Journal of
Mathematical
Inequalities
Volume 3, Number 1 (2009), 21–29

OPERATOR FUNCTION ASSOCIATED WITH AN ORDER PRESERVING OPERATOR INEQUALITY

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Dedicated to
Professor Sterling K. Berberian
with respect and affection

(*communicated by M. Fujii*)

Abstract. A capital letter means a bounded linear operator on a Hilbert space H. The celebrated Löwner-Heinz inequality asserts that $A \geqslant B \geqslant 0$ ensures $A^{\alpha} \geqslant B^{\alpha}$ for any $\alpha \in [0,1]$, but $A^p \geqslant B^p$ does not always hold for $p > 1$. From this point of view, we obtained: *If* $A \geqslant B \geqslant 0$ *with* $A > 0$, *then for* $t \in [0,1]$ *and* $p \geqslant 1$,

$$F_{A,B}(r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is a decreasing function for $r \geqslant t$ *and* $s \geqslant 1$, *and* $F_{A,A}(r,s) \geqslant F_{A,B}(r,s)$ *holds, that is,*

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $t \in [0,1]$, $p \geqslant 1$, $r \geqslant t$ *and* $s \geqslant 1$.

We shall prove the following further extension. *Let* $A \geqslant B \geqslant 0$ *with* $A > 0$, $t \in [0,1]$ *and* $p_1, p_2, \dots, p_{2n} \geqslant 1$ *for natural number* n. *Then*

$$G_{A,B}[r,p_{2n}]$$
$$= A^{\frac{-r}{2}}\{A^{\frac{r}{2}}[\underbrace{A^{\frac{-t}{2}}\{A^{\frac{t}{2}}\dots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\substack{\leftarrow A^{\frac{-t}{2}}\ n \text{ times and}\\ A^{\frac{t}{2}}\ n-1 \text{ times by turns}}}B^{p_1}\underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\dots\}A^{\frac{-t}{2}}]^{p_{2n}}}_{\substack{\rightarrow A^{\frac{-t}{2}}\ n \text{ times and}\\ A^{\frac{t}{2}}\ n-1 \text{ times by turns}}}A^{\frac{r}{2}}\}^{\frac{1+r-t}{q[2n]+r-t}}A^{\frac{-r}{2}}$$

is a decreasing function of $p_{2n} \geqslant 1$ *and* $r \geqslant t$, *and the following inequality holds*: $G_{A,A}[r,p_{2n}] \geqslant G_{A,B}[r,p_{2n}]$, *that is,*

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}[\underbrace{A^{\frac{-t}{2}}\{A^{\frac{t}{2}}\dots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\substack{\leftarrow A^{\frac{-t}{2}}\ n \text{ times and}\\ A^{\frac{t}{2}}\ n-1 \text{ times by turns}}}B^{p_1}\underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\dots\}A^{\frac{-t}{2}}]^{p_{2n}}}_{\substack{\rightarrow A^{\frac{-t}{2}}\ n \text{ times and}\\ A^{\frac{t}{2}}\ n-1 \text{ times by turns}}}A^{\frac{r}{2}}\}^{\frac{1-t+r}{q[2n]+r-t}}$$

where $q[2n] = q[2n; p_1, p_2, \dots, p_{2n}] = \underbrace{\{\dots[\{[(p_1-t)p_2+t]p_3-t\}p_4+t]p_5-\dots-t\}p_{2n}+t}_{-t \text{ and } t \text{ alternately } n \text{ times appear}}$.

Mathematics subject classification (2000): 47A63.
Keywords and phrases: Löwner-Heinz inequality, order preserving operator inequality.
This work was supported by Grant-in-Aid for Scientific Research (C).

Paper JMI-03-03

1. Introduction

An operator T is said to be *positive* (denoted by $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all $x \in H$, and T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

THEOREM LH. (Löwner-Heinz inequality, denoted by (LH) briefly).

