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NOTE ON THE DISCRETE OSTROWSKI–GRÜSS TYPE INEQUALITY

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(*communicated by N. Ujević*)

Abstract. In the present note, we establish several new discrete Ostrowski-Grüss type inequalities which extend some known results.

1. Introduction

In 1935, Grüss (see [11, p. 296]) proved the following integral inequality which gives an approximation for the integral of a product of two functions in terms of the product of integrals of the two functions.

GRÜSS INEQUALITY. *Let f and g be two bounded functions defined on $[a,b]$ with $\gamma_1 \leqslant f(x) \leqslant \Gamma_1$ and $\gamma_2 \leqslant g(x) \leqslant \Gamma_2$, where $\gamma_1, \gamma_2, \Gamma_1, \Gamma_2$ are four constants. Then we have:*

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \frac{1}{b-a} \int_a^b g(x)dx \right| \leqslant \frac{1}{4}(\Gamma_1 - \gamma_1)(\Gamma_2 - \gamma_2),$$

and the inequality is sharp, in the sense that the constant $1/4$ can't be replaced by a smaller one.

In 1938, Ostrowski [12] (see also [11, p. 469]) gave us the following estimate for the deviation of the values of a smooth function from its mean value.

OSTROWSKI INEQUALITY. *If $f : [a,b] \to \mathbb{R}$ is a differentiable function with bounded derivative, then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leqslant \left[\frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for every $x \in [a,b]$. Moreover the constant $1/4$ is the best possible.

In the years thereafter, numerous generalizations, extensions and variants of Ostrowski-Grüss inequality have appeared in the literature (see [1, 2, 4, 5, 6, 7, 8, 9]). The purpose of the present note is to establish some new discrete Ostrowski-Grüss type inequalities.

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2. Main results

In this section, we shall state our main results and give their proofs.

THEOREM 2.1. *Let* $\{u_k\},\{v_k\},\{w_k\},\{\tilde{w}_k\}$ *for* $k=1,\dots,n$ *be four finite sequences of real numbers with* $\sum_{k=1}^n w_k = 1$ *and* $\sum_{k=1}^n \tilde{w}_k = 1$. *Then we have*

$$\begin{aligned}\left|u_kv_k-\frac{1}{2}\left(v_k\sum_{i=1}^n w_iu_i+u_k\sum_{i=1}^n \tilde{w}_iv_i\right)\right| \\ \leqslant \frac{1}{2}\left(|v_k|\max_{1\leqslant i\leqslant n-1}|u_{i+1}-u_i|\Phi(n,k)+|u_k|\max_{1\leqslant i\leqslant n-1}|v_{i+1}-v_i|\tilde{\Phi}(n,k)\right)\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\left|u_kv_k-\left(v_k\sum_{i=1}^n w_iu_i+u_k\sum_{i=1}^n \tilde{w}_iv_i\right)+\left(\sum_{i=1}^n w_iu_i\right)\left(\sum_{i=1}^n \tilde{w}_iv_i\right)\right| \\ \leqslant \max_{1\leqslant i\leqslant n-1}|u_{i+1}-u_i|\max_{1\leqslant i\leqslant n-1}|v_{i+1}-v_i|\Phi(n,k)\tilde{\Phi}(n,k),\end{aligned} \tag{2.2}$$

where

$$\Phi(n,k) := \sum_{j=1}^{k-1}\left|\sum_{i=1}^{j} w_i\right| + \sum_{j=k}^{n-1}\left|\sum_{i=j+1}^{n} w_i\right|$$

and

$$\tilde{\Phi}(n,k) := \sum_{j=1}^{k-1}\left|\sum_{i=1}^{j} \tilde{w}_i\right| + \sum_{j=k}^{n-1}\left|\sum_{i=j+1}^{n} \tilde{w}_i\right|.$$

Proof. For any $k=1,\dots,n$, we have the following known equality [10],

$$\begin{aligned}u_k-\sum_{i=1}^n w_iu_i &= \sum_{i=1}^{k-1} w_i(u_k-u_i)+\sum_{i=k+1}^{n} w_i(u_k-u_i)\\ &= \sum_{i=1}^{k-1} w_i\sum_{j=i}^{k-1}(u_{j+1}-u_j)+\sum_{i=k+1}^{n} w_i\sum_{j=k}^{i-1}(u_j-u_{j+1})\\ &= \sum_{j=1}^{k-1}(u_{j+1}-u_j)\sum_{i=1}^{j} w_i+\sum_{j=k}^{n-1}(u_j-u_{j+1})\sum_{i=j+1}^{n} w_i.\end{aligned} \tag{2.3}$$

