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A NOTE ON A GAMMA FUNCTION INEQUALITY

PÉTER IVÁDY

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Abstract. The aim of this article is to present new inequalities which improve some gamma function inequalities of H. Alzer, Á. Baricz and N. Elezović et al.

1. Introduction

For real and positive values of x the Euler gamma function and its logarithmic derivative Ψ the so-called digamma function, are defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad \Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

Several authors have studied different properties of the gamma function which are collected, for example in [2, 3]. In particular, there exists an extensive literature on gamma functions inequalities. For more details please refer to [7] and the references therein. Among the various kinds of inequalities concerning the gamma function we will focus our attention on a special inequality of D. Kershaw. In order to improve the following inequality of W. Gautschi [6]

$$n^{1-s} < \frac{\Gamma(n+1)}{\Gamma(n+s)} < \exp\left[(1-s)\Psi(n+1)\right],$$

where $0 < s < 1$ and $n = 1, 2 \ldots$ Kershaw [9] established among others that

$$\left(x + \frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left[x - \frac{1}{2} + \left(\frac{1}{4} + s\right)^{1/2}\right]^{1-s}, \tag{1.1}$$

holds for $x > 0$ and $0 < s < 1$. Recently N.Elezović et al.[5] considered the following inequalities:

$$\frac{x}{2} < \Gamma(x)^{-\frac{1}{1-x}} < -\frac{1}{2} + \sqrt{\frac{1}{4} + x}, \quad 0 < x < 1 \tag{1.2}$$

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which can be regarded as an esimation of the value of the gamma function. In their paper they asked for “other bounds in terms of elementary functions”. The main purpose of this paper is to present a special lower and upper bound in the above sense of the authors, in terms of very simple rational functions in the $0 < x < 1$ interval. In particular, we shall show that

$$\frac{x}{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}+x}\right)^{1-x}} < \frac{x^2+1}{x+1} < \Gamma(x+1) < \frac{x^2+2}{x+2} < x\left(\frac{2}{x}\right)^{1-x}, \quad 0 < x < 1$$

holds. Finally, as a simple consequence of the theorem we derive refinements of some inequalities of H. Alzer and Á. Baricz. The numerical values given in the following sections have been calculated via Maple V Release 10.0.

2. Lemmas

In order to establish the main result of this paper we only use some simple theorems from basic algebra, such as Descartes' rule of signs and Sturm's theorem [10]. Moreover, we apply some inequalities to the logarithmic function in special intervals. Furthermore, we use well-known estimations for the Ψ function and its derivative Ψ'.

LEMMA 2.1. *Let* $0 \leqslant t \leqslant 1/5$. *Then we have*

$$p_1(t) := -2t^7 - 4t^6 - 7t^5 + t^4 - 2t^2 - 3t + 1 > 0.$$

Proof. It is easy to check that $p_1(t) > p_2(t) := -12t^3 - 2t^2 - 3t + 1$ holds. Since p_2 is strictly decreasing on $[0, 1/5]$, we get $p_2(t) > p_2(1/5) = 0.224 > 0$. □

LEMMA 2.2. *If* $1/5 \leqslant t \leqslant 1/2$, *then we have*

$$p_3(t) := 150t^9 + 270t^8 + 165t^7 + 70t^6 - 101t^5 - 150t^4 - 82t^3 - 21t^2 - 2t + 1 < 0.$$

Proof. Since $p_3(1/5) = -1.160... < 0$ it is sufficient to show that p_3 is decreasing on $[1/5, 1/2]$. Building the first two derivatives we obtain

$$p_3'(t) = 1350t^8 + 2160t^7 + 1155t^6 + 420t^5 - 505t^4 - 600t^3 - 246t^2 - 42t - 2,$$

and

$$p_3''(t) = 10\,800t^7 + 15\,120t^6 + 6930t^5 + 2100t^4 - 2020t^3 - 1800t^2 - 492t - 42.$$

According to Descartes' rule of signs, we infer that $p_3''(t)$ has at most one positive real root. Since $p_3(t)$ is continuous, moreover $p_3''(1/2) = -322.063 < 0$ and $p_3''(1) = 30\,596 > 0$, there exists a unique $t_0 \in (1/2, 1)$, such that $p_3''(t_0) = 0$. Thus, we have on $[1/5, 1/2]$, $p_3''(t) < 0$. Hence $p_3'(t)$ is strictly decreasing. Clearly, $p_3'(1/5) = -25.608 \cdots < 0$ which imply $p_3'(t) < 0$. □

LEMMA 2.3. *For all* $t \geqslant 0$, *we have*

$$\log(1+t) \leqslant \frac{t\left(t^2+21t+30\right)}{9t^2+36t+30} \leqslant \frac{t(t+6)}{4t+6}. \tag{2.1}$$

For a proof of (2.1) we refer to [12, p.667].