If $A \geqslant B \geqslant 0$ *holds, then* $A^{\alpha} \geqslant B^{\alpha}$ *for any* $\alpha \in [0,1]$. (LH)

This was originally proved in [17] and then in [14]. Many nice proofs of (LH) are known. We mention [18] and [2]. Although (LH) asserts that $A \geqslant B \geqslant 0$ ensures $A^{\alpha} \geqslant B^{\alpha}$ for any $\alpha \in [0,1]$, unfortunately $A^{\alpha} \geqslant B^{\alpha}$ does not always hold for $\alpha > 1$. The following result has been obtained from this point of view.

THEOREM A.
If $A \geqslant B \geqslant 0$, *then for each* $r \geqslant 0$,

(i) $$(B^{\frac{r}{2}}A^{p}B^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}}B^{p}B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^{p}A^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}}B^{p}A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geqslant 0$ *and* $q \geqslant 1$ *with* $(1+r)q \geqslant p+r$.

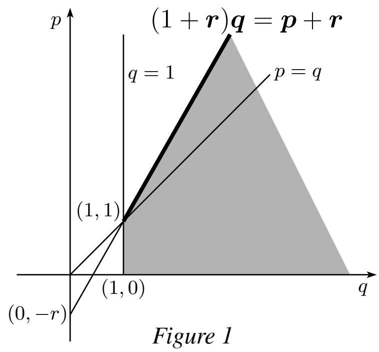

Figure 1

The original proof of Theorem A is shown in [6], an elementary one-page proof is in [7] and alternative ones are in [3], [15]. It is shown in [19] that the conditions p, q and r in FIGURE 1 are best possible.

THEOREM B. *If* $A \geqslant B \geqslant 0$ *with* $A > 0$, *then for* $t \in [0,1]$ *and* $p \geqslant 1$,

$$F_{A,B}(r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is a decreasing function for $r \geqslant t$ *and* $s \geqslant 1$, *and* $F_{A,A}(r,s) \geqslant F_{A,B}(r,s)$ *holds, that is,*

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{1.1}$$

holds for $t \in [0,1]$, $p \geqslant 1$, $r \geqslant t$ *and* $s \geqslant 1$.

The original proof of Theorem B is in [8], and an alternative one is in [4]. An elementary one-page proof of (1.1) is in [9]. Further extensions of Theorem B and related results are in [10], [12], [13], [16] and [22]. It is originally shown in [20] that the exponent value $\dfrac{1-t+r}{(p-t)s+r}$ of the right hand of (1.1) is best possible and alternative ones are in [5], [21]. It is known that the operator inequality (1.1) interpolates Theorem A and an inequality equivalent to the main result of Ando-Hiai log majorization [1] by the parameter $t \in [0,1]$.

2. Definitions of $C_{A,B}[2n]$ and $q[2n]$ and preparation

Let $A > 0$, $B \geqslant 0$, $t \in [0,1]$ and $p_1, p_2, \ldots, p_n, \ldots, p_{2(n-1)}, p_{2n-1}, p_{2n} \geqslant 1$ for a natural number n. Let $C_{A,B}[2n]$ be defined by:

$$C_{A,B}[2n] = C_{A,B}[2n; p_1, p_2, \ldots, p_{2(n-1)}, p_{2n-1}, p_{2n}]$$

$$= \underbrace{A^{\frac{t}{2}}\{A^{\frac{-t}{2}}[A^{\frac{t}{2}}\ldots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow\ A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}\ldots A^{\frac{t}{2}}]^{p_{2n-1}}A^{\frac{-t}{2}}\}^{p_{2n}}A^{\frac{t}{2}}}_{\rightarrow\ A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}}. \tag{2.1}$$

For examples,

$$C_{A,B}[2] = A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}$$

and

$$C_{A,B}[4] = A^{\frac{t}{2}}\left[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}\right]^{p_4}A^{\frac{t}{2}}.$$

Particularly put $A = B$ in $C_{A,B}[2n]$ in (2.1). Then

$$C_{A,A}[2n] = C_{A,A}[2n; p_1, p_2, \ldots, p_{2(n-1)}, p_{2n-1}, p_{2n}]$$

$$= \underbrace{A^{\frac{t}{2}}\{A^{\frac{-t}{2}}[A^{\frac{t}{2}}\ldots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow\ A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}} A^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}\ldots A^{\frac{t}{2}}]^{p_{2n-1}}A^{\frac{-t}{2}}\}^{p_{2n}}A^{\frac{t}{2}}}_{\rightarrow\ A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}} \tag{2.2}$$

$$= A^{\{\ldots[\{[(p_1-t)p_2+t]p_3-t\}p_4+t]p_5-\ldots+t]p_{2n-1}-t\}p_{2n}+t}. \tag{2.3}$$

