

ESTIMATES OF THE INTEGRAL REMAINDERS IN SEVERAL NUMERICAL INTEGRAL FORMULAS USING THE HENSTOCK–KURZWEIL INTEGRAL

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Abstract. Some integral remainders of Trapezoidal, Corrective Trapezoidal and Simpson formulas are given. The Hölder inequality and Henstock-Kurzweil integral are used to estimate the remainders in terms of Alexiewicz and Lebesgue p -norms.

1. Introduction

In 1957-1958, R. Henstock and J. Kurzweil, independently, gave a Riemann-type integral called the Henstock-Kurzweil integral (or sometimes called Henstock integral or generalized Riemann integral). With Henstock-Kurzweil definition, the locally fine partition leads to a more generalized Riemann integral and such definition enables us to integrate some non-absolute integrals. In short, the Henstock-Kurzweil integral extends the Riemann, improper Riemann, Lebesgue and Newton integrals. In recent years a number of authors considered generalizations of some classical and some new quadrature rules. In this paper we use the Henstock-Kurzweil integral with Lebesgue p -norm and Alexiewicz norm to estimate the integral remainders by using following three approximation rules, namely, Trapezoidal, Corrective Trapezoidal and Simpson Rules. This is the first time that Henstock-Kurzweil integral is introduced into these three estimations of respective error bounds.

Let $[a, b]$ be a compact interval in \mathbb{R} and $f : [a, b] \rightarrow \mathbb{R}$. For $n = 1$ or 3 and when these three approximation rules are used to compute the integral value of f on $[a, b]$, the remainders are estimated using the Lebesgue p -norm of $f^{(n+1)}$ when $f^{(n)}$ is AC or $f^{(n+1)}$ is Lebesgue integrable. When the assumption is that $f^{(n)}$ is ACG^* or $f^{(n+1)}$ is Henstock-Kurzweil integrable, the Alexiewicz norm is applied to the estimation of the remainders. Under these two different norms of estimations, although we get the same order of error estimations, the second estimates are useful since the smoothness imposed on the integrand is less restrictive than the one required by the first condition. This is very useful for the error estimation of numerical integration in the area of finite element methods.

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2. Preliminaries

Let $[a, b]$ be a compact interval in \mathbb{R} , μ stand for the Lebesgue measure.

A *partial partition* $D = \{(I, \xi)\}$ in $[a, b]$ is a finite collection of interval-point pairs (I, ξ) with non-overlapping intervals $I \subset [a, b]$, $\xi \in [a, b]$ being the associated point of I .

We write $D = \{(I, \xi)\}$.

A *partial partition* $D = \{(I, \xi)\}$ in $[a, b]$ is a *partition* of $[a, b]$ if the union of all the intervals I from D equals $[a, b]$.

Let δ be a positive function defined on the interval $[a, b]$ called a *gauge* on $[a, b]$. A partial partition $D = \{(I, \xi)\}$ is said to be δ -fine if for each interval-point pair $(I, \xi) \in D$ we have $I \subset B(\xi, \delta(\xi))$ where $B(\xi, \delta(\xi)) = (\xi - \delta(\xi), \xi + \delta(\xi))$.

Given a partition $D = \{(I, \xi)\}$ we write

$$f(D) = (D) \sum f(\xi) \mu(I)$$

for integral sum over D , whenever $f : [a, b] \rightarrow \mathbb{R}$ and $\mu(I)$ is the length of the interval I .

DEFINITION 1. A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be *Henstock-Kurzweil integrable* on $[a, b]$ if there exists an S_f such that for every $\varepsilon > 0$, there exists a gauge δ on $[a, b]$ such that for every δ -fine partition $D = \{(I, \xi)\}$ of $[a, b]$, we have

$$|(D) \sum f(\xi) \mu(I) - S_f| < \varepsilon.$$

We denote the Henstock-Kurzweil integral I by (HK) $\int_a^b f(x) dx$. If a subset $E \subseteq I$ is given, then we write (HK) $\int_E f = (HK) \int_a^b f \chi_E = F(E)$ for the Henstock-Kurzweil integral of f on E and F is the primitive of f .

DEFINITION 2. ([6]) Let $F : [a, b] \rightarrow \mathbb{R}$ and let $E \subseteq [a, b]$.

(a) The variation of F on $[a, b]$ is defined by

$$V(F, [a, b]) = \sup \left\{ \sum_{i=1}^n |F(d_i) - F(c_i)| \right\}$$

where the supremum is over all finite collections $\{[c_i, d_i] : 1 \leq i \leq n\}$ of non-overlapping intervals in $[a, b]$. The function F is of bounded variation on $[a, b]$ if $V(F, [a, b])$ is finite.

(b) The oscillation of F on the interval $[a, b]$ is defined by

$$\omega(F, [a, b]) = \sup\{|F(y) - F(x)| : a \leq x < y \leq b\}.$$

(c) The function F is absolutely continuous on E (F is AC on E) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^n |F(d_i) - F(c_i)| < \varepsilon$ whenever $\{[c_i, d_i] : 1 \leq i \leq n\}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^n (d_i - c_i) < \delta$.

(d) The function F is absolutely continuous in the restricted sense on E (F is AC* on E) if F is bounded on an interval that contains E and for each $\varepsilon > 0$ there

exists $\delta > 0$ such that $\sum_{i=1}^n \omega(F, [c_i, d_i]) < \varepsilon$ whenever $\{[c_i, d_i] : 1 \leq i \leq n\}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^n (d_i - c_i) < \delta$.

