ON SOME INEQUALITIES FOR CONVEX FUNCTIONS

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*Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday*

Abstract. In this paper we derive new inequalities for convex functions. The results presented here are an extension of the inequalities obtained by S. S. Dragomir, J. Pečarić and L.-E. Persson.

1. Introduction

Let C be a convex subset of the real linear space X and $f : C \to \mathbb{R}$ a convex function on C. If $x_i \in C$ and $p_i \in (0,1)$ with $\sum_{i=1}^{n} p_i = 1$, then the following well-known form of Jensen's discrete inequality holds:

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leqslant \sum_{i=1}^{n} p_i f(x_i). \tag{1.1}$$

In [2] S.S. Dragomir, J. Pečarić and L.E. Persson proved the following refinement of Jensen's inequality in the general setting of linear spaces

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right)$$

$$\geqslant \max\left\{p_i f(x_i) + p_j f(x_j) - (p_i + p_j) f\left(\frac{p_i x_i + p_j x_j}{p_i + p_j}\right)\right\}. \tag{1.2}$$

In 2006 S.S. Dragomir ([1]) proved the following result:

$$\max_{1 \leqslant i \leqslant n}\left\{\frac{p_i}{q_i}\right\}\left[\sum_{j=1}^{n} q_j f(x_j) - f\left(\sum_{j=1}^{n} q_j x_j\right)\right] \geqslant \sum_{j=1}^{n} p_j f(x_j) - f\left(\sum_{j=1}^{n} p_j x_j\right)$$

$$\geqslant \min_{1 \leqslant i \leqslant n}\left\{\frac{p_i}{q_i}\right\}\left[\sum_{j=1}^{n} q_j f(x_j) - f\left(\sum_{j=1}^{n} q_j x_j\right)\right] \tag{1.3}$$

provided $f : C \to \mathbb{R}$ is convex on the convex subset C of the linear space X and p_i, q_i, $i \in \{1, 2, \ldots, n\}$ are probability sequences with $q_i > 0$ for each $i \in \{1, 2, \ldots, n\}$.

In particular, from (1.3) the following result is obtained:

$$n \max_{1 \leqslant i \leqslant n} \{p_i\} \left[\frac{1}{n} \sum_{j=1}^{n} f(x_j) - f\left(\frac{1}{n} \sum_{j=1}^{n} x_j \right) \right] \geqslant \sum_{j=1}^{n} p_j f(x_j) - f\left(\sum_{j=1}^{n} p_j x_j \right)$$

$$\geqslant n \min_{1 \leqslant i \leqslant n} \{p_i\} \left[\frac{1}{n} \sum_{j=1}^{n} f(x_j) - f\left(\frac{1}{n} \sum_{j=1}^{n} x_j \right) \right]. \tag{1.4}$$

In this paper some new results in connection with the inequalities (1.2)-(1.4) are given.

2. Main results

Let $\mathscr{F}$ be a linear set of functions defined on the interval I, $I \subseteq \mathbb{R}$ and A be a linear positive functional defined on $\mathscr{F}$. We suppose that $C(I) \subset \mathscr{F}$ and for every $X \subset I$ the characteristic function of X denoted by h_X belongs to $\mathscr{F}$ and for every continuous function f, $h_X \cdot f$ belongs to $\mathscr{F}$, too. In the following we suppose that A is a normalized functional. This means that

$$A(e_0) = 1,$$

where we denote by e_i, $i \in \mathbb{N}$ the monomial function, $e_i : I \to \mathbb{R}$,

$$e_i(x) = x^i, \quad x \in I.$$

THEOREM 2.1. *Let X be a subset of I such that $A(h_X) > 0$, and f a convex function defined on I. Then for every linear positive normalized functional we have:*

$$A(f) - f(a_1) \geqslant A(f h_X) - A(h_X) f\left(\frac{A(e_1 h_X)}{A(h_X)} \right), \tag{2.1}$$

where $a_1 = A(e_1)$.

Proof. From the equality

$$f = f h_X + f h_{I-X}$$

we get

$$A(f) - f(a_1) = A(f h_X) + A(f h_{I-X}) - A(h_X) f\left(\frac{A(e_1 h_X)}{A(h_X)} \right) - f(a_1)$$

$$+ A(h_X) f\left(\frac{A(e_1 h_X)}{A(h_X)} \right). \tag{2.2}$$

Let $B : C(I) \to \mathbb{R}$ be the linear functional defined by

$$B(f) = A(fh_{I-X}) + A(h_X)f\left(\frac{A(e_1 h_X)}{A(h_X)}\right). \tag{2.3}$$

We note that B is a linear positive functional.
We have

$$B(e_0) = A(h_{I-X}) + A(h_X) = A(e_0) = 1$$

$$B(e_1) = A(e_1 h_{I-X}) + A(e_1 h_X) = A(e_1) = a_1.$$

So, B is a normalized functional and $B(e_1) = a_1$.
If f is a convex function on I, by Jensen's inequality we obtain:

$$B(f) \geqslant f(a_1)$$

or

$$A(fh_{I-X}) + A(h_X)f\left(\frac{A(e_1 h_X)}{A(h_X)}\right) - f(a_1) \geqslant 0. \tag{2.4}$$