Let $q[2n]$ be defined by

$$\begin{aligned} q[2n] &= q[2n; p_1, p_2, \ldots, p_n, \ldots, p_{2(n-1)}, p_{2n-1}, p_{2n}] \\ &= \text{the exponential power of } A \text{ in (2.3)} \\ &= \underbrace{\{\ldots[\{[(p_1-t)p_2+t]p_3-t\}p_4+t]p_5-\ldots-t\}p_{2n}+t}_{-t \text{ and } t \text{ alternately } n \text{ times appear}}. \end{aligned} \tag{2.4}$$

For examples,

$$q[2] = (p_1-t)p_2+t$$

and

$$q[4] = \left[\{(p_1-t)p_2+t\}p_3-t\right]p_4+t.$$

For the sake of convenience, we define

$$C_{A,B}[0] = B \quad \text{and} \quad q[0] = 1 \tag{2.5}$$

and these definitions in (2.5) may be naturally defined by (2.1) and (2.4).

THEOREM C. [11] *Let $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \ldots, p_{2n} \geqslant 1$ for natural number n. Then the following inequality holds for $r \geqslant t$:*

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}[\underbrace{A^{\frac{-t}{2}}\{A^{\frac{t}{2}}\ldots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\substack{\leftarrow A^{\frac{-t}{2}}\ n \text{ times and} \\ A^{\frac{t}{2}}\ n-1 \text{ times by turns}}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\ldots\}A^{\frac{-t}{2}}}_{\substack{\rightarrow A^{\frac{-t}{2}}\ n \text{ times and} \\ A^{\frac{t}{2}}\ n-1 \text{ times by turns}}}]^{p_{2n}}A^{\frac{r}{2}}\}^{\frac{1-t+r}{q[2n]-t+r}} \tag{2.6}$$

where $q[2n]$ is defined in (2.4).

We need the following lemmas.

LEMMA A. [8, Lemma 1] *Let X be a positive invertible operator and Y be an invertible operator. For any real number λ,*

$$(YXY^*)^{\lambda} = YX^{\frac{1}{2}}(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}})^{\lambda-1}X^{\frac{1}{2}}Y^*.$$

The following lemma is easily shown by (2.1), (2.4) and (2.5).

LEMMA 2.1. *For $A > 0$, $B \geqslant 0$ and any natural number n, the following* (i) *and* (ii) *hold:*

(i)
$$C_{A,B}[2n] = A^{\frac{t}{2}}\{A^{\frac{-t}{2}}C_{A,B}[2(n-1)]^{p_{2n-1}}A^{\frac{-t}{2}}\}^{p_{2n}}A^{\frac{t}{2}} \tag{2.7}$$

and

(ii)
$$q[2n] = \{q[2(n-1)]p_{2n-1} - t\}p_{2n} + t \tag{2.8}$$

where $C_{A,B}[0] = B$ and $q[0] = 1$.

3. Further extension of Theorem B

We shall state further extension of Theorem B.

THEOREM 3.1. *Let $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, \ldots, p_{2n} \geqslant 1$ for natural number n. Then*

$$G_{A,B}[r, p_{2n}]$$
$$= A^{\frac{-r}{2}}\{A^{\frac{r}{2}}[\underbrace{A^{\frac{-t}{2}}\{A^{\frac{t}{2}}\ldots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\substack{\leftarrow A^{\frac{-t}{2}}\ n \textit{ times and} \\ A^{\frac{t}{2}}\ n-1 \textit{ times by turns}}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\ldots\}A^{\frac{-t}{2}}}_{\substack{\rightarrow A^{\frac{-t}{2}}\ n \text{ times and} \\ A^{\frac{t}{2}}\ n-1 \text{ times by turns}}}]^{p_{2n}}A^{\frac{r}{2}}\}^{\frac{1+r-t}{q[2n]+r-t}}A^{\frac{-r}{2}} \tag{3.1}$$

