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CAUCHY–RASSIAS STABILITY OF HOMOMORPHISMS ASSOCIATED TO A PEXIDERIZED CAUCHY–JENSEN TYPE FUNCTIONAL EQUATION

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Abstract. We use a fixed point method to prove the Cauchy–Rassias stability of homomorphisms associated to the Pexiderized Cauchy–Jensen type functional equation

$$rf\left(\frac{x+y}{r}\right)+sg\left(\frac{x-y}{s}\right)=2h(x), \qquad r,s\in\mathbb{R}\setminus\{0\}$$

in Banach algebras.

1. Introduction

The stability problem of functional equations originated from a question of S. M. Ulam [28] concerning the stability of group homomorphisms : *Let* $(G_1,*)$ *be a group and let* $(G_2,\diamond,d)$ *be a metric group with the metric* $d(\cdot,\cdot)$. *Given* $\varepsilon>0$, *does there exist* $\delta(\varepsilon)>0$ *such that if a mapping* $h:G_1\to G_2$ *satisfies the inequality*

$$d(h(x*y),h(x)\diamond h(y))<\delta$$

for all $x,y\in G_1$, *then there is a homomorphism* $H:G_1\to G_2$ *with*

$$d(h(x),H(x))<\varepsilon$$

for all $x\in G_1$?

In other words, we are looking for situations where homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a homomorphism near it. D. H. Hyers [6] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces: *Assume that* $f:X\to Y$ *satisfies*

$$\|f(x+y)-f(x)-f(y)\|\leqslant\varepsilon$$

for some $\varepsilon\geqslant 0$ *and all* $x,y\in X$. *Then there exists a unique additive mapping* $T:X\to Y$ *such that*

$$\|f(x)-T(x)\|\leqslant\varepsilon$$

for all $x\in X$.

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T. Aoki [2] and Th. M. Rassias [25] provided a generalization of the Hyers' theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded. The following theorem which is called the Cauchy–Rassias stability is a generalized solution to the stability problem.

THEOREM 1.1. (Th. M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ε and p are constants with $\varepsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leqslant \frac{2\varepsilon}{2-2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) *holds for $x, y \neq 0$ and* (1.2) *for $x \neq 0$. Also, if for each $x \in E$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.*

The inequality (1.1) is called *Cauchy–Rassias inequality* and the stability of the functional equation is called *Cauchy–Rassias stability* [5, 21, 27]. The inequality (1.1) has provided a lot of influence in the development of what is now known as a *generalized Hyers–Ulam stability or HyersUlamRassias stability* of functional equations. A generalization of the Th.M. Rassias' theorem was obtained by P. Găvruta [5]. We refer the readers to [8], [9], [14], [16]–[24] and references therein for more detailed results on the stability problems of various functional equations and mappings and their Pexider types. We also refer the readers to the books [4], [7], [10] and [26].

Let E be a set. A function $d : E \times E \to [0,\infty]$ is called a *generalized metric* on E if d satisfies

(i) $d(x,y) = 0$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x, y \in E$;

(iii) $d(x,z) \leqslant d(x,y) + d(y,z)$ for all $x, y, z \in E$.

We recall the following theorem by Margolis and Diaz.

THEOREM 1.2. [13] *Let (E,d) be a complete generalized metric space and let $J : E \to E$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in E$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all non-negative integers n or there exists a non-negative integer n_0 such that

1. $d(J^n x, J^{n+1}x) < \infty$ *for all* $n \geqslant n_0$*;*

2. the sequence $\{J^n x\}$ *converges to a fixed point* y^* *of* J*;*

3. y^* *is the unique fixed point of* J *in the set* $Y = \{y \in E : d(J^{n_0}x, y) < \infty\}$*;*

4. $d(y, y^*) \leqslant \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$*.*

Throughout this paper, $\mathscr{A}$ denotes a (complex) normed algebra and $\mathscr{B}$ represents a (complex) Banach algebra. In addition, we assume r, s to be fixed non-zero real numbers.

In this paper using the fixed point method (see [1, 3, 11, 15]), we prove the Cauchy–Rassias stability of homomorphisms associated to the Pexiderized Cauchy–Jensen type functional equation

$$rf\left(\frac{x+y}{r}\right) + sg\left(\frac{x-y}{s}\right) = 2h(x), \qquad r, s \in \mathbb{R} \setminus \{0\}$$

in Banach algebras.