LEMMA 2.4. *For* $1/100 < t < 1$, *we have*

$$t\frac{33t^2+24t-57}{10t^3+57t^2+24t-1} < \log t.$$

Proof. Let us define the function h by

$$h(t) := t\frac{33t^2+24t-57}{10t^3+57t^2+24t-1} - \log t.$$

Differentiation gives

$$h'(t) = \frac{(1-t)^5(100t-1)}{t(10t^3+57t^2+24t-1)^2} > 0.$$

Since h is strictly increasing and $h(1)=0$ we have $h(t)<0$. □

LEMMA 2.5. *For* $7/20 \leqslant t \leqslant 1$, *we have*

$$p_4(t) := 253t^7+690t^6+338t^5-424t^4-390t^3+278t^2-17t-8>0.$$

In order to check that $p_4(t)$ has no real roots in $[7/20,1]$ we used the built-in functions: **sturmseq** and **sturm** of Maple V. Release 10.0. Since $p_4(1/2)=1.070$, thus $p_4(t)>0$. □

LEMMA 2.6. *For all* $t>0$, *we have*

$$\Psi(t) < \log t - \frac{1}{2t} - \frac{1}{12\left(t+1/14\right)^2}, \tag{2.2}$$

and

$$\frac{1}{t}+\frac{1}{2t^2}+\frac{1}{6\left(t+\frac{1}{14}\right)^3} < \Psi'(t) < \frac{1}{t}+\frac{1}{2t^2}+\frac{1}{6t^3}. \tag{2.3}$$

A proof of (2.2) and (2.3) is given in [8].

LEMMA 2.7. *For* $0 \leqslant t \leqslant 0.7$, *we have*

$$p_5(t) := -6t^6-21t^5-52t^4-42t^3+10t^2+27t+8>0.$$

Proof. Descartes's rule of signs reveals, that $p_5(t)$ has at most one positive real root. Since $p_5(t)$ is continuous, $p_5(0.7) = 0.673\ldots > 0$ and $p_5(0.75) = -6.348\ldots < 0$, there exists only one $t_1 \in (0.7, 0.75)$, with $p_5(t_1) = 0$. Thus we have on $[0, 0.7]$ $p_5(t) > 0$. $\square$

LEMMA 2.8. *Let* $0 < t \leqslant 1$. *Then we have*

$$\frac{24\,(1-t)}{23t^2+14t+11} \leqslant \log\frac{t^2+1}{t\,(t+1)}. \tag{2.4}$$

Proof. The proof can be given by standard calculus. $\square$

LEMMA 2.9. *For* $0.7 \leqslant t \leqslant 1$, *we have*

$$\log(t+2) - \frac{1}{2\,(t+2)} < \frac{2t}{t^2+1}. \tag{2.5}$$

Proof. Setting $t = u+1$, in the second inequality of (2.1) we obtain the following relation:

$$\begin{aligned}\log(u+2) - \frac{1}{2\,(u+2)} - \frac{2u}{u^2+1} &< \frac{(u+1)\,(u+7)}{4u+10} - \frac{1}{2\,(u+2)} - \frac{2u}{u^2+1}\\ &= \frac{p_6\,(u)}{2\,(u+2)^2\,(u^2+1)^2},\end{aligned}$$

where $p_6(u) := 2u^5 + 9u^4 + 20u^3 + 22u^2 - 14u - 11$. We shall show that p_6 is convex on $[0.7, 1]$. First, we find $p_6(0.7) = -5.2589\ldots < 0$ and $p_6(1) = -2$. For checking the convexity of p_6 it is sufficient to verify that $p_6''(u) > 0$, which is easy to see. If we set $f = p_6$ in Jensen's inequality [11, p.15] for convex function f:

$$f\,(\lambda a + (1-\lambda)\,b) \leqslant \lambda f\,(a) + (1-\lambda)\,f\,(b), \quad \text{where} \quad \lambda \in [0,1], \tag{2.6}$$

we conclude on $[0.7, 1]$, $p_6(u) < 0$. $\square$

LEMMA 2.10. *For* $1 \leqslant t \leqslant 2$, *we have*

$$p_7\,(t) := 2t^5 - t^4 - 2t^3 - 12t^2 + 5 < 0. \tag{2.7}$$

Proof. We get $p_7'(t) = 2t(5t^3 - 2t^2 - 3t - 12) =: 2t\,p_8(t)$, where the $p_8(t)$ polynomial has at most one positive real root. Clearly, $p_8(1) = -12$ and $p_8(2) = 14$, thus there exists a $t_2 \in (1, 2)$ with $p_8(t_2) = 0$. For $t \in (1, t_2)$ we therefore have $p_8(t) < 0$ such that p_7 is decreasing. For $t \in (t_2, 2)$ we obtain $p_7'(t) > 0$, hence p_7 is strictly increasing. Since $p_7(1) = -8$ and $p_7(2) = -11$, we have $p_7(t) < 0$. $\square$

3. Main results

We are now in a position to establish our main theorem.

THEOREM. *For* $0 < x \leqslant 1$, *we have*

$$\frac{x}{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}+x}\right)^{1-x}} < \frac{x^2+1}{x+1} \leqslant \Gamma(x+1), \tag{3.1}$$

and

$$\Gamma(x+1) \leqslant \frac{x^2+2}{x+2} \leqslant x\left(\frac{2}{x}\right)^{1-x}. \tag{3.2}$$

Inequalities (3.1) and the first part of (3.2) become equalities if $x=0$ *or* $x=1$. *The second inequality in (3.2) is strict for* $x=0$.

Proof. We start with the proof of the first inequality of (3.1). Although, the inequality seems to be a relatively simple one, the proof we present here is somewhat cumbersome. Writing it in the equivalent form, we get

$$f(x) := (1-x)\log\left(-\frac{1}{2}+\sqrt{\frac{1}{4}+x}\right)+\log\frac{x^2+1}{x^2+x} \geqslant 0, \quad 0 < x \leqslant 1. \tag{3.3}$$

In order to establish (3.3) we distinguish three cases.

Case 1. $0 < x \leqslant 1/5$. It is not difficult to see that in this case we can write

$$f(x) > f_1(x) := (1-x)\log\left(x-x^2\right)+\log\frac{x^2+1}{x^2+x}.$$

We shall show that $f_1(x) > 0$ holds true. Clearly, $\lim_{x\to 0} f_1(x) = 0$. A simple calculation gives $f_1(1/5) = 0.00027\ldots > 0$. Now we show that f_1 is concave on $[0,1/5]$, from which the assertion follows, since in (2.6) the reversed inequality holds. Differentiation yields

$$f_1''(x) = \frac{P(x)}{x(x-1)(x+1)^2(x^2+1)^2},$$

where $P(x) = -2x^7-4x^6-7x^5+x^4-2x^2-3x+1$. According to Lemma 2.1 we get $P(x) > 0$, i.e. $f_1''(x) < 0$.

Case 2. $1/5 \leqslant x \leqslant 1/2$. Now we can write

$$f(x) > (1-x)f_2(x) := (1-x)\left[\log\frac{2x^2+x}{x^2+3x+1}+\frac{1}{1-x}\log\frac{x^2+1}{x^2+x}\right],$$

and must show that $f_2(x) > 0$ holds true. Since $f_2(1/2) = 0.0100\ldots > 0$, it is sufficient to show that f_2 is strictly decreasing i.e. $f_2'(x) < 0$. Differentiation gives

$$f_2'(x) = \frac{1}{1-x}\left[-\frac{5x^5+2x^4-2x^3+11x^2+11x+3}{(x+1)(2x+1)(x^2+1)(x^2+3x+1)}+\frac{1}{1-x}\log\frac{x^2+1}{x^2+x}\right]$$