(e) The function F is generalized absolutely continuous in the restricted sense on E (F is ACG^* on E) if $F|_E$ is continuous on E and E can be written as the union of countable sets on each of which F is AC^* .

From Definition 2, we know that $AC \subsetneq ACG^*$. The set of absolutely continuous functions is a proper subset of ACG^* .

THEOREM 3. ([5]) $f : [a, b] \rightarrow \mathbb{R}$ is Henstock-Kurzweil integrable on $[a, b]$ if and only if there exist an ACG^* function F on $[a, b]$ such that $F' = f$ almost everywhere and $\int_a^x f = F(x) - F(a)$ for any $x \in [a, b]$.

We denote the space of Lebesgue and Hestock-Kurzweil integrable functions on $[a, b]$ by L and HK , respectively. Let n be a positive integer. If $f^{(n)} \in AC$ then $f^{(n+1)} \in L$. If $f^{(n)} \in ACG^*$ then $f^{(n+1)} \in HK$.

Unless otherwise stated, all undefined terms and notations can be found [5]. For a detailed discussion of Henstock-Kurzweil integral see [4], [5], [6] and [10].

If $f : [a, b] \rightarrow \mathbb{R}$ is Riemann or Henstock-Kurzweil integrable, we use $I(f) = \int_a^b f(t)dt$ and $I_{HK}(f) = (HK) \int_a^b f(t)dt$ to denote its Riemann and Henstock-Kurzweil integrals respectively.

In the literature [7], the author gave the following formulas:

2.1 Trapezoidal Formula. Let $f : [a, b] \rightarrow \mathbb{R}$. If f is twice differentiable on (a, b) and the second derivative is Riemann integrable on $[a, b]$, then we have

$$I(f) = T(f) + e^T, \quad (2.1)$$

where

$$T(f) = \frac{b-a}{2} [f(a) + f(b)],$$

$$e^T = \frac{1}{2} \int_a^b (t-a)(t-b)f''(t)dt.$$

2.2 Corrective Trapezoidal Formula. Let $f : [a, b] \rightarrow \mathbb{R}$. If f is four-time differentiable on (a, b) and the fourth derivative is Riemann integrable on $[a, b]$, and both $f'(a)$ and $f'(b)$ exist, then we have

$$I(f) = CT(f) + e^{CT}, \quad (2.2)$$

where

$$CT(f) = \frac{b-a}{2} [f(a) + f(b)] + \frac{(b-a)^2}{12} [f'(a) - f'(b)],$$

$$e^{CT} = \frac{1}{4!} \int_a^b (t-a)^2(t-b)^2 f^{(4)}(t)dt.$$

2.3 Simpson Formula. Let $f : [a, b] \rightarrow \mathbb{R}$. If f is four-time differentiable on (a, b) and the fourth derivative is Riemann integrable on $[a, b]$, then we have

$$I(f) = S(f) + e^S, \tag{2.3}$$

where

$$S(f) = \frac{b-a}{6} [f(a) + 4f(\frac{a+b}{2}) + f(b)],$$

$$e^S = \frac{1}{6 \cdot 3!} \left\{ \int_a^{\frac{a+b}{2}} (t-a)^3 [(t-b) + \frac{t-a}{2}] f^{(4)}(t) dt \right.$$

$$\left. + \int_{\frac{a+b}{2}}^b (t-b)^3 [(t-a) + \frac{t-b}{2}] f^{(4)}(t) dt \right\}.$$

REMARK 4. The equation (2.1) given in [8] is applied on Riemann integrable functions. By applying integration by parts technique on e^T , equation (2.2) follows and by using the Corrective Trapezoidal Formula for $\int_a^b f(t) dt = \int_a^{\frac{a+b}{2}} f(t) dt + \int_{\frac{a+b}{2}}^b f(t) dt$ we also have equation (2.3).

3. The Main Results

For a real-valued function $g : [a, b] \rightarrow \mathbb{R}$, if g is Lebesgue integrable; for simplicity, we denote its integral over $[a, b]$ by $\int_a^b g(t) dt$ with no confusion. We further define

$$\|g\|_{[a,b];p} := \left| \int_a^b |g(t)|^p dt \right|^{\frac{1}{p}}, p \geq 1,$$

and

$$\|g\|_{[a,b];\infty} := \operatorname{ess\,sup}_{t \in [a,b]} |g(t)|.$$

We state the following theorems by using the Lebesgue p -norm and Hölder's inequality.

THEOREM 5. Let $f : [a, b] \rightarrow \mathbb{R}$. Assume the equation (2.1) holds for $f^{(2)} \in L_p[a, b]$, with $1 \leq p \leq +\infty$. Then we have

$$|e^T| \leq \begin{cases} \frac{(b-a)^2}{8} \|f^{(2)}(t)\|_{[a,b];1} & \text{if } f^{(2)}(t) \in L_1[a, b]; \\ \frac{1}{2} \frac{(b-a)^{2+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|f^{(2)}(t)\|_{[a,b];p} & \text{if } f^{(2)}(t) \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^3}{12} \|f^{(2)}(t)\|_{[a,b];\infty} & \text{if } f^{(2)}(t) \in L_\infty[a, b]. \end{cases} \tag{3.4}$$

Proof. It follows from equation (2.1) that

$$|e^T| = \left| \frac{1}{2} \int_a^b (t-a)(t-b)f^{(2)}(t)dt \right| \leq \frac{1}{2} \left| \int_a^b |t-a||t-b||f^{(2)}(t)|dt \right| := M$$

If $f^{(2)}(t) \in L_1[a, b]$, then

$$M \leq \frac{1}{2} \sup_{t \in [a, b]} |t-a||t-b| \left| \int_a^b |f^{(2)}(t)|dt \right| = \frac{(b-a)^2}{8} \|f^{(2)}(t)\|_{[a, b]; 1},$$

and this gives the first inequality of equation (3.4).