For convenience, we use the following abbreviation for given mappings $f, g, h : X \to Y$,

$$D_\mu(f, g, h)(x, y) := rf\left(\frac{\mu x + \mu y}{r}\right) + sg\left(\frac{\mu x - \mu y}{s}\right) - 2\mu h(x)$$

for all $x, y \in X$ and all $\mu \in \mathbb{T}^1$, where X and Y are linear spaces and $\mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$.

2. Main Results

We will use the following Lemma in this paper:

LEMMA 2.1. [23] *Let* $f : A \to B$ *be an additive mapping such that* $f(\mu x) = \mu f(x)$ *for all* $x \in A$ *and all* $\mu \in \mathbb{T}^1$*. Then the mapping* f *is* $\mathbb{C}$*-linear.*

PROPOSITION 2.2. *Let* $f, g, h : X \to Y$ *be mappings with* $f(0) = g(0) = 0$ *such that*

$$D_\mu(f, g, h)(x, y) = 0 \tag{2.1}$$

for all $x, y \in X$ *and all* $\mu \in \mathbb{T}^1$*. Then the mappings* f, g, h *are* $\mathbb{C}$*-linear and* $f = g = h$*.*

Proof. Letting $\mu = 1$ and $y = x$ in (2.1), we get $rf\left(\frac{2x}{r}\right) = 2h(x)$ for all $x \in X$. So

$$rf\left(\frac{x}{r}\right) = 2h\left(\frac{x}{2}\right) \tag{2.2}$$

for all $x \in X$. Similarly, letting $\mu = 1$ and $y = -x$ in (2.1), we get

$$sg\left(\frac{x}{s}\right) = 2h\left(\frac{x}{2}\right) \tag{2.3}$$

for all $x \in X$. Hence we get from (2.1), (2.2) and (2.3) that

$$h\left(\frac{x+y}{2}\right)+h\left(\frac{x-y}{2}\right)=h(x) \tag{2.4}$$

for all $x,y \in X$. Therefore h is additive, i.e., $h(x+y)=h(x)+h(y)$ for all $x,y \in X$. Replacing x by μx in (2.2) and (2.3) and using (2.1), we get that

$$h\left(\frac{\mu x+\mu y}{2}\right)+h\left(\frac{\mu x-\mu y}{2}\right)=\mu h(x) \tag{2.5}$$

for all $x,y \in X$ and all $\mu \in \mathbb{T}^1$. It follows from (2.4) and (2.5) that $h(\mu x)=\mu h(x)$ for all $x \in X$ and all $\mu \in \mathbb{T}^1$. By Lemma 2.1 the mapping h is $\mathbb{C}$-linear. Since h is $\mathbb{C}$-linear, we get from (2.2) and (2.3) that

$$f(x)=\frac{2}{r}h\left(\frac{rx}{2}\right)=h(x), \quad g(x)=\frac{2}{s}h\left(\frac{sx}{2}\right)=h(x)$$

for all $x \in X$. So $f=g=h$. $\square$

Now we prove the Cauchy–Rassias stability of homomorphisims in Banach alebras.

THEOREM 2.3. *Let $f,g,h:\mathscr{A}\to\mathscr{B}$ be mappings with $f(0)=g(0)=0$ for which there exist functions $\varphi,\phi:\mathscr{A}\times\mathscr{A}\to[0,\infty)$ such that*

$$\lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)=0, \quad \lim_{n\to\infty} 4^n\phi\left(\frac{x}{2^n},\frac{y}{2^n}\right)=0, \tag{2.6}$$

$$\|D_\mu(f,g,h)(x,y)\| \leqslant \varphi(x,y), \tag{2.7}$$

$$\|f(xy)-g(x)h(y)\| \leqslant \phi(x,y) \tag{2.8}$$

for all $x,y \in \mathscr{A}$ and all $\mu \in \mathbb{T}^1$. If there exists a constant $L<1$ such that the function

$$x \mapsto \psi(x) := \varphi(x,0)+\varphi\left(\frac{x}{2},\frac{x}{2}\right)+\varphi\left(\frac{x}{2},-\frac{x}{2}\right)$$

has the property

$$2\psi(x) \leqslant L\psi(2x)$$

for all $x \in \mathscr{A}$, then there exists a unique homomorphism $H:\mathscr{A}\to\mathscr{B}$ such that

$$\|h(x)-H(x)\| \leqslant \frac{1}{2-2L}\psi(x),$$

$$\|f(x)-H(x)\| \leqslant \frac{1}{r}\varphi\left(\frac{rx}{2},\frac{rx}{2}\right)+\frac{1}{r-rL}\psi\left(\frac{rx}{2}\right),$$

$$\|g(x)-H(x)\| \leqslant \frac{1}{s}\varphi\left(\frac{sx}{2},-\frac{sx}{2}\right)+\frac{1}{s-sL}\psi\left(\frac{sx}{2}\right)$$

for all $x \in \mathscr{A}$.

