Journal of Mathematical Inequalities
Volume 3, Number 3 (2009), 329–339

ON SUPERQUADRACITY

S. ABRAMOVICH

Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. There are two classes of functions called Superquadratic Functions. In some cases these classes coincide but not always. In this paper this subject is discussed.

Our definition of a superquadratic function is: A function $\varphi : [0.\infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geqslant 0$ there is a constant $C(x) \in \mathbb{R}$ such that

$$\varphi(y) \geqslant \varphi(x) + C(x)(y-x) + \varphi(|y-x|)$$

for all $y \geqslant 0$.

This definition was used in many papers since 2004.

On the other hand, Kominek and Troczka (2006) used W. Smajdor (1987) definition of superquadracity, and in particular, for functions defined on $\mathbb{R}$ their definition is as follows:

The function $\varphi : \mathbb{R} \to \mathbb{R}$ is superquadratic if

$$\varphi(x+y) + \varphi(x-y) \geqslant 2\varphi(x) + 2\varphi(y),$$

is satisfied for all $x, y \in \mathbb{R}$.

After discussing the differences and similarities of these definitions, we show that the class of superquadratic functions as we defined, lead to many applications. Some of these applications we show here.

1. Introduction

There are two classes of functions called Superquadratic Functions. In some cases these classes coincide but not always. In this paper this subject is discussed.

A sample of papers dedicated to superquadracity, as defined in Definition 1, and published in the recent years are listed in the references of this paper. As we show, there are many applications of superquadracity and we encourage interested researchers to continue the investigation of this subject.

Our definition of a superquadratic function is:

DEFINITION 1. [1] A function $\varphi : [0,\infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geqslant 0$ there is a constant $C(x) \in \mathbb{R}$ such that

$$\varphi(y) \geqslant \varphi(x) + C(x)(y-x) + \varphi(|y-x|) \tag{1.1}$$

for all $y \geqslant 0$.

Mathematics subject classification (2000): Primary 26D15.
Keywords and phrases: convex functions, superquadratic functions, Hardy's inequalities, Fejer's inequality.

Paper JMI-03-33

It was proved in [1], that iff φ is superquadratic, the inequality

$$\varphi\left(\int f d\mu\right) \leqslant \int \varphi(f(s)) - \varphi\left(\left|f(s) - \int f d\mu\right|\right) d\mu \tag{1.2}$$

holds for all probability measures μ and all non-negative, μ-integrable functions f.

The equivalent discrete version of (1.2) is

$$\varphi\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leqslant \sum_{i=1}^{n} \lambda_i \varphi(x_i) - \sum_{i=1}^{n} \lambda_i \varphi\left(\left|x_i - \sum_{j=1}^{n} \lambda_j x_j\right|\right) \tag{1.3}$$

for $x_i, \lambda_i \geqslant 0,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} \lambda_i = 1,$ which in case $n = 2\ \ \lambda_1, \lambda_2 \geqslant 0,\ \lambda_1 + \lambda_2 = 1$ is

$$\varphi(\lambda_1 x_1 + \lambda_2 x_2) \leqslant \lambda_1 (\varphi(x_1) - \varphi(\lambda_2 |x_1 - x_2|)) + \lambda_2 (\varphi(x_2) - \varphi(\lambda_1 |x_1 - x_2|)), \quad x_1, x_2 \geqslant 0. \tag{1.4}$$

In [1] and in [9] it is shown that (1.4) holds for all $x_1, x_2, \lambda_1, \lambda_2 \geqslant 0$ such that $\lambda_1 + \lambda_2 = 1$, if and only if φ is superquadratic according to Definition 1. Therefore we may use (1.4) instead of (1.1) as the definition of superquadratic functions.

In the special case that $\lambda_1 = \lambda_2 = 1/2$, we get that a superquadratic function satisfies

$$\varphi\left(\frac{x_1 + x_2}{2}\right) + \varphi\left(\frac{|x_1 - x_2|}{2}\right) \leqslant \frac{\varphi(x_1) + \varphi(x_2)}{2}, \quad x_1, x_2 \geqslant 0. \tag{1.5}$$

If $\varphi(x) = x^2$, the condition (1.1) reduces to identity where $C(x) = 2x$. Also, if $\varphi(x)$ is superquadratic and $a, b \geqslant 0$ then $\varphi(x) - (ax + b)$ is also superquadratic. Any function $\varphi(x)$ satisfying $-2 \leqslant \varphi(x) \leqslant -1$ for all $x \geqslant 0$ is superquadratic.

However non negative superquadratic functions are much better behaved as we see in the following lemma.

LEMMA 1. [1, Lemma 2.1] *Let φ be a superquadratic function with $C(x)$ as in (1.1),*

(i) Then $\varphi(0) \leqslant 0$

(ii) If $\varphi(0) = \varphi'(0) = 0$, then $C(x) = \varphi'(x)$ whenever φ is differentiable at $x > 0$.

(iii) If $\varphi \geqslant 0$, then φ is convex and $\varphi(0) = \varphi'(0) = 0$.