Next we apply Hölder inequality and the Second Mean Value theorem on $f^{(2)}(t) \in L_p[a, b]$, with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$\begin{aligned} M &\leq \frac{1}{2} \left| \int_a^b |t-a|^q |t-b|^q dt \right|^{\frac{1}{q}} \left| \int_a^b |f^{(2)}(t)|^p dt \right|^{\frac{1}{p}} \\ &= \frac{1}{2} \left| \int_a^b (t-a)^q d(b-t)^{q+1} \frac{1}{q+1} \right|^{\frac{1}{q}} \left| \int_a^b |f^{(2)}(t)|^p dt \right|^{\frac{1}{p}} \\ &= \frac{1}{2} \left| \frac{(b-a)^q}{q+1} \int_{\xi}^b d(b-t)^{q+1} \right|^{\frac{1}{q}} \left| \int_a^b |f^{(2)}(t)|^p dt \right|^{\frac{1}{p}} \\ &= \frac{1}{2} \left| \frac{(b-a)^q (b-\xi)^{q+1}}{q+1} \right|^{\frac{1}{q}} \left| \int_a^b |f^{(2)}(t)|^p dt \right|^{\frac{1}{p}} \\ &\leq \sup_{\xi \in [a, b]} \frac{1}{2} \left| \frac{(b-a)^q (b-\xi)^{q+1}}{q+1} \right|^{\frac{1}{q}} \left| \int_a^b |f^{(2)}(t)|^p dt \right|^{\frac{1}{p}} \\ &= \frac{1}{2} \frac{(b-a)^{2+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \|f^{(2)}(t)\|_{[a, b]; p}, \end{aligned}$$

and the second inequality is proved.

Finally, if $f^{(2)} \in L_{\infty}[a, b]$, then

$$\begin{aligned} M &\leq \operatorname{ess\,sup}_{t \in [a, b]} |f^{(2)}(t)| \left| \frac{1}{2} \int_a^b (t-a)(t-b)dt \right| \\ &= \frac{(b-a)^3}{12} \|f^{(2)}(t)\|_{[a, b]; \infty} \end{aligned}$$

and the theorem is proved.

For Corrective Trapezoidal Formula and Simpson Formula we have similar estimations as follows:

THEOREM 6. Let $f : [a, b] \rightarrow \mathbb{R}$. Assume the equation (2.2) holds for $f^{(4)} \in L_p[a, b]$, with $1 \leq p \leq +\infty$. Then we have

$$|e^{CT}| \leq \begin{cases} \frac{(b-a)^4}{16 \cdot 4!} \|f^{(4)}(t)\|_{[a,b];1} & \text{if } f^{(4)}(t) \in L_1[a, b]; \\ \frac{1}{4!} \frac{(b-a)^{4+\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}} \|f^{(4)}(t)\|_{[a,b];p} & \text{if } f^{(4)}(t) \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^5}{30 \cdot 4!} \|f^{(4)}(t)\|_{[a,b];\infty} & \text{if } f^{(4)}(t) \in L_\infty[a, b]. \end{cases} \quad (3.5)$$

The proof is similar to that of Theorem 5, which we omit here.

THEOREM 7. Let $f : [a, b] \rightarrow \mathbb{R}$. Assume the equation (2.3) holds for $f^{(4)} \in L_p[a, b] (1 \leq p \leq +\infty)$. Then we have

$$|e^S| \leq \begin{cases} \frac{(b-a)^4}{2^5 \cdot 6 \cdot 3!} \|f^{(4)}(t)\|_{[a,b];1} & \text{if } f^{(4)}(t) \in L_1[a, b]; \\ \frac{(b-a)^{4+\frac{1}{q}} \cdot (1-4^{q+1})^{\frac{1}{q}}}{96 \cdot 3! \cdot 6^{\frac{1}{q}} (q+1)^{\frac{1}{q}}} \|f^{(4)}(t)\|_{[a,b];p} & \text{if } f^{(4)}(t) \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(b-a)^5}{960 \cdot 3!} \|f^{(4)}(t)\|_{[a,b];\infty} & \text{if } f^{(4)}(t) \in L_\infty[a, b]. \end{cases} \quad (3.6)$$

Proof. It follows from the equation (2.3) that

$$|e^S| \leq \frac{1}{6 \cdot 3!} \left\{ \left| \int_a^{\frac{a+b}{2}} |t-a|^3 \cdot |t-b + \frac{t-a}{2}| \cdot |f^{(4)}(t)| dt \right| + \left| \int_{\frac{a+b}{2}}^b |t-b|^3 \cdot |t-a + \frac{t-b}{2}| \cdot |f^{(4)}(t)| dt \right| \right\} := N.$$

If $f^{(4)}(t) \in L_1[a, b]$, then

$$\begin{aligned} N &\leq \frac{1}{6 \cdot 3!} \left\{ \sup_{t \in [a, \frac{a+b}{2}]} |t-a|^3 \cdot |t-b + \frac{t-a}{2}| \cdot \left| \int_a^{\frac{a+b}{2}} |f^{(4)}(t)| dt \right| \right. \\ &\quad \left. + \sup_{t \in [\frac{a+b}{2}, b]} |t-b|^3 \cdot |t-a + \frac{t-b}{2}| \cdot \left| \int_{\frac{a+b}{2}}^b |f^{(4)}(t)| dt \right| \right\} \\ &= \frac{1}{6 \cdot 3!} \left\{ \frac{(b-a)^4}{2^5} \|f^{(4)}(t)\|_{[a, \frac{a+b}{2}];1} + \frac{(b-a)^4}{2^5} \|f^{(4)}(t)\|_{[\frac{a+b}{2}, b];1} \right\} \\ &= \frac{(b-a)^4}{2^5 \cdot 6 \cdot 3!} \|f^{(4)}(t)\|_{[a,b];1}, \end{aligned}$$

and this gives the first inequality of equation (3.6).