Proof. Letting $\mu = 1$ and $y = 0, \pm x$ in (2.7), we get the following inequalities

$$\left\| rf\left(\frac{x}{r}\right) + sg\left(\frac{x}{s}\right) - 2h(x) \right\| \leqslant \varphi(x,0), \tag{2.9}$$

$$\left\| rf\left(\frac{x}{r}\right) - 2h\left(\frac{x}{2}\right) \right\| \leqslant \varphi\left(\frac{x}{2}, \frac{x}{2}\right), \tag{2.10}$$

$$\left\| sg\left(\frac{x}{s}\right) - 2h\left(\frac{x}{2}\right) \right\| \leqslant \varphi\left(\frac{x}{2}, -\frac{x}{2}\right) \tag{2.11}$$

for all $x \in \mathscr{A}$. So it follows from (2.9), (2.10) and (2.11) that

$$\left\| h(x) - 2h\left(\frac{x}{2}\right) \right\| \leqslant \frac{1}{2}\psi(x) \tag{2.12}$$

for all $x \in \mathscr{A}$. Let $\mathscr{X} := \{F : \mathscr{A} \to \mathscr{B} \mid F(0) = 0\}$. We introduce a generalized metric on $\mathscr{X}$ as follows:

$$d(F,G) := \inf\{C \in [0,\infty] : \|F(x) - G(x)\| \leqslant C\psi(x) \text{ for all } x \in \mathscr{A}\}.$$

It is easy to show that $(\mathscr{X}, d)$ is a generalized complete metric space [3].

Now we consider the mapping $\Lambda : \mathscr{X} \to \mathscr{X}$ defined by

$$(\Lambda F)(x) = 2F\left(\frac{x}{2}\right), \quad \text{for all } F \in \mathscr{X} \text{ and } x \in \mathscr{A}.$$

Let $F, G \in \mathscr{X}$ and let $C \in [0,\infty]$ be an arbitrary constant with $d(F,G) \leqslant C$. From the definition of d, we have

$$\|F(x) - G(x)\| \leqslant C\psi(x)$$

for all $x \in \mathscr{A}$. By the assumption and last inequality, we have

$$\|(\Lambda F)(x) - (\Lambda G)(x)\| = 2\left\| F\left(\frac{x}{2}\right) - G\left(\frac{x}{2}\right) \right\| \leqslant 2C\psi\left(\frac{x}{2}\right) \leqslant CL\psi(x)$$

for all $x \in \mathscr{A}$. So

$$d(\Lambda F, \Lambda G) \leqslant Ld(F,G)$$

for any $F, G \in \mathscr{X}$. It follows from (2.12) that $d(\Lambda h, h) \leqslant \frac{1}{2}$. Therefore according to Theorem 1.2, the sequence $\{\Lambda^n h\}$ converges to a fixed point H of Λ, i.e.,

$$H : \mathscr{A} \to \mathscr{B}, \quad H(x) = \lim_{n\to\infty} (\Lambda^n h)(x) = \lim_{n\to\infty} 2^n h\left(\frac{x}{2^n}\right)$$

and $H(2x) = 2H(x)$ for all $x \in \mathscr{A}$. Also H is the unique fixed point of Λ in the set $\mathscr{X}^* = \{F \in \mathscr{X} : d(h,F) < \infty\}$ and

$$d(H,h) \leqslant \frac{1}{1-L} d(\Lambda h, h) \leqslant \frac{1}{2-2L},$$

i.e., the inequality

$$\|h(x) - H(x)\| \leqslant \frac{1}{2-2L}\psi(x) \tag{2.13}$$

holds true for all $x \in \mathscr{A}$. It follows from the definition of H, (2.6), (2.10) and (2.11) that

$$\lim_{n\to\infty} 2^n rf\left(\frac{x}{2^n r}\right) = \lim_{n\to\infty} 2^n sg\left(\frac{x}{2^n s}\right) = H(x) \tag{2.14}$$