is a decreasing function of $p_{2n} \geqslant 1$ and $r \geqslant t$, and the following inequality holds

$$G_{A,A}[r, p_{2n}] \geqslant G_{A,B}[r, p_{2n}],$$

that is,

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}[\underbrace{A^{\frac{-t}{2}}\{A^{\frac{t}{2}}\ldots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\substack{\leftarrow A^{\frac{-t}{2}}\ n \text{ times and} \\ A^{\frac{t}{2}}\ n-1 \text{ times by turns}}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{t}{2}}\ldots\}A^{\frac{-t}{2}}}_{\substack{\rightarrow A^{\frac{-t}{2}}\ n \text{ times and} \\ A^{\frac{t}{2}}\ n-1 \text{ times by turns}}}]^{p_{2n}}A^{\frac{r}{2}}\}^{\frac{1-t+r}{q[2n]+r-t}} \tag{3.2}$$

where $q[2n]$ is defined by (2.4).

COROLLARY 3.2. *If $A \geqslant B \geqslant 0$ with $A > 0$, $t \in [0,1]$ and $p_1, p_2, p_3, p_4 \geqslant 1$,*

$$G_{A,B}[r,p_4] = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{r}{2}}\}^{\frac{1-t+r}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r}}A^{\frac{-r}{2}}$$

is a decreasing function of $p_4 \geqslant 1$ and $r \geqslant t$, and the following inequality holds $G_{A,A}[r,p_4] \geqslant G_{A,B}[r,p_4]$, that is,

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}A^{\frac{r}{2}}\}^{\frac{1-t+r}{[\{(p_1-t)p_2+t\}p_3-t]p_4+r}}$$

holds for $t \in [0,1]$, $r \geqslant t$ and $p_1, p_2, p_3, p_4 \geqslant 1$.

REMARK 3.1. Theorem 3.1 yields Corollary 3.2 by putting $n = 2$ and also Corollary 3.2 yields Theorem B by putting $p_2 = p_3 = 1$.

Proof of Theorem 3.1. (3.2) is (2.6) itself of Theorem C. Recall that (3.2) can be described as by (2.1):

$$A^{1+r-t} \geqslant (A^{\frac{r-t}{2}}C_{A,B}[2n]A^{\frac{r-t}{2}})^{\frac{1+r-t}{q[2n]+r-t}}. \tag{3.3}$$

By (2.7) we have

$$A^{\frac{r-t}{2}}C_{A,B}[2n]A^{\frac{r-t}{2}} = A^{\frac{r}{2}}\{A^{\frac{-t}{2}}C_{A,B}[2(n-1)]^{p_{2n-1}}A^{\frac{-t}{2}}\}^{p_{2n}}A^{\frac{r}{2}}. \tag{3.4}$$

Put $D = A^{\frac{-t}{2}}C_{A,B}[2(n-1)]^{p_{2n-1}}A^{\frac{-t}{2}}$ in (3.4) briefly in the proofs **(a)** and **(b)** under below.

(a) *Proof of the result that $G_{A,B}[r,p_{2n}]$ is a decreasing function of r.*

Raise each side of (3.3) to the power $\dfrac{v}{1+r-t} \in [0,1]$ for $r \geqslant v \geqslant 0$, we have

$$A^{v} \geqslant (A^{\frac{r-t}{2}}C_{A,B}[2n]A^{\frac{r-t}{2}})^{\frac{v}{q[2n]+r-t}}$$

$$= (A^{\frac{r}{2}}D^{p_{2n}}A^{\frac{r}{2}})^{\frac{v}{q[2n]+r-t}} \quad \text{for } r \geqslant v \geqslant 0 \quad \text{by (3.4)} \tag{3.5}$$