If $f^{(4)}(t) \in L_p[a, b]$, where $p > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, by using Hölder inequality and the Second Mean Value theorem, we deduce that

$$\begin{aligned}
N &\leq \frac{1}{6 \cdot 3!} \left\{ \left| \int_a^{\frac{a+b}{2}} |t-a|^{3q} \cdot |t-b + \frac{t-a}{2}|^q dt \right|^{\frac{1}{q}} \left| \int_a^{\frac{a+b}{2}} |f^{(4)}(t)|^p dt \right|^{\frac{1}{p}} \right. \\
&\quad \left. + \left| \int_{\frac{a+b}{2}}^b |t-b|^{3q} |t-a + \frac{t-b}{2}|^q dt \right|^{\frac{1}{q}} \left| \int_{\frac{a+b}{2}}^b |f^{(4)}(t)|^p dt \right|^{\frac{1}{p}} \right\} \\
&= \frac{1}{6 \cdot 3!} \left\{ \left| \int_a^{\frac{a+b}{2}} (t-a)^{3q} \cdot |t-b + \frac{t-a}{2}|^q dt \right|^{\frac{1}{q}} \left| \int_a^{\frac{a+b}{2}} |f^{(4)}(t)|^p dt \right|^{\frac{1}{p}} \right. \\
&\quad \left. + \left| \int_{\frac{a+b}{2}}^b (b-t)^{3q} |t-a + \frac{t-b}{2}|^q dt \right|^{\frac{1}{q}} \left| \int_{\frac{a+b}{2}}^b |f^{(4)}(t)|^p dt \right|^{\frac{1}{p}} \right\} \\
&= \frac{1}{6 \cdot 3!} \left\{ \left| \int_a^{\frac{a+b}{2}} (t-a)^{3q} d\left(\frac{a+2b-3t}{2}\right)^{q+1} \frac{2}{3(q+1)} \right|^{\frac{1}{q}} \left| \int_a^{\frac{a+b}{2}} |f^{(4)}(t)|^p dt \right|^{\frac{1}{p}} \right. \\
&\quad \left. + \left| \int_{\frac{a+b}{2}}^b (b-t)^{3q} d\left(\frac{3t-2a-b}{2}\right)^{q+1} \frac{2}{3(q+1)} \right|^{\frac{1}{q}} \left| \int_{\frac{a+b}{2}}^b |f^{(4)}(t)|^p dt \right|^{\frac{1}{p}} \right\} \\
&= \frac{1}{6 \cdot 3!} \left\{ \left| \left(\frac{b-a}{2}\right)^{3q} \int_{\xi}^{\frac{a+b}{2}} d\left(\frac{a+2b-3t}{2}\right)^{q+1} \frac{2}{3(q+1)} \right|^{\frac{1}{q}} \left| \int_a^{\frac{a+b}{2}} |f^{(4)}(t)|^p dt \right|^{\frac{1}{p}} \right. \\
&\quad \left. + \left| \left(\frac{b-a}{2}\right)^{3q} \int_{\frac{a+b}{2}}^{\eta} d\left(\frac{3t-2a-b}{2}\right)^{q+1} \frac{2}{3(q+1)} \right|^{\frac{1}{q}} \left| \int_{\frac{a+b}{2}}^b |f^{(4)}(t)|^p dt \right|^{\frac{1}{p}} \right\} \\
&= \frac{1}{6 \cdot 3!} \left\{ \left| \left(\frac{b-a}{2}\right)^{3q} \frac{2}{3(q+1)} \left[\left(\frac{b-a}{4}\right)^{q+1} - \left(\frac{a+2b-3\xi}{2}\right)^{q+1} \right] \right|^{\frac{1}{q}} \left| \int_a^{\frac{a+b}{2}} |f^{(4)}(t)|^p dt \right|^{\frac{1}{p}} \right. \\
&\quad \left. + \left| \left(\frac{b-a}{2}\right)^{3q} \frac{2}{3(q+1)} \left[\left(\frac{3\eta-2a-b}{2}\right)^{q+1} - \left(\frac{b-a}{4}\right)^{q+1} \right] \right|^{\frac{1}{q}} \left| \int_{\frac{a+b}{2}}^b |f^{(4)}(t)|^p dt \right|^{\frac{1}{p}} \right\} \\
&\leq \frac{1}{6 \cdot 3!} \sup_{\substack{\xi \in [a, \frac{a+b}{2}] \\ \eta \in [\frac{a+b}{2}, b]}} \left\{ \left| \left(\frac{b-a}{2}\right)^{3q} \frac{2}{3(q+1)} \left[\left(\frac{b-a}{4}\right)^{q+1} - \left(\frac{a+2b-3\xi}{2}\right)^{q+1} \right] \right|^{\frac{1}{q}} \left| \int_a^{\frac{a+b}{2}} |f^{(4)}(t)|^p dt \right|^{\frac{1}{p}} \right. \\
&\quad \left. + \left| \left(\frac{b-a}{2}\right)^{3q} \frac{2}{3(q+1)} \left[\left(\frac{3\eta-2a-b}{2}\right)^{q+1} - \left(\frac{b-a}{4}\right)^{q+1} \right] \right|^{\frac{1}{q}} \left| \int_{\frac{a+b}{2}}^b |f^{(4)}(t)|^p dt \right|^{\frac{1}{p}} \right\} \\
&\leq \frac{1}{6 \cdot 3!} \left| \left(\frac{b-a}{2}\right)^{3q} \cdot \frac{2}{3(q+1)} \cdot \frac{(b-a)^{q+1}}{4^{q+1}} (1-q^{q+1}) \right|^{\frac{1}{q}} \left\{ \left| \int_a^{\frac{a+b}{2}} |f^{(4)}(t)|^p dt \right|^{\frac{1}{p}} \right. \\
&\quad \left. + \left| \int_{\frac{a+b}{2}}^b |f^{(4)}(t)|^p dt \right|^{\frac{1}{p}} \right\} \\
&\leq \frac{1}{6 \cdot 3!} \left| \left(\frac{b-a}{2}\right)^{3q} \cdot \frac{2}{3(q+1)} \cdot \frac{(b-a)^{q+1}}{4^{q+1}} (1-q^{q+1}) \right|^{\frac{1}{q}} \cdot 2 \cdot \left| \int_a^b |f^{(4)}(t)|^p dt \right|^{\frac{1}{p}} \\
&\leq \frac{(b-a)^{4+\frac{1}{q}} \cdot (1-4^{q+1})^{\frac{1}{q}}}{96 \cdot 3! \cdot 6^{\frac{1}{q}} (q+1)^{\frac{1}{q}}} \|f^{(4)}\|_{[a,b];p}.
\end{aligned}$$