for all $x \in \mathscr{A}$. Hence we get get from (2.6) and (2.7) that

$$H(\mu x + \mu y) + H(\mu x - \mu y) = 2\mu H(x)$$

for all $x, y \in \mathscr{A}$ and all $\mu \in \mathbb{T}^1$. By Proposition 2.2 the mapping $H : \mathscr{A} \to \mathscr{B}$ is $\mathbb{C}$-linear. So we get from (2.10) and (2.13) that

$$\begin{aligned}\|f(x) - H(x)\| &\leqslant \left\|f(x) - \frac{2}{r}h\left(\frac{rx}{2}\right)\right\| + \frac{2}{r}\left\|h\left(\frac{rx}{2}\right) - H\left(\frac{rx}{2}\right)\right\| \\ &\leqslant \frac{1}{r}\varphi\left(\frac{rx}{2}, \frac{rx}{2}\right) + \frac{1}{r - rL}\psi\left(\frac{rx}{2}\right)\end{aligned}$$

for all $x \in \mathscr{A}$. In a similar way we obtain the following inequality

$$\|g(x) - H(x)\| \leqslant \frac{1}{s}\varphi\left(\frac{sx}{2}, -\frac{sx}{2}\right) + \frac{1}{s - sL}\psi\left(\frac{sx}{2}\right)$$

for all $x \in \mathscr{A}$. Since H is $\mathbb{C}$-linear, it follows from (2.14) that

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = \lim_{n\to\infty} 2^n g\left(\frac{x}{2^n}\right) = H(x)$$

for all $x \in \mathscr{A}$. Hence we get from (2.6) and (2.8) that $H(xy) = H(x)H(y)$ for all $x, y \in \mathscr{A}$. So the mapping $H : \mathscr{A} \to \mathscr{B}$ is a homomorphism. Finally it remains to prove the uniqueness of H. Let $P : \mathscr{A} \to \mathscr{B}$ be another homomorphism satisfying $d(h, P) \leqslant \frac{1}{2-2L}$. So $P \in \mathscr{X}^*$ and $(\Lambda P)(x) = 2P(x/2) = P(x)$ for all $x \in \mathscr{A}$, i.e., P is a fixed point of Λ. Since H is the unique fixed point of Λ in $\mathscr{X}^*$, we infer that $P = H$. □

COROLLARY 2.4. *Let $p > 1, q > 2$ and θ be non-negative real numbers and let $f, g, h : \mathscr{A} \to \mathscr{B}$ be mappings with $f(0) = g(0) = 0$ satisfying the inequalities*

$$\|D_\mu(f, g, h)(x, y)\| \leqslant \theta(\|x\|^p + \|y\|^p), \quad \|f(xy) - g(x)h(y)\| \leqslant \theta(\|x\|^q + \|y\|^q)$$

for all $x, y \in \mathscr{A}$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique homomorphism $H : \mathscr{A} \to \mathscr{B}$ such that

$$\begin{aligned}\|h(x) - H(x)\| &\leqslant \frac{4 + 2^p}{2(2^p - 2)}\theta\|x\|^p, \\ \|f(x) - H(x)\| &\leqslant \frac{3r^p}{(2^p - 2)r}\theta\|x\|^p, \\ \|g(x) - H(x)\| &\leqslant \frac{3s^p}{(2^p - 2)s}\theta\|x\|^p\end{aligned}$$

for all $x \in \mathscr{A}$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p), \quad \phi(x,y) := \theta(\|x\|^q + \|y\|^q)$$

for all $x,y \in \mathscr{A}$. Then we can choose $L = 2^{1-p}$ and we get the desired results. □

THEOREM 2.5. *Let $f,g,h : \mathscr{A} \to \mathscr{B}$ be mappings with $f(0) = g(0) = 0$ for which there exist functions $\Phi, \Psi : \mathscr{A} \times \mathscr{A} \to [0,\infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{2^n}\Psi(2^n x, 2^n y) = 0, \quad \lim_{n\to\infty} \frac{1}{4^n}\Phi(2^n x, 2^n y) = 0,$$

$$\|D_\mu(f,g,h)(x,y)\| \leqslant \Psi(x,y), \quad \|f(xy) - g(x)h(y)\| \leqslant \Phi(x,y)$$

for all $x,y \in \mathscr{A}$ and all $\mu \in \mathbb{T}^1$. If there exists a constant $L < 1$ such that the function

$$x \mapsto \psi(x) := \Psi(x,0) + \Psi\left(\frac{x}{2}, \frac{x}{2}\right) + \Psi\left(\frac{x}{2}, -\frac{x}{2}\right)$$