and we have

$$\begin{aligned}
G_{A,B}[r,p_{2n}] &= A^{\frac{-r}{2}}\{A^{\frac{r-t}{2}}C_{A,B}[2n]A^{\frac{r-t}{2}}\}^{\frac{1+r-t}{q[2n]+r-t}}A^{\frac{-r}{2}} \quad \text{by (3.1) and (2.1)}\\
&= A^{\frac{-r}{2}}(A^{\frac{r}{2}}D^{p_{2n}}A^{\frac{r}{2}})^{\frac{1+r-t}{q[2n]+r-t}}A^{\frac{-r}{2}} \quad \text{by (3.4)}\\
&= D^{\frac{p_{2n}}{2}}(D^{\frac{p_{2n}}{2}}A^{r}D^{\frac{p_{2n}}{2}})^{\frac{1-q[2n]}{q[2n]+r-t}}D^{\frac{p_{2n}}{2}} \quad \text{by Lemma A}\\
&= D^{\frac{p_{2n}}{2}}\left\{(D^{\frac{p_{2n}}{2}}A^{r}D^{\frac{p_{2n}}{2}})^{\frac{q[2n]+r+v-t}{q[2n]+r-t}}\right\}^{\frac{1-q[2n]}{q[2n]+r+v-t}}D^{\frac{p_{2n}}{2}}\\
&= D^{\frac{p_{2n}}{2}}\left\{D^{\frac{p_{2n}}{2}}A^{\frac{r}{2}}(A^{\frac{r}{2}}D^{p_{2n}}A^{\frac{r}{2}})^{\frac{v}{q[2n]+r-t}}A^{\frac{r}{2}}D^{\frac{p_{2n}}{2}}\right\}^{\frac{1-q[2n]}{q[2n]+r+v-t}}D^{\frac{p_{2n}}{2}} \quad \text{by Lemma A}\\
&\geqslant D^{\frac{p_{2n}}{2}}(D^{\frac{p_{2n}}{2}}A^{r+v}D^{\frac{p_{2n}}{2}})^{\frac{1-q[2n]}{q[2n]+r+v-t}}D^{\frac{p_{2n}}{2}}\\
&= G_{A,B}[r+v,p_{2n}]
\end{aligned} \tag{3.6}$$

and the last inequality follows by LH because (3.5) and $\frac{1-q[2n]}{q[2n]+r+v-t} \in [-1,0]$ holds and taking inverses of both sides, so that $G_{A,B}[r,p_{2n}]$ is decreasing function of r by (3.6).

(b) *Proof of the result that* $G_{A,B}[r,p_{2n}]$ *is a decreasing function of* p_{2n}.
Raise each side of (3.3) to the power $\dfrac{r}{1+r-t} \in [0,1]$, we have

$$A^r \geqslant (A^{\frac{r-t}{2}} C_{A,B}[2n] A^{\frac{r-t}{2}})^{\frac{r}{q[2n]+r-t}} = (A^{\frac{r}{2}} D^{p_{2n}} A^{\frac{r}{2}})^{\frac{r}{q[2n]+r-t}} \quad \text{by (3.4)} \tag{3.7}$$

and applying Lemma A to (3.7) and taking inverses of both sides, we have

$$(D^{\frac{p_{2n}}{2}} A^r D^{\frac{p_{2n}}{2}})^{\frac{q[2n]-t}{q[2n]+r-t}} \geqslant D^{p_{2n}}. \tag{3.8}$$

(3.8) and (2.8) imply

$$(D^{\frac{p_{2n}}{2}} A^r D^{\frac{p_{2n}}{2}})^{\frac{(q[2(n-1)]p_{2n-1}-t)p_{2n}}{(q[2(n-1)]p_{2n-1}-t)p_{2n}+r}} \geqslant D^{p_{2n}}, \tag{3.9}$$

and raise each side of (3.9) to the power $\dfrac{v}{p_{2n}} \in [0,1]$ for $p_{2n} \geqslant v \geqslant 0$, we have

$$(D^{\frac{p_{2n}}{2}} A^r D^{\frac{p_{2n}}{2}})^{\frac{(q[2(n-1)]p_{2n-1}-t)v}{(q[2(n-1)]p_{2n-1}-t)p_{2n}+r}} \geqslant D^v \quad \text{for } p_{2n} \geqslant v \geqslant 0. \tag{3.10}$$