Finally, we have for $f^{(4)} \in L_\infty[a, b]$, that

$$\begin{aligned} N &\leq \frac{1}{6 \cdot 3!} \left\{ \operatorname{ess\,sup}_{t \in [a, \frac{a+b}{2}]} |f^{(4)}(t)| \cdot \left| \int_a^{\frac{a+b}{2}} |t-a|^3 |t-b + \frac{t-a}{2}| dt \right| \right. \\ &\quad \left. + \operatorname{ess\,sup}_{t \in [\frac{a+b}{2}, b]} |f^{(4)}(t)| \cdot \left| \int_{\frac{a+b}{2}}^b |t-b| |t-a + \frac{t-b}{2}| dt \right| \right\} \\ &\leq \frac{1}{6 \cdot 3!} \frac{(b-a)^5}{160} \|f^{(4)}(t)\|_{[a,b];\infty} \end{aligned}$$

and theorem is proved.

REMARK 8. We note that Theorems 5, 6 and 7 give the estimations of the remainders e^T, e^{CT} and e^S respectively in terms of Lebesgue p -norm on $f^{(n+1)}$ for $n = 1$ or 3 , or when $f^{(n)} \in AC$ and $f^{(n+1)} \in L_p (1 \leq p \leq \infty)$. We note that for $f^{(n)} \in ACG^* \setminus AC$ (or $f^{(n+1)}$ is Henstock but not Lebesgue integrable) with $1 \leq p \leq \infty$ we don't have $f^{(n+1)} \in L_p$.

For example, we consider the function

$$f(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}.$$

The primitive of function f is F given by

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}.$$

F is not AC on $[0, 1]$, so f is not Lebesgue integrable on $[0, 1]$. But F is ACG^* on $[0, 1]$, and therefore f is Henstock-Kurzweil integrable on $[0, 1]$.

For Henstock-Kurzweil integral, we consider Trapezoidal, Corrective Trapezoidal and Simpson Formulas as follows. First we state the following three Lemmas.

LEMMA 9. Let $f : [a, b] \rightarrow \mathbb{R}$, n be a positive integer. If f is n -time differentiable on (a, b) and $f^{(n)}$ is Henstock-Kurzweil integrable on $[a, b]$, then $f, \dots, f^{(n-1)}$ (We use $f^{(0)} = f$) are also Henstock-Kurzweil integrable on $[a, b]$.

Proof. Since $f^{(n)}$ is Henstock-Kurzweil integrable on $[a, b]$, the primitive $f^{(n-1)}$ of $f^{(n)}$ are ACG^* and continuous on $[a, b]$. Therefore we have that $f^{(n-1)}$ is Henstock-Kurzweil integrable on $[a, b]$. Similarly, we have that $f^{(n-2)}, \dots, f$ are Henstock-Kurzweil integrable on $[a, b]$. Then the lemma is proved.