has the property

$$\psi(2x) \leqslant 2L\psi(x)$$

for all $x \in \mathscr{A}$, then there exists a unique homomorphism $H : \mathscr{A} \to \mathscr{B}$ such that

$$\begin{aligned}
\|h(x) - H(x)\| &\leqslant \frac{L}{2-2L}\psi(x),\\
\|f(x) - H(x)\| &\leqslant \frac{1}{r}\Psi\left(\frac{rx}{2}, \frac{rx}{2}\right) + \frac{L}{r-rL}\psi\left(\frac{rx}{2}\right),\\
\|g(x) - H(x)\| &\leqslant \frac{1}{s}\Psi\left(\frac{sx}{2}, -\frac{sx}{2}\right) + \frac{L}{s-sL}\psi\left(\frac{sx}{2}\right)
\end{aligned} \tag{2.15}$$

for all $x \in \mathscr{A}$.

Proof. Using the same method as in the proof of Theorem 2.3, we have

$$\left\|\frac{1}{2}h(2x) - h(x)\right\| \leqslant \frac{1}{4}\psi(2x) \leqslant \frac{L}{2}\psi(x) \tag{2.16}$$

for all $x \in \mathscr{A}$. We introduce the same definitions for $\mathscr{X}$ and d as in the proof of Theorem 2.3 such that $(\mathscr{X}, d)$ becomes a generalized complete metric space. Let $\Lambda : \mathscr{X} \to \mathscr{X}$ be the mapping defined by

$$(\Lambda F)(x) = \frac{1}{2}F(2x), \quad \text{for all } F \in \mathscr{X} \text{ and } x \in \mathscr{A}.$$

One can show that $d(\Lambda F, \Lambda G) \leqslant Ld(F,G)$ for any $F,G \in \mathscr{X}$. It follows from (2.16) that $d(\Lambda h, h) \leqslant \frac{L}{2}$. Due to Theorem 1.2, the sequence $\{\Lambda^n h\}$ converges to a fixed point H of Λ, i.e.,

$$H : \mathscr{A} \to \mathscr{B}, \quad H(x) = \lim_{n\to\infty}(\Lambda^n h)(x) = \lim_{n\to\infty}\frac{1}{2^n}h(2^n x)$$

and $H(2x) = 2H(x)$ for all $x \in \mathscr{A}$. Also

$$d(H,h) \leqslant \frac{1}{1-L} d(\Lambda h, h) \leqslant \frac{L}{2-2L},$$

i.e.,

$$\|h(x) - H(x)\| \leqslant \frac{L}{2-2L} \psi(x)$$

holds true for all $x \in \mathscr{A}$. Similar to the proof of Theorem 2.3 we obtain the inequalities (2.15).

The rest of the proof is similar to the proof of Theorem 2.3 and we omit the details. □

COROLLARY 2.6. *Let $0 < p < 1, 0 < q < 2$ and θ, δ be non-negative real numbers and let $f, g, h : \mathscr{A} \to \mathscr{B}$ be mappings with $f(0) = g(0) = h(0) = 0$ satisfying the inequalities*

$$\|D_\mu(f,g,h)(x,y)\| \leqslant \delta + \theta(\|x\|^p + \|y\|^p),$$
$$\|f(xy) - g(x)h(y)\| \leqslant \delta + \theta(\|x\|^q + \|y\|^q)$$

for all $x, y \in \mathscr{A}$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique homomorphism $H : \mathscr{A} \to \mathscr{B}$ such that

$$\|h(x) - H(x)\| \leqslant \frac{3 \times 2^p}{2(2-2^p)} \delta + \frac{4+2^p}{2(2-2^p)} \theta \|x\|^p,$$
$$\|f(x) - H(x)\| \leqslant \frac{2(1+2^p)}{(2-2^p)r} \delta + \frac{(8-2^p)r^p}{2^p(2-2^p)r} \theta \|x\|^p,$$
$$\|g(x) - H(x)\| \leqslant \frac{2(1+2^p)}{(2-2^p)s} \delta + \frac{(8-2^p)s^p}{2^p(2-2^p)s} \theta \|x\|^p$$

for all $x \in \mathscr{A}$.

Proof. The proof follows from Theorem 2.5 by taking

$$\Psi(x,y) := \delta + \theta(\|x\|^p + \|y\|^p), \quad \Phi(x,y) := \delta + \theta(\|x\|^q + \|y\|^q)$$

for all $x, y \in \mathscr{A}$. Then we can choose $L = 2^{p-1}$ and we get the desired results. □

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