$$=: \frac{1}{1-x} f_3(x).$$

In order to get a suitable upper bound for f_3, setting $t = (1-x)/(x^2+x)$, and applying the first inequality of (2.1) we obtain

$$\begin{aligned} f_3(x) &< -\frac{5x^5+2x^4-2x^3+11x^2+11x+3}{(x+1)(2x+1)(x^2+1)(x^2+3x+1)} \\ &\quad + \frac{30x^4+39x^3+31x^2+19x+1}{3(x^2+x)(10x^4+8x^3+13x^2+6x+3)} \\ &= \frac{(1-x)P_1(x)}{3x(x+1)(2x+1)(x^2+1)(x^2+3x+1)(10x^4+8x^3+13x^2+6x+3)}, \end{aligned}$$

where $P_1(x) = 150x^9+270x^8+165x^7+70x^6-101x^5-150x^4-82x^3-21x^2-2x+1$. By using Lemma 2.2 we infer $f_3(x) < 0$, hence $f_2'(x) < 0$.

Case 3. $1/2 \leqslant x \leqslant 1$. Applying (2.4) we obtain

$$f(x) > (1-x)\left[\log\left(-\frac{1}{2}+\sqrt{\frac{1}{4}+x}\right) + \frac{24}{23x^2+14x+11}\right].$$

Substituting $-1/2+\sqrt{1/4+x} = u$, $(0.3660\ldots = (\sqrt{3}-1)/2 \leqslant u \leqslant (\sqrt{5}-1)/2 = 0.6180\ldots)$, and define the auxiliary function f_4 by:

$$f_4(u) := \log u + \frac{24}{23(u^2+u)^2+14(u^2+u)+11}.$$

It is sufficient to verify that $f_4(u) > 0$, holds. Application of Lemma 2.4 leads to

$$\begin{aligned} f_4(u) &> u\frac{33u^2+24u-57}{10u^3+57u^2+24u-1} + \frac{24}{23(u^2+u)^2+14(u^2+u)+11} \\ &= \frac{3P_2(u)}{(10u^3+57u^2+24u-1)\left(23(u^2+u)^2+14(u^2+u)+11\right)}, \end{aligned}$$

where $P_2(u) = 253u^7+690u^6+338u^5-424u^4-390u^3+278u^2-17u-8$. In view of Lemma 2.5 we get $f_4(u) > 0$, thus the proof of the first inequality of (3.1) is complete.

In order to establish the second inequality of (3.1) we consider two cases.

Case 1. $0 \leqslant x \leqslant 0.7$. Let the function g be defined by

$$g(x) := \log\frac{x^2+1}{x+1} - \log\Gamma(x+1).$$

We will show that $g(x)$ is convex on that interval. According to (2.6) the inequality $g(x) \leqslant 0$ will follow. A short calculation gives $g(0) = 0$ and $g(0.7) = -0.036\ldots$. For checking the convexity of g it is sufficient to show that $g'' > 0$. Differentiation yields

$$g''(x) = \frac{-x^4-4x^3+2x^2+4x+3}{(x+1)^2(x^2+1)^2} - \Psi'(x+1).$$

Setting in the second inequality of (2.3) $t = x+1$, we obtain

$$g''(x) > \frac{-x^4+4x^3+2x^2+4x+3}{(x+1)^2(x^2+1)^2} - \left(\frac{1}{x+1} + \frac{1}{2(x+1)^2} + \frac{1}{6(x+1)^3}\right)$$
$$= \frac{P_3(x)}{6(x+1)^3(x^2+1)^3},$$

where $P_3(x) := -6x^6 - 21x^5 - 52x^4 - 42x^3 + 10x^2 + 27x + 8$. Applying Lemma 2.7 we conclude $g''(x) > 0$, which proves the assertion.

Case 2. $0.7 \leqslant x \leqslant 1$. Let g_1 be defined by

$$g_1(x) := \log\left(x^2+1\right) - \log\left(\Gamma(x+2)\right).$$

Since $g_1(1) = 0$, it is sufficient to show, that $g_1'(x) > 0)$ on $(0.7, 1]$. Differentiation yields

$$g_1'(x) = \frac{2x}{x^2+1} - \Psi(x+2).$$

Using a weaker form of (2.2) – setting $t = x+2$, and applying (2.5) – leads to

$$\Psi(x+2) < \log(x+2) - \frac{1}{2(x+2)} \leqslant \frac{2x}{x^2+1},$$

i.e. $g_1'(x) > 0$, hence the proof of (3.1) is complete.