Similarly, we have

$$v_k-\sum_{i=1}^n \tilde{w}_iv_i = \sum_{j=1}^{k-1}(v_{j+1}-v_j)\sum_{i=1}^{j} \tilde{w}_i+\sum_{j=k}^{n-1}(v_j-v_{j+1})\sum_{i=j+1}^{n} \tilde{w}_i. \tag{2.4}$$

From the above two equations, it follows that

$$
\begin{aligned}
&\left|u_k v_k - \frac{1}{2}\left(v_k \sum_{i=1}^{n} w_i u_i + u_k \sum_{i=1}^{n} \tilde{w}_i v_i\right)\right| \\
&\quad = \frac{1}{2}\left|v_k \sum_{j=1}^{k-1}(u_{j+1}-u_j)\sum_{i=1}^{j} w_i + v_k \sum_{j=k}^{n-1}(u_j - u_{j+1})\sum_{i=j+1}^{n} w_i \right. \\
&\qquad \left. + u_k \sum_{j=1}^{k-1}(v_{j+1}-v_j)\sum_{i=1}^{j} \tilde{w}_i + u_k \sum_{j=k}^{n-1}(v_j - v_{j+1})\sum_{i=j+1}^{n} \tilde{w}_i\right| \\
&\quad \leqslant \frac{1}{2}\left(|v_k| \max_{1\leqslant i\leqslant n-1}|u_{i+1}-u_i|\Phi(n,k) + |u_k| \max_{1\leqslant i\leqslant n-1}|v_{i+1}-v_i|\tilde{\Phi}(n,k)\right),
\end{aligned}
\tag{2.5}
$$

which implies the inequality (2.1). Multiplying (2.3) and (2.4), we have

$$
\begin{aligned}
&\left|u_k v_k - \left(v_k \sum_{i=1}^{n} w_i u_i + u_k \sum_{i=1}^{n} \tilde{w}_i v_i\right) + \left(\sum_{i=1}^{n} w_i u_i\right)\left(\sum_{i=1}^{n} \tilde{w}_i v_i\right)\right| \\
&\quad = \left|\sum_{j=1}^{k-1}(u_{j+1}-u_j)\sum_{i=1}^{j} w_i + \sum_{j=k}^{n-1}(u_j - u_{j+1})\sum_{i=j+1}^{n} w_i\right| \\
&\qquad \times \left|\sum_{j=1}^{k-1}(v_{j+1}-v_j)\sum_{i=1}^{j} \tilde{w}_i + \sum_{j=k}^{n-1}(v_j - v_{j+1})\sum_{i=j+1}^{n} \tilde{w}_i\right| \\
&\quad \leqslant \max_{1\leqslant i\leqslant n-1}|u_{i+1}-u_i| \max_{1\leqslant i\leqslant n-1}|v_{i+1}-v_i|\Phi(n,k)\tilde{\Phi}(n,k).
\end{aligned}
\tag{2.6}
$$

The proof of this theorem is completed.
Summing both sides of (2.5) and (2.6) over k from 1 to n, we have

THEOREM 2.2. *Let $\{u_k\},\{v_k\},\{w_k\}$ for $k=1,\dots,n$ be three finite sequences of real numbers with $\sum_{k=1}^{n} w_k = 1$. Then we have*

$$
\begin{aligned}
&\left|\sum_{k=1}^{n} w_k u_k v_k - \left(\sum_{i=1}^{n} w_i u_i\right)\left(\sum_{i=1}^{n} w_i v_i\right)\right| \\
&\quad \leqslant \frac{1}{2}\sum_{k=1}^{n}\left(|v_k w_k| \max_{1\leqslant i\leqslant n-1}|u_{i+1}-u_i| + |u_k w_k| \max_{1\leqslant i\leqslant n-1}|v_{i+1}-v_i|\right)\Phi(n,k)
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
&\left|\sum_{k=1}^{n} w_k u_k v_k - \left(\sum_{i=1}^{n} w_i u_i\right)\left(\sum_{i=1}^{n} w_i v_i\right)\right| \\
&\quad \leqslant \max_{1\leqslant i\leqslant n-1}|u_{i+1}-u_i| \max_{1\leqslant i\leqslant n-1}|v_{i+1}-v_i| \sum_{k=1}^{n}|w_k|(\Phi(n,k))^2.
\end{aligned}
\tag{2.8}
$$