Then we have

$$\begin{aligned}
f(r,p_{2n}) &= (A^{\frac{r}{2}} D^{p_{2n}} A^{\frac{r}{2}})^{\frac{1+r-t}{q[2n]+r-t}} \\
&= (A^{\frac{r}{2}} D^{p_{2n}} A^{\frac{r}{2}})^{\frac{1+r-t}{(q[2(n-1)]p_{2n-1}-t)p_{2n}+r}} \quad \text{by (2.8)} \\
&= \{(A^{\frac{r}{2}} D^{p_{2n}} A^{\frac{r}{2}})^{\frac{(q[2(n-1)]p_{2n-1}-t)(p_{2n}+v)+r}{(q[2(n-1)]p_{2n-1}-t)p_{2n}+r}}\}^{\frac{1+r-t}{(q[2(n-1)]p_{2n-1}-t)(p_{2n}+v)+r}} \\
&= \{A^{\frac{r}{2}} D^{\frac{p_{2n}}{2}} (D^{\frac{p_{2n}}{2}} A^r D^{\frac{p_{2n}}{2}})^{\frac{(q[2(n-1)]p_{2n-1}-t)v}{(q[2(n-1)]p_{2n-1}-t)p_{2n}+r}} D^{\frac{p_{2n}}{2}} A^{\frac{r}{2}}\}^{\frac{1+r-t}{(q[2(n-1)]p_{2n-1}-t)(p_{2n}+v)+r}} \\
&\qquad \text{(by Lemma A)} \\
&\geqslant (A^{\frac{r}{2}} D^{p_{2n}+v} A^{\frac{r}{2}})^{\frac{1+r-t}{(q[2(n-1)]p_{2n-1}-t)(p_{2n}+v)+r}} \\
&= f(r,p_{2n}+v)
\end{aligned} \tag{3.11}$$

and the last inequality follows by LH because (3.10) and $\frac{1-t+r}{(q[2(n-1)]p_{2n-1}-t)(p_{2n}+v)+r} \in [0,1]$, so that $G_{A,B}[r,p_{2n}] = A^{\frac{-r}{2}} f(r,p_{2n}) A^{\frac{-r}{2}}$ is decreasing function of $p_{2n} \geqslant 1$ by (3.11).

Finally $G_{A,B}[r,p_{2n}]$ is a decreasing function of both $r \geqslant t$ and $p_{2n} \geqslant 1$ by (a) and (b). □

4. Satellite inequalities as an application of Theorem 3.1

PROPOSITION 4.1. *$F[A,B;p_1,p_2,\dots p_{2(n-1)},p_{2n-1},p_{2n}]$ is defined as follows for natural number n;*

$$F[A,B;p_1,p_2,\dots p_{2(n-1)},p_{2n-1},p_{2n}] = (C_{A,B}[2n])^{\frac{1}{q[2n]}}. \tag{4.1}$$

If $p_{2n}=1$, then $(C_{A,B}[2n])^{\frac{1}{q[2n]}} = (C_{A,B}[2(n-1)])^{\frac{1}{q[2(n-1)]}}$ holds, that is,

$$F[A,B;p_1,p_2,\dots p_{2(n-1)},p_{2n-1},p_{2n}] = F[A,B;p_1,p_2,\dots,p_{2(n-1)}]. \tag{4.2}$$

Proof. If $p_{2n}=1$, then

$$\begin{aligned} C_{A,B}[2n] &= C_{A,B}[2n;p_1,p_2,\dots p_{2(n-1)},p_{2n-1},p_{2n}] \\ &= A^{\frac{t}{2}}\{A^{\frac{-t}{2}}C_{A,B}[2(n-1)]^{p_{2n-1}}A^{\frac{-t}{2}}\}^1A^{\frac{t}{2}} \quad \text{by putting } p_{2n}=1 \text{ in (2.7)} \\ &= C_{A,B}[2(n-1)]^{p_{2n-1}} \end{aligned} \tag{4.3}$$