From (2.2) and (2.4) we get inequality (2.1). $\quad\square$

REMARK 2.2. Let A be the linear positive normalized functional defined by:

$$A(f) = \sum_{i=1}^{n} p_i f(x_i)$$

where $p_i \in (0,1)$, $i = \overline{1,n}$ and $\sum_{i=1}^{n} p_i = 1$.
For a given convex function f, let k and s be the natural number, $k,s \in \{0,1,2,\ldots,n\}$ for which:

$$\max\left\{p_i f(x_i) + p_j f(x_j) - (p_i + p_j)f\left(\frac{p_i x_i + p_j x_j}{p_i + p_j}\right)\right\}$$

$$= p_k f(x_k) + p_s f(x_s) - (p_k + p_s)f\left(\frac{p_k x_k + p_s x_s}{p_k + p_s}\right).$$

Let us consider $X = \{x_k, x_s\}$.
Then

$$A(h_X) = p_k + p_s$$

and

$$A(e_1 h_X) = x_k p_k + p_s x_k. \tag{2.5}$$

From (2.1) and (2.5) we get (1.2).
Let I be an interval of the real axis $\mathbb{R}$ and $\{X_i\}_{i=1}^{n}$ a partition of the interval I.

THEOREM 2.2. *Let f be a continuous convex function defined on I and $p_i, q_i \in (0,1)$, $\sum\limits_{i=1}^{n} p_i = \sum\limits_{i=1}^{n} q_i = 1$. Then for every partition $\{X_i\}_{i=1}^{n}$ of the interval I such that $A(h_{X_i}) > 0$, $i = 1, \ldots, n$ the following inequalities hold:*

$$\sum_{i=1}^{n} p_i \frac{A(fh_{X_i})}{A(h_{X_i})} - f\left(\sum_{i=1}^{n} p_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right)$$

$$\leqslant \max_{i=\overline{1,n}} \left\{ \frac{p_i}{q_i} \right\} \left[\sum_{i=1}^{n} q_i \frac{A(fh_{X_i})}{A(h_{X_i})} - f\left(\sum_{i=1}^{n} q_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right) \right] \tag{2.6}$$

$$\sum_{i=1}^{n} p_i \frac{A(fh_{X_i})}{A(h_{X_i})} - f\left(\sum_{i=1}^{n} p_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right)$$

$$\geqslant \min \left\{ \frac{p_i}{q_i} \right\} \left[\sum_{i=1}^{n} q_i \frac{A(fh_{X_i})}{A(h_{X_i})} - f\left(\sum_{i=1}^{n} q_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right) \right]. \tag{2.7}$$

Proof. Let k be a natural number, $k \in \{1, \ldots, n\}$ such that

$$\frac{p_k}{q_k} = \max_{i=\overline{1,n}} \left\{ \frac{p_i}{q_i} \right\}.$$

Inequality (2.6) is equivalent with the following inequality:

$$\sum_{i=1}^{n} \left(q_i \frac{p_k}{q_k} - p_i \right) \frac{A(fh_{X_i})}{A(h_{X_i})} + f\left(\sum_{i=1}^{n} p_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right)$$

$$\geqslant \frac{p_k}{q_k} f\left(\sum_{i=1}^{n} q_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right). \tag{2.8}$$

The last inequality can be written in the following form:

$$\sum_{i=1}^{n} q_i \left(\frac{p_k}{q_k} - \frac{p_i}{q_i} \right) \left[\frac{A(fh_{X_i})}{A(h_{X_i})} - f\left(\frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right) \right]$$

$$+ \sum_{i=1}^{n} q_i \left(\frac{p_k}{q_k} - \frac{p_i}{q_i} \right) f\left(\frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right) + f\left(\sum_{i=1}^{n} p_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right)$$

$$\geqslant \frac{p_k}{q_k} f\left(\sum_{i=1}^{n} q_i \frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right). \tag{2.9}$$

Since A is a linear positive functional we have

$$\frac{A(fh_{X_i})}{A(h_{X_i})} - f\left(\frac{A(e_1 h_{X_i})}{A(h_{X_i})} \right) \geqslant 0, \quad i = 1, \ldots, n. \tag{2.10}$$

From (2.10) we obtain:

$$\sum_{i=1}^{n}\left(q_i\frac{p_k}{q_k}-p_i\right)\left[\frac{A(fh_{X_i})}{A(h_{X_i})}-f\left(\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)\right]\geqslant 0. \tag{2.11}$$

Now, inequality (2.9) follows from (2.11) and from Jensen's inequality:

$$\sum_{i=1}^{n+1}\lambda_i f(x_i)\geqslant f\left(\sum_{i=1}^{n+1}\lambda_i x_i\right)$$

where

$$\lambda_i=\left(q_i\frac{p_k}{q_k}-p_i\right)\frac{q_k}{p_k},\quad x_i=\frac{A(e_1h_{X_i})}{A(h_{X_i})},\quad i=1,\ldots,n$$

$$\lambda_{n+1}=\frac{q_k}{p_k},\quad x_{n+1}=\sum_{i=1}^{n}p_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}.$$

Let us prove inequality (2.7).