REMARK 2.3. In fact, the bounds in Theorem 2.1 and 2.2 are sharp. Here we only show that the estimates in Theorem 2.2 are all sharp and the ones in Theorem 2.1 are similar. Let $n=2$, $w_1=w_2=1/2$, $u_1<0<u_2$, $v_1<0<v_2$, then

$$\left|\sum_{k=1}^{2} w_k u_k v_k - \left(\sum_{i=1}^{2} w_i u_i\right)\left(\sum_{i=1}^{2} w_i v_i\right)\right| = \frac{1}{4}(u_2-u_1)(v_2-v_1)$$

and

$$\begin{aligned}
&\frac{1}{2}\sum_{k=1}^{2}\left(|v_k w_k||u_{i+1}-u_i| + |u_k w_k||v_{i+1}-v_i|\right)\Phi(n,k)\\
=&\frac{1}{8}\left(|v_1||u_2-u_1| + |u_1||v_2-v_1| + |v_2||u_2-u_1| + |u_2||v_2-v_1|\right)\\
=&\frac{1}{8}\left[(|v_1|+|v_2|)|u_2-u_1| + (|u_1|+|u_2|)|v_2-v_1|\right]\\
=&\frac{1}{4}|v_2-v_1||u_2-u_1| = \frac{1}{4}(u_2-u_1)(v_2-v_1),
\end{aligned}$$

which implies the equality (2.7) holds. By similar proof, we can obtained the sharpness of the inequality (2.8).

3. Further remarks

Let X be a discrete random variable taking values $x_1,x_2,\dots,x_n$ with $P(X=x_k)=w_k$ for all $k=1,\dots,n$, where w_k denotes the probability of the event $\{X=x_k\}$. Furthermore, let EX denote the mathematical expectation of X, then we have the following

THEOREM 3.1. *Let X and Y be two discrete random variables taking values $u_1,\dots,u_n$ and $v_1,\dots,v_n$ respectively. In addition, assume that $P(X=u_k)=P(Y=v_k)=w_k$ for all $k=1,\dots,n$. Then we have*

$$\begin{aligned}
&\left|u_k v_k - \frac{1}{2}(v_k EX + u_k EY)\right|\\
\leqslant&\frac{1}{2}\left(|v_k|\max_{1\leqslant i\leqslant n-1}|u_{i+1}-u_i| + |u_k|\max_{1\leqslant i\leqslant n-1}|v_{i+1}-v_i|\right)\Phi(n,k)
\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}
&|u_k v_k - (v_k EX + u_k EY) + EXEY|\\
&\leqslant \max_{1\leqslant i\leqslant n-1}|u_{i+1}-u_i|\max_{1\leqslant i\leqslant n-1}|v_{i+1}-v_i|(\Phi(n,k))^2.
\end{aligned}\tag{3.2}$$

THEOREM 3.2. *Under the assumptions of Theorem 3.1, we have*

$$\begin{aligned}
&|EXY - EXEY|\\
\leqslant&\frac{1}{2}\sum_{k=1}^{n}\left(|v_k|\max_{1\leqslant i\leqslant n-1}|u_{i+1}-u_i| + |u_k|\max_{1\leqslant i\leqslant n-1}|v_{i+1}-v_i|\right)w_k\Phi(n,k)
\end{aligned}\tag{3.3}$$

and

$$|EXY - EXEY| \leqslant \max_{1\leqslant i\leqslant n-1} |u_{i+1} - u_i| \max_{1\leqslant i\leqslant n-1} |v_{i+1} - v_i| \sum_{k=1}^{n} w_k (\Phi(n,k))^2. \tag{3.4}$$

REMARK 3.3. Here we notice that the above results give some estimate bounds for the covariance of two discrete random variables.

REMARK 3.4. By taking $w_k = 1/n$ for all $k = 1,\ldots,n$, Theorem 3.1 and Theorem 3.2 can be reduced to the results in [13].

REMARK 3.5. By taking $v_k = 1$ for all $k = 1,\ldots,n$ in Theorem 3.1, it follows that

$$|u_k - EX| \leqslant \left(\max_{1\leqslant i\leqslant n-1} |u_{i+1} - u_i| \right) \sum_{k=1}^{n} w_k \Phi(n,k).$$

Furthermore, by taking $w_k = 1/n$, we have

$$\begin{aligned} \Phi(n,k) &= \frac{1}{2n}(n^2 - 2kn - 2k + 2k^2 + n) \\ &= \frac{1}{n}\left[\frac{n^2-1}{4} + \left(k - \frac{n+1}{2} \right)^2 \right] \end{aligned}$$

and

$$\left| u_k - \frac{1}{n}\sum_{i=1}^{n} u_i \right| \leqslant \frac{1}{n}\left[\frac{n^2-1}{4} + \left(k - \frac{n+1}{2} \right)^2 \right] \max_{1\leqslant i\leqslant n-1} |u_{i+1} - u_i|.$$

In fact, the above inequality was established by Dragomir [3, Theorem 3.1] in a norm linear space.

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