LEMMA 10. ([5]) Let $f : [a, b] \rightarrow \mathbb{R}$, $g : [a, b] \rightarrow \mathbb{R}$. If f is Henstock-Kurzweil integrable on $[a, b]$ and g is of bounded variation on $[a, b]$, then the product fg is Henstock-Kurzweil integrable on $[a, b]$

LEMMA 11. (Integration By Parts)([2]) If g and h have Henstock-Kurzweil Primitives G and H , respectively, on $[a, b]$, then gH is Henstock-Kurzweil integrable on $[a, b]$. Moreover

$$(HK) \int_a^b g(t)H(t)dt = G(b)H(b) - G(a)H(a) - (HK) \int_a^b G(t)h(t)dt$$

THEOREM 12. *Let $f : [a, b] \rightarrow \mathbb{R}$. If f is twice differentiable on (a, b) and the second derivative is Henstock-Kurzweil integrable on $[a, b]$, then we have*

$$I_{HK}(f) = T_{HK}(f) + e_{HK}^T, \quad (3.7)$$

where

$$T_{HK}(f) = \frac{b-a}{2}[f(a) + f(b)], \quad (3.8)$$

$$e_{HK}^T = \frac{1}{2}(HK) \int_a^b (t-a)(t-b)f''(t)dt. \quad (3.9)$$

Proof. Since f is twice differentiable on (a, b) and the second derivative is Henstock-Kurzweil integrable on $[a, b]$, by lemma 9 we have f is Henstock-Kurzweil integrable on $[a, b]$.

The function $(t-a)(t-b)$ is bounded variation on $[a, b]$, so $(t-a)(t-b)f''$ is Henstock-Kurzweil integrable on $[a, b]$.

Using integration by parts formula for e_{HK}^T , we have

$$\begin{aligned} e_{HK}^T &= \frac{1}{2}(HK) \int_a^b (t-a)(t-b)f''(t)dt \\ &= \frac{1}{2}(HK) \int_a^b (t-a)(t-b)df'(t) \\ &= -(HK) \int_a^b f'(t)\left(t - \frac{a+b}{2}\right)dt \\ &= -(HK) \int_a^b \left(t - \frac{a+b}{2}\right)df(t) \\ &= -\frac{b-a}{2}[f(a) + f(b)] + (HK) \int_a^b f(t)dt. \end{aligned}$$

Hence we have equation (3.7)

Similarly, we have following Theorems.

THEOREM 13. *Let $f : [a, b] \rightarrow \mathbb{R}$. If f is four-time differentiable on (a, b) and the fourth derivative is Henstock-Kurzweil integrable on $[a, b]$ and $f'(a), f'(b)$ exist, then we have*

$$I_{HK}(f) = CT_{HK}(f) + e_{HK}^{CT}, \quad (3.10)$$

where

$$CT_{HK}(f) = \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(a) - f'(b)], \quad (3.11)$$

$$e_{HK}^{CT} = \frac{1}{4!}(HK) \int_a^b (t-a)^2(t-b)^2f^{(4)}(t)dt. \quad (3.12)$$

THEOREM 14. *Let $f : [a, b] \rightarrow \mathbb{R}$. If f is four-time differentiable on (a, b) and the fourth derivative is Henstock-Kurzweil integrable on $[a, b]$, then we have*

$$I_{HK}(f) = S_{HK}(f) + e_{HK}^S, \quad (3.13)$$

where

$$S_{HK}(f) = \frac{b-a}{6} [f(a) + 4f(\frac{a+b}{2}) + f(b)], \quad (3.14)$$

$$e_{HK}^S = \frac{1}{6 \cdot 3!} \left\{ (HK) \int_a^{\frac{a+b}{2}} (t-a)^3 [(t-b) + \frac{t-a}{2}] f^{(4)}(t) dt \right. \quad (3.15)$$

$$\left. + (HK) \int_{\frac{a+b}{2}}^b (t-b)^3 [(t-a) + \frac{t-b}{2}] f^{(4)}(t) dt \right\}. \quad (3.16)$$

By applying Lemma 15, we can deduce Theorems 16, 17 and 18 easily which we omit their proofs here, these Theorems will be useful in compound numerical integration especially when dealing with some non-absolute Henstock-Kurzweil integrable functions.

LEMMA 15. ([6]) *Let $f : [a, b] \rightarrow \mathbb{R}$. If f is Henstock-Kurzweil integrable on each interval $[c, d] \subseteq (a, b)$ and $(HK) \int_c^d f$ converges to a finite limit as $c \rightarrow a^+$ and $d \rightarrow b^-$, then f is Henstock-Kurzweil integrable on $[a, b]$ and*

$$(HK) \int_a^b f = \lim_{\substack{c \rightarrow a^+ \\ d \rightarrow b^-}} (HK) \int_c^d f.$$

THEOREM 16. *Let $f : [a, b] \rightarrow \mathbb{R}$, and let x_n be a decreasing sequence with $x_1 = b$, and $\lim_{n \rightarrow \infty} x_n = a$. If f is twice differentiable on (x_{n+1}, x_n) and the second derivative is Henstock-Kurzweil integrable on $[x_{n+1}, x_n]$. If $\lim_{n \rightarrow \infty} (HK) \int_{x_n}^b f''$ exists, then f is Henstock-Kurzweil integrable on $[a, b]$. Moreover, if we denote*

$$I_n(f) = (HK) \int_{x_{n+1}}^{x_n} f,$$

$$T_n(f) = \frac{x_n - x_{n+1}}{2} [f(x_n) + f(x_{n+1})], \text{ and}$$

$$e_n(f) = \frac{1}{2} (HK) \int_{x_{n+1}}^{x_n} (t - x_{n+1})(t - x_n) f''(t) dt.$$