Let s be a natural number, $s\in\{1,2,\ldots,n\}$ for which we have

$$\frac{p_s}{q_s}=\min_{i=\overline{1,n}}\left\{\frac{p_i}{q_i}\right\}.$$

Inequality (2.7) is equivalent with the inequality:

$$\sum_{i=1}^{n}\left(p_i-\frac{p_s}{q_s}q_i\right)\frac{A(fh_{X_i})}{A(h_{X_i})}+\frac{p_s}{q_s}f\left(\sum_{i=1}^{n}q_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)$$

$$\geqslant f\left(\sum_{i=1}^{n}p_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right) \tag{2.12}$$

or

$$\sum_{i=1}^{n}\left(p_i-\frac{p_s}{q_s}q_i\right)\left[\frac{A(fh_{X_i})}{A(h_{X_i})}-f\left(\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)\right]$$

$$+\sum_{i=1}^{n}\left(p_i-\frac{p_s}{q_s}q_i\right)f\left(\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)+\frac{p_s}{q_s}f\left(\sum_{i=1}^{n}q_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right)$$

$$\geqslant f\left(\sum_{i=1}^{n}p_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}\right). \tag{2.13}$$

We note that:

$$\sum_{i=1}^{n}\left(p_i-\frac{p_s}{q_s}q_i\right)+\frac{p_s}{q_s}=1$$

and

$$\sum_{i=1}^{n}\left(p_i-\frac{p_s}{q_s}q_i\right)\frac{A(e_1h_{X_i})}{A(h_{X_i})}+\frac{p_s}{q_s}\sum_{i=1}^{n}q_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}=\sum_{i=1}^{n}p_i\frac{A(e_1h_{X_i})}{A(h_{X_i})}.$$

Jensen's inequality leads to the inequality:

$$\sum_{i=1}^{n}\left(p_i - \frac{p_s}{q_s}q_i\right)f\left(\frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right) + \frac{p_s}{q_s}f\left(\sum_{i=1}^{n}q_i\frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right)$$

$$\geqslant f\left(\sum_{i=1}^{n}p_i\frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right). \tag{2.14}$$

From (2.14), and using the fact that

$$\frac{A(f h_{X_i})}{A(h_{X_i})} - f\left(\frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right) \geqslant 0$$

we obtain inequality (2.13).

The proof of the theorem is finished. □

COROLLARY 2.3. *Let $(X_i)_{i=1}^{n}$ be a partition of the interval I and A be a linear positive normalized functional, such that $A(h_{X_i}) > 0$, for every $i \in \{1,\dots,n\}$. If $q_i \in (0,1)$, $i = \overline{1,n}$ and $\sum_{i=1}^{n} q_i = 1$, then for every convex function f, $f \in C(I)$ we have:*

$$A(f) - f(a_1) \leqslant \max_{i=\overline{1,n}}\left\{\frac{A(h_{X_i})}{q_i}\right\}\left[\sum_{i=1}^{n}q_i\frac{A(f h_{X_i})}{A(h_{X_i})} - f\left(\sum_{i=1}^{n}q_i\frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right)\right],$$

$$A(f) - f(a_1) \geqslant \min_{i=\overline{1,n}}\left\{\frac{A(h_{X_i})}{q_i}\right\}\left[\sum_{i=1}^{n}q_i\frac{A(f h_{X_i})}{A(h_{X_i})} - f\left(\sum_{i=1}^{n}q_i\frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right)\right]. \tag{2.15}$$

Proof. Let p_i $(i = \overline{1,n})$ be positive numbers defined by:

$$p_i = A(h_{X_i}), \quad i = \overline{1,n}.$$

We note that

$$\sum_{i=1}^{n} p_i = A\left(\sum_{i=1}^{n} h_{X_i}\right) = A(e_0) = 1.$$

Now, the inequalities from Corollary 2.3 follow by Theorem 2.2. □

COROLLARY 2.4. *Let $(X_i)_{i=1}^{n}$ be a partition of the interval I and A be a linear positive normalized functional such that $A(h_{X_i}) > 0$ for every $i \in \{1,2,\dots,n\}$. Then for every convex function f the following inequalities are true:*

$$A(f) - f(a_1) \leqslant n\max\{A(h_{X_i})\}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{A(f h_{X_i})}{A(h_{X_i})} - f\left(\frac{1}{n}\sum_{i=1}^{n}\frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right)\right],$$

$$A(f) - f(a_1) \geqslant n\min\{A(h_{X_i})\}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{A(f h_{X_i})}{A(h_{X_i})} - f\left(\frac{1}{n}\sum_{i=1}^{n}\frac{A(e_1 h_{X_i})}{A(h_{X_i})}\right)\right]. \tag{2.16}$$

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