bounded and compact from $l_{q',u^{-1}}$ into $l_{p',v^{-1}}$, respectively, where $v^{-1}=\{v_i^{-1}\}_{i=1}^{\infty}$. If we change q' by p, p' by q, u^{-1} by v, and v^{-1} by u from Theorems 1 and 2 we, respectively, have

THEOREM 3. *Let $1<p\leqslant q<\infty$. The inequality*

$$\left(\sum_{j=1}^{\infty}\left|\sum_{i=j}^{\infty}g_iA_{n-1,1}(i,j)\right|^q u_j^q\right)^{\frac{1}{q}}\leqslant C\left(\sum_{j=1}^{\infty}|v_ig_i|^p\right)^{\frac{1}{p}} \tag{2.34}$$

holds if and only if $\mathfrak{B}^n = \max\limits_{1\leqslant m\leqslant n} \sup\limits_{1\leqslant j<\infty} (\mathfrak{B}^n_m)_j < \infty$*, where*

$$(\mathfrak{B}^n_m)_j = \left(\sum_{i=j}^{\infty} A^{p'}_{m-1,1}(i,j)v_i^{-p'}\right)^{\frac{1}{p'}} \left(\sum_{\tau=1}^{j} u_j^q A^q_{n-1,m}(j,\tau)\right)^{\frac{1}{q}}, \quad j \geqslant 1.$$

Moreover, $\mathfrak{B}^n \approx C$*, where* C *is the best constant in (2.34).*

THEOREM 4. *Let* $1 < p \leqslant q < \infty$*. The operator (2.33) is compact from* $l_{p,v}$ *into* $l_{q,u}$ *if and only if*

$$\lim_{j\to\infty} \max_{1\leqslant m\leqslant n} (\mathfrak{B}^n_m)_j = 0.$$

Acknowledgement. The authors thank Aigerim Kalybay for discussion of the presented results. Moreover, they also thank the Referee for help in improving the paper.

REFERENCES

[1] K. F. ANDERSEN AND H. HEINIG, *Weighted norm inequalities for certain integral operators*, SIAM J. Math. Anal., **14**, no. 4 (1983), 834–844.

[2] G. BENNETT, *Some elementary inequalities*, Quart. J. Math. Oxford Ser. (2), **38**, no. 152 (1987), 401–425.

[3] G. BENNETT, *Some elementary inequalities II*, Quart. J. Math. Oxford Ser. (2), **39**, no. 156 (1988), 385–400.

[4] G. BENNETT, *Some elementary inequalities III*, Quart. J. Math. Oxford Ser. (2), **42**, no. 166 (1991), 149–174.

[5] M. SH. BRAVERMAN AND V. D. STEPANOV, *On the discrete Hardy inequality*, Bull. London Math. Soc., **26** (1994), 283–287.

[6] A. O. BAIARYSTANOV, *Two-weighted estimate of operator of multiple integration with weights*, Vestnik KazGU, Ser. Math., Mech. and Inf., **12** (1998), 13–21 (in Russian).

[7] A. O. BAIARYSTANOV, *On a generalization of V. D. Stepanov's inequality and its applications*, Deposit manuscript at All-Soviet Institute of Scientific and Technical Information, 3975 Ka93, 1993.

[8] R. OINAROV AND S. H. SHALGYNBAEVA, *Weighted additive estimate of a class of matrix operators*, Izvestiya NAN RK, Ser. Phys.-Math., **1** (2004), 39–49 (in Russian).

[9] R. OINAROV, C. A. OKPOTI AND L.-E. PERSSON, *Weighted inequalities of Hardy type for matrix operators: the case* $q < p$, Math. Inequal. Appl., **10**, no. 4 (2007), 843–861.

[10] M. STIEGLITZ, H. TIETZ *Matrixtransformationen von Folgenräumen Eine Ergebnisübersicht*, Math. Z., **154** (1977), 1–16.

[11] *Reference book Mathematical Library. Functional Analysis*. Edited by S. G. Krein, Nauka, Moscow, 1972 (in Russian).

(Received May 30, 2008)

R. Oinarov
Eurasian National University
Munaytpasov st., 5
010008 Astana, Kazakhstan
e-mail: o_ryskul@mail.ru

A. Temirkhanova
Eurasian National University
Munaytpasov st., 5
010008 Astana, Kazakhstan
e-mail: ainura-t@yandex.ru