on the other hand,

$$\begin{aligned} q[2n] &= q[2n;p_1,p_2,\dots p_{2(n-1)},p_{2n-1},p_{2n}] \\ &= (q[2(n-1)]p_{2n-1}-t)1+t \quad \text{by putting } p_{2n}=1 \text{ in (2.8)} \\ &= q[2(n-1)]p_{2n-1}, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} F[A,B;p_1,p_2,\dots p_{2(n-1)},p_{2n-1},p_{2n}] &= (C_{A,B}[2n])^{\frac{1}{q[2n]}} \\ &= (C_{A,B}[2(n-1)]^{p_{2n-1}})^{\frac{1}{q[2(n-1)]p_{2n-1}}} \quad \text{by (4.3) and (4.4)} \\ &= (C_{A,B}[2(n-1)])^{\frac{1}{q[2(n-1)]}} \\ &= F[A,B;p_1,p_2,\dots,p_{2(n-1)}] \end{aligned}$$

and (4.2) is shown. $\square$

THEOREM 4.2. *If $A \geqslant B \geqslant 0$ with $A>0$, $t\in[0,1]$ and $p_1,p_2,\dots,p_{2n}\geqslant 1$, then the following inequality holds:*

$$\begin{aligned} &A \geqslant B \\ &\geqslant \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{\frac{1}{(p_1-t)p_2+t}} \\ &\geqslant \left\{A^{\frac{t}{2}}\left[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^{p_1}A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}\right]^{p_4}A^{\frac{t}{2}}\right\}^{\frac{1}{\{[(p_1-t)p_2+t]p_3-t\}p_4+t}} \\ &\dots \\ &\geqslant \\ &\dots \end{aligned}$$

$$\geqslant \Big[\underbrace{A^{\frac{t}{2}}\{A^{\frac{-t}{2}}[A^{\frac{t}{2}}\ldots[A^{\frac{-t}{2}}\{A^{\frac{t}{2}}(A^{\frac{-t}{2}}}_{\leftarrow A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}} B^{p_1} \underbrace{A^{\frac{-t}{2}})^{p_2}A^{\frac{t}{2}}\}^{p_3}A^{\frac{-t}{2}}]^{p_4}\ldots A^{\frac{t}{2}}]^{p_{2n-1}}A^{\frac{-t}{2}}\}^{p_{2n}}A^{\frac{t}{2}}}_{\rightarrow A^{\frac{-t}{2}} \text{ and } A^{\frac{t}{2}} \text{ alternately } n \text{ times}} \Big]^{\frac{1}{q[2n]}} \tag{4.5}$$

where $q[2n]$ *is defined in* (2.4).

Proof. First of all, we recall the following relation by (4.1) and (3.1):

$$F[A,B;p_1,p_2,\ldots p_{2(n-1)},p_{2n-1},p_{2n}] = C_{A,B}[2n]^{\frac{1}{q[2n]}} = A^{\frac{t}{2}}G_{A,B}[t,p_{2n}]A^{\frac{t}{2}}. \tag{4.6}$$

Since $G_{A,B}[t,p_{2n}]$ is a decreasing function of $p_{2n} \geqslant 1$ by Theorem 3.1, (4.6) yields

$$F[A,B;p_1,p_2,\ldots p_{2(n-1)},p_{2n-1},p_{2n}] \quad \text{is also a decreasing function of } p_{2n} \geqslant 1. \tag{4.7}$$

By (4.7) and Proposition 4.1, we have

$$\begin{aligned} A \geqslant B &= F[A,B;p_1,1] \\ &\geqslant F[A,B;p_1,p_2] \\ &\ldots \\ &\geqslant \\ &\ldots \\ &\geqslant F[A,B;p_1,p_2,\ldots,p_{2(n-1)}] \\ &= F[A,B;p_1,p_2,\ldots,p_{2(n-1)},p_{2n-1},1] \\ &\geqslant F[A,B;p_1,p_2,\ldots,p_{2(n-1)},p_{2n-1},p_{2n}] \end{aligned}$$

and the proof of (4.5) is complete. □

REMARK 4.1. Corollary 2 in § 3.2.5 of [10] states that *if* $A \geqslant B > 0$, *then*

$$A \geqslant B \geqslant \{A^{\frac{t}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{t}{2}}\}^{\frac{1}{(p-t)s+t}} \quad \text{holds for each } t \in [0,1] \text{ and } p,s \geqslant 1 \tag{4.8}$$

and Theorem 4.2 is further extension of (4.8).

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(Received November 24, 2008)

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