We now proceed to prove the relations in (3.2). First we consider the left side of (3.2). Let g_2 be defined by

$$g_2(x) := \log\left(\Gamma(x+1)\right) - \log\frac{x^2+2}{x+2}.$$

Clearly, $g_2(0) = 0$ and $g_2(1) = 0$. We shall show that g_2 is convex on $[0,1]$, thus in view of (2.6), $g_2(x) \leqslant 0$ will follow. After some computations we get

$$g_2''(x) = \Psi'(x+1) + \frac{x^4+8x^3-16x-20}{(x+2)^2(x^2+2)^2}.$$

To prove that g_2'' is positive we set in the first part of (2.3) $t = x+1$, and show that

$$g_2''(x) > \frac{1}{x+1} + \frac{1}{2(x+1)^2} + \frac{1}{6\left(x+1+1/14\right)^3} + \frac{x^4+8x^3-16x-20}{(x+2)^2(x^2+2)^2}$$
$$= \frac{P_4(x)}{6(x+1)^2(x+2)^2(x^2+2)^2(14x+15)^3} > 0,$$

holds, where

$$P_4(x) := 16464x^{10} + 159936x^9 + 798560x^8 + 2416224x^7 + 4513177x^6$$

$$+4\,942\,620x^5+2\,536\,146x^4-194\,496x^3-630\,428x^2+34\,512x+124\,904.$$

We can reduce the obvious complexity of $P_4(x)$. A little calculation reveals that $P_4(x)>16\,384xr(x)$, where we have $r(x)=154x^3-12x^2-39x+9$. Determining the minimum value of $r(x)$ we get $r(x)\geqslant r(x_0)=\left(47\,323-1009\sqrt{2018}\right)/5929=0.3367\ldots>0$, where $x_0=1/154\left(4+\sqrt{2018}\right)=0.3176\ldots>0$, which proves $g_2''(x)>0$. To end the proof of (3.2) we consider the function

$$g_3(x):=\log\frac{x^2+2}{x\,(x+2)}-(1-x)\log\frac{2}{x}.$$

Since $g_3(1)=0$ and $g_3'(x)>0$ the assertion will follow. A direct computation gives

$$g_3'(x)=-\frac{x^3+x^2-2x+6}{(x+2)\,(x^2+2)}+\log\frac{2}{x}$$

and

$$g_3''(x)=-\frac{x^6+5x^5+16x^4+16x^3+4x^2-4x+16}{x\,(x+2)^2\,(x^2+2)^2}.$$

It is clear that for $x\in(0,1]\,,g_3''(x)<0$ holds. Therefore $g_3'(x)$ is strictly decreasing on $(0,1]$. Since $g_3'(1)=-2/3+\log 2=0.0264\ldots>0$, we obtain $g_3'(x)>0$. Let

$$L(x)=\frac{x^2+2}{x+2}\qquad\text{and}\qquad R(x)=x\left(\frac{2}{x}\right)^{1-x}.$$

It is not difficult to see that

$$L(0)=1\qquad\text{and}\qquad R(0)=\lim_{x\to 0}R(x)=2.$$

So, the second inequality in (3.2) should be strict for $x=0$.

This completes the proof of the Theorem. □

In the next section we will give some simple consequences of the theorem.

4. Refinements

Alzer [1] proved that for all $x>0,(x-1/2)\log x-x+\log\sqrt{2\pi}<\log\Gamma(x)$ holds. The first corollary shows, that the logarithmic version of the right side of (1.2) gives a set of improvements — according the Theorem — of the previously mentioned inequality of Alzer's for interval $(0,1]$.

Recently Á.Baricz [4] proves — among others — that if $x\in(0,1]$, then

$$\frac{1-x}{1+x}\frac{e^{(1-\gamma)x}}{x}\leqslant\Gamma(x),\tag{4.1}$$

holds, where $\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n\right)=0.57721\ldots$ is the Euler-Mascheroni constant. The second corollary gives an improvement of (4.1).