Then we have

$$(HK) \int_a^b f = \sum_{n=1}^{\infty} I_n(f),$$

and

$$(HK) \int_a^b f = \sum_{n=1}^{\infty} T_n(f) + \sum_{n=1}^{\infty} e_n(f).$$

THEOREM 17. Let $f : [a, b] \rightarrow \mathbb{R}$, and let x_n be a decreasing sequence with $x_1 = b$, and $\lim_{n \rightarrow \infty} x_n = a$. If f is four-time differentiable on (x_{n+1}, x_n) , $f'(a)$, $f'(b)$ exist and the fourth derivative is Henstock-Kurzweil integrable on $[x_{n+1}, x_n]$. If

$$\lim_{n \rightarrow \infty} (HK) \int_{x_n}^b f^{(4)}$$

exists, then f is Henstock-Kurzweil integrable on $[a, b]$. Moreover, if we denote

$$I_n(f) = (HK) \int_{x_{n+1}}^{x_n} f,$$

$$T_n(f) = \frac{x_n - x_{n+1}}{2} [f(x_n) + f(x_{n+1})] + \frac{(x_n - x_{n+1})^2}{12} [f'(x_{n+1}) - f'(x_n)], \text{ and}$$

$$e_n(f) = \frac{1}{4!} (HK) \int_{x_{n+1}}^{x_n} (t - x_{n+1})^2 (t - x_n)^2 f^{(4)}(t) dt.$$

Then we have

$$(HK) \int_a^b f = \sum_{n=1}^{\infty} I_n(f),$$

and

$$(HK) \int_a^b f = \sum_{n=1}^{\infty} T_n(f) + \sum_{n=1}^{\infty} e_n(f).$$

THEOREM 18. Let $f : [a, b] \rightarrow \mathbb{R}$, and let x_n be a decreasing sequence with $x_1 = b$, and $\lim_{n \rightarrow \infty} x_n = a$. If f is four-time differentiable on (x_{n+1}, x_n) and the fourth derivative is Henstock-Kurzweil integrable on $[x_{n+1}, x_n]$. If $\lim_{n \rightarrow \infty} (HK) \int_{x_n}^b f^{(4)}$ exists, then f is Henstock-Kurzweil integrable on $[a, b]$. Moreover, if we denote

$$I_n(f) = (HK) \int_{x_{n+1}}^{x_n} f,$$

$$T_n(f) = \frac{x_n - x_{n+1}}{6} [f(x_n) + 4f(\frac{x_n + x_{n+1}}{2}) + f(x_{n+1})], \text{ and}$$

$$e_n(f) = \frac{1}{6 \cdot 3!} \left\{ (HK) \int_{x_{n+1}}^{\frac{x_n + x_{n+1}}{2}} (t - x_{n+1})^3 \left[(t - x_n) + \frac{t - x_{n+1}}{2} \right] f^{(4)}(t) dt \right. \\ \left. + (HK) \int_{\frac{x_n + x_{n+1}}{2}}^{x_n} (t - x_n)^3 \left[(t - x_{n+1}) + \frac{t - x_n}{2} \right] f^{(4)}(t) dt \right\}.$$

Then we have

$$(HK) \int_a^b f = \sum_{n=1}^{\infty} I_n(f),$$

and

$$(HK) \int_a^b f = \sum_{n=1}^{\infty} T_n(f) + \sum_{n=1}^{\infty} e_n(f).$$

We shall apply Alexiewicz norm properly to obtain the error bounds for a Henstock-Kurzeil integrable function.

DEFINITION 19. *Let f is a Henstock-Kurzeil integrable function on $[a, b]$, then the Alexiewicz norm of f is*

$$\|f\| = \sup_{[c,d] \subset [a,b]} \left| (HK) \int_c^d f \right|$$

This leads us to consider the estimates mentioned in Theorems 21, 22 and 23 by applying Alexiewicz norm. Before giving these theorems we need the following lemma.

LEMMA 20. (The Second Mean Value Theorem)([4]) *If f is Henstock-Kurzeil integrable and g is monotonic and bounded on $[a, b] \subset \mathbb{R}$ then gf is Henstock-Kurzeil integrable and there exists $\xi \in [a, b]$ such that*

$$(HK) \int_a^b gf = g(a)(HK) \int_a^\xi f + g(b)(HK) \int_\xi^b f.$$

THEOREM 21. *Let $f : [a, b] \rightarrow \mathbb{R}$. Assume the equation (3.7) holds for f . Then we have*

$$|e_{HK}^T| \leq \frac{(b-a)^2}{4} \|f^{(2)}\| \tag{3.17}$$

Proof. Using Lemma 20 for e_{HK}^T in equation (3.7), we have

$$\begin{aligned} |e_{HK}^T| &= \left| \frac{1}{2} (HK) \int_a^b (t-a)(t-b)f''(t)dt \right| \\ &\leq \frac{1}{2} \left\{ \left| (HK) \int_a^{\frac{a+b}{2}} (t-a)(t-b)f''(t)dt \right| + \left| (HK) \int_{\frac{a+b}{2}}^b (t-a)(t-b)f''(t)dt \right| \right\} \\ &\leq \frac{1}{2} \left\{ \left| \frac{(b-a)^2}{4} (HK) \int_\xi^{\frac{a+b}{2}} f''(t)dt \right| + \left| \frac{(b-a)^2}{4} (HK) \int_{\frac{a+b}{2}}^\eta f''(t)dt \right| \right\} \\ &\leq \frac{(b-a)^2}{8} \sup_{\substack{\xi \in [a, \frac{a+b}{2}] \\ \eta \in [\frac{a+b}{2}, b]}} \left\{ \left| (HK) \int_\xi^{\frac{a+b}{2}} f''(t)dt \right| + \left| (HK) \int_{\frac{a+b}{2}}^\eta f''(t)dt \right| \right\} \\ &\leq \frac{(b-a)^2}{8} \cdot 2 \cdot \|f^{(2)}\| \leq \frac{(b-a)^2}{4} \|f^{(2)}\|, \end{aligned}$$

and the theorem is proved.