COROLLARY 4.1 *For* $0<x\leqslant 1$, *we have*

$$F(x):=\left(x-\frac{1}{2}\right)\log x-x+\frac{1}{2}\log 2\pi+(1-x)\log\left(-\frac{1}{2}+\sqrt{\frac{1}{4}+x}\right)<0. \qquad (4.2)$$

Proof. The proof of (4.2) will be split up into two cases.

Case 1. $0<x\leqslant 11/50$. Some computation gives

$$F'(x)=\frac{2\left(2x-2x^2\right)+\sqrt{4x+1}-(4x+1)}{2x\left(4x+1-\sqrt{4x+1}\right)}+\log\frac{2x}{\sqrt{4x+1}-1}. \qquad (4.3)$$

Setting $\sqrt{4x+1}=u$, $(1<u\leqslant u_0=\sqrt{44/50+1}=1.3711\ldots)$, we get from (4.3)

$$F_1(u):=\frac{-u^3-u^2+u+5}{2u^3-2u}+\log\frac{u+1}{2}.$$

Differentiation yields

$$F_1'(u)=\frac{2u^5-u^4-2u^3-12u^2+5}{2(u+1)^2\left(u^2-u\right)^2}.$$

Applying Lemma 2.10 we get $F_1'(u)<0$, i.e. F_1 is strictly decreasing on $(1,u_0)$. Since $F_1(u_0)=0.9631\ldots>0$, we obtain $F_1(u)>0$, i.e. $F'(x)>0$, thus F is strictly increasing. Since $F(11/50)=-0.1908\ldots<0$, we have $F(x)<0$.

Case 2. $11/50<x\leqslant 1$. Using the arithmetic-geometric mean inequality, we obtain: $-1/2+\sqrt{1/4+x}\leqslant(1+4x)/8$. This leads to the sharper relation

$$F(x)<\left(x-\frac{1}{2}\right)\log x-x+\log\sqrt{2\pi}+(1-x)\log\frac{4x+1}{8}=:F_2(x)\ \text{say}.$$

We will show that $F_2(x)<0$ holds. Next, we obtain

$$F_2'(x)=-\frac{8x^2-4x+1}{8x^2+2x}+\log\frac{8x}{4x+1}.$$

It is obviously that for $11/50<x\leqslant 1/4, F_2'(x)<0$ holds. The same is valid also for $1/4<x\leqslant 1$. In that interval we have – after setting in the right side of (2.1) $t=(4x-1)/(4x+1)$ – the following estimation:

$$\begin{aligned}F_2'(x)&<-\frac{8x^2-4x+1}{8x^2+2x}+\frac{(4x-1)(28x+5)}{2(4x+1)(20x+1)}\\&=\frac{-48x^3+64x^2-21x-1}{2x(4x+1)(20x+1)}.\end{aligned}$$

According to the arithmetic-geometric mean inequality, the numerator is negative, thus F_2 is strictly decreasing on $[11/50,1]$. Since $F_2(11/50)=-0.0066\ldots<0$, we infer $F_2(x)<0$, which proves Corollary 4.1. □

COROLLARY 4.2 *For* $0 < x \leqslant 1$, *we have*

$$\frac{1-x}{1+x}\frac{e^{(1-\gamma)x}}{x} < \frac{x^2+1}{x(x+1)}.$$

Proof. Let $G(x) := \log(1-x) + (1-\gamma)x - \log\left(x^2+1\right)$. Differentiation gives

$$G'(x) = 1-\gamma - \frac{-x^2+2x+1}{(1-x)(x^2+1)} < \frac{1}{2} - \frac{-x^2+2x+1}{(1-x)(x^2+1)}$$
$$= -\frac{x^3-3x^2+5x+1}{2(1-x)(x^2+1)} < 0,$$

i.e. G is strictly decreasing on $(0,1]$. Since $G(0)=0$, thus $G(x)<0$. □

REMARK. We mention that the logarithmic version of (4.1) and Alzer's inequality on $(0,1]$ are not comparable to each other. The same is also true for the right side of (1.2) and the left side of (4.1).

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Péter Ivády
Tímár u.21
1034 Budapest
Hungary
e-mail: ivadyp@t-online.hu
ivadyp@invitel.hu