THEOREM 22. *Let $f : [a, b] \rightarrow \mathbb{R}$. Assume the equation (3.10) holds for f . Then we have*

$$|e_{HK}^{CT}| \leq \frac{(b-a)^4}{8 \cdot 4!} \|f^{(4)}\| \tag{3.18}$$

Proof. We apply Lemma 20 on e_{HK}^{CT} in equation (3.10), we have

$$\begin{aligned}
 |e_{HK}^{CT}| &= \left| \frac{1}{4!} (HK) \int_a^b (t-a)^2 (t-b)^2 f^{(4)}(t) dt \right| \\
 &\leq \frac{1}{4!} \left\{ \left| (HK) \int_a^{\frac{a+b}{2}} (t-a)^2 (t-b)^2 f^{(4)}(t) dt \right| + \left| (HK) \int_{\frac{a+b}{2}}^b (t-a)^2 (t-b)^2 f^{(4)}(t) dt \right| \right\} \\
 &\leq \frac{1}{4!} \left\{ \left| \frac{(b-a)^4}{16} (HK) \int_{\xi}^{\frac{a+b}{2}} f^{(4)}(t) dt \right| + \left| \frac{(b-a)^4}{16} (HK) \int_{\frac{a+b}{2}}^{\eta} f^{(4)}(t) dt \right| \right\} \\
 &\leq \frac{(b-a)^4}{16 \cdot 4!} \sup_{\substack{\xi \in [a, \frac{a+b}{2}] \\ \eta \in [\frac{a+b}{2}, b]}} \left\{ \left| (HK) \int_{\xi}^{\frac{a+b}{2}} f^{(4)}(t) dt \right| + \left| (HK) \int_{\frac{a+b}{2}}^{\eta} f^{(4)}(t) dt \right| \right\} \\
 &\leq \frac{(b-a)^4}{8 \cdot 4!} \|f^{(4)}\|,
 \end{aligned}$$

and the theorem is proved.

THEOREM 23. *Let $f : [a, b] \rightarrow \mathbb{R}$. Assume the equation (3.13) holds for f . Then we have*

$$|e_{HK}^S| \leq \frac{(b-a)^4}{2^4 \cdot 6 \cdot 3!} \|f^{(4)}\| \quad (3.19)$$

Proof. Applying Lemma 20 on e_{HK}^S in equation (3.13), we obtain

$$\begin{aligned}
 |e_{HK}^S| &= \frac{1}{6 \cdot 3!} \left\{ \left| (HK) \int_a^{\frac{a+b}{2}} (t-a)^3 \left[(t-b) + \frac{t-a}{2} \right] f^{(4)}(t) dt \right. \right. \\
 &\quad \left. \left. + (HK) \int_{\frac{a+b}{2}}^b (t-b)^3 \left[(t-a) + \frac{t-b}{2} \right] f^{(4)}(t) dt \right| \right\} \\
 &\leq \frac{1}{6 \cdot 3!} \left\{ \left| (HK) \int_a^{\frac{a+b}{2}} (t-a)^3 \left[(t-b) + \frac{t-a}{2} \right] f^{(4)}(t) dt \right| \right. \\
 &\quad \left. + \left| (HK) \int_{\frac{a+b}{2}}^b (t-b)^3 \left[(t-a) + \frac{t-b}{2} \right] f^{(4)}(t) dt \right| \right\} \\
 &\leq \frac{1}{6 \cdot 3!} \left\{ \left| \frac{(b-a)^4}{2^5} (HK) \int_{\xi}^{\frac{a+b}{2}} f^{(4)}(t) dt \right| + \left| \frac{(b-a)^4}{2^5} (HK) \int_{\frac{a+b}{2}}^{\eta} f^{(4)}(t) dt \right| \right\} \\
 &\leq \frac{(b-a)^4}{2^5 \cdot 6 \cdot 3!} \sup_{\substack{\xi \in [a, \frac{a+b}{2}] \\ \eta \in [\frac{a+b}{2}, b]}} \left\{ \left| (HK) \int_{\xi}^{\frac{a+b}{2}} f^{(4)}(t) dt \right| + \left| (HK) \int_{\frac{a+b}{2}}^{\eta} f^{(4)}(t) dt \right| \right\} \\
 &\leq \frac{(b-a)^4}{2^4 \cdot 6 \cdot 3!} \|f^{(4)}\|,
 \end{aligned}$$

and the theorem is proved.

Let $h = b - a$ be a infinitesimal, we get following remainders' estimations of the inequalities in equations (3.4),(3.5) and (3.6); specifically,

$$|e^T| = o(h^2), |e^{CT}| = o(h^4), \text{ and } |e^S| = o(h^4). \quad (3.20)$$

Similarly, we observe from Theorems 21, 22 and 23 that we also have

$$|e_{HK}^T| = o(h^2), |e_{HK}^{CT}| = o(h^4), \text{ and } |e_{HK}^S| = o(h^4). \quad (3.21)$$

REMARK 24. We observe from equations (3.20) and (3.21) that these corresponding remainders have the same order of remainder terms. However, equation (3.20) holds under the assumption that the primitive functions of their respective integrands to be AC in $[a, b]$, and yet the equation (3.21) holds under the assumption that the primitive functions of their respective integrands to be ACG^* in $[a, b]$. In other words, the later integrands have less restriction on the integrands than that of the former one.

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