In [11] and [12] the following scale of convexity was introduced for continuously differentiable function φ satisfying $\varphi(0) = \varphi'(0) = 0$:

k1: $\varphi'(x)$ convex

k2: $\varphi(x)/x$ convex

k3: $\varphi'(x)/x$ non-decreasing

k4: $\varphi'(x)$ superadditive $\quad (\varphi'(x+y) \geqslant \varphi'(x) + \varphi'(y))$

k5: $\varphi(x)/x^2$ non-decreasing

k6: $\varphi(x)/x$ superadditive.

There it was proved that k1 $\Rightarrow$ k2 $\Rightarrow$ k3 $\Rightarrow$ k4 $\Rightarrow$ k5 $\Rightarrow$ k6.

Among continuously differential functions φ satisfying $\varphi(0) = \varphi'(0) = 0$, it is shown in [1, Lemma 3.1, Lemma 3.2] that the superquadratic functions fall between k4 and k5, and the following examples [1, Example 3.3, Example 3.4] ensure that superquadratic functions satisfying $\varphi(0) = \varphi'(0) = 0$, fall strictly between k4 and k5:

The function φ where $\varphi(0) = 0$ and

$$\varphi'(x) = \begin{cases} 0, & x \leqslant 1 \\ 1 + (x-2)^3, & x \geqslant 1 \end{cases}$$

does not satisfy k4 but is superquadratic.

The function

$$\varphi(x) = \begin{cases} \left(3x - x^3\right) x^2, & x \leqslant 1 \\ 2x^2, & x > 1 \end{cases}$$

satisfies k5 but is not superquadratic.

2. Equivalence problem

The definition of superquadracity as stated here appeared first in 2004 in papers [1] and [2] and since then this terminology was used by several authors in many papers and journals.

The users of this definition were not aware that the term Superquadracity appeared in a different context in [16] (2006) by Kominek and Troczka, who used W. Smajdor's [15] (1987) definition of superquadracity.

In [16] it is stated:

DEFINITION 2. Let X be a real linear space and $\mathbb{R}$ be the set of all reals. Then every function $\varphi : X \longrightarrow \mathbb{R}$ satisfying the inequality

$$\varphi(x+y) + \varphi(x-y) \geqslant 2\varphi(x) + 2\varphi(y), \quad x, y \in X$$

is called superquadratic.

If we extend our superquadratic function according to Definition 1 to $-\infty < x < \infty$ as an even function, (1.5), which is a special case of (1.4) for $\lambda = 1/2$, is equivalent to

$$\varphi(x+y) + \varphi(x-y) \geqslant 2\varphi(x) + 2\varphi(y), \qquad -\infty < x, y < \infty. \tag{2.1}$$

On the other hand, inequality (2.1) is the definition of superquadracity according to [15] and [16] for the special case $\varphi : \mathbb{R} \to \mathbb{R}$.

It is of interest to clarify the relations and the differences between the classes of functions satisfying these definitions. Lately, A. Gilányi [13] raised the question whether inequality (1.5) is equivalent to inequality (1.4) for $x_1, x_2 \geqslant 0$, in other words, are the two definitions of superquadratic functions equivalent?

In general the answer to this question is negative, even for continuous functions, as we show in the following first example. The second example deals with non-continuous functions and again we show that (1.5) and (1.4) are not equivalent. The third example shows that an even extension of superquadratic function (satisfying (1.4) on $\mathbb{R}^+$) satisfies (2.1) but not (1.4) on $\mathbb{R}$.

The first two examples are of superquadratic functions which are non-positive on $x \geqslant 0$. However as many interesting superquadratic functions (satisfying (1.4)) are non-negative for $x \geqslant 0$ the following question is still unresolved: Does a non-negative function φ which satisfy (1.5), satisfy also (1.4) and therefore (1.1) and (1.2) too? In other words, it is not known yet if every function $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ which is superquadratic according to Definition 2 is also superquadratic according to our Definition 1.

EXAMPLE 1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}^-$ be

$$f(x) = \begin{cases} 2|x| - 3, & -1 \leqslant x \leqslant 1 \\ -1, & |x| > 1 \end{cases} .$$

This is an example of an even continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}^-$ that satisfies inequality (2.1) and therefore is superquadratic according to Definition 2 but f does not satisfy the inequality

$$\lambda f(a) + (1-\lambda) f(b) \geqslant f(\lambda a + (1-\lambda) b) + \lambda f((1-\lambda)(b-a)) + (1-\lambda) f(\lambda (b-a)) \tag{2.2}$$

for all $0 \leqslant \lambda \leqslant 1,\ a,\ b,\ b-a > 0$, and therefore is not superquadratic according to Definition 1.

Proof. A. First we show that f does not satisfy (2.2) (which is a consequence of (1.4)) for all $0 \leqslant \lambda \leqslant 1,\ a, b, b-a \geqslant 0$:

For $a = 0,\ b,\ \lambda b,\ (1-\lambda) b > 1$ we get that for (2.2) to be satisfied, the inequality

$$\lambda(-3) + (1-\lambda)(-1) \geqslant (-1) + \lambda(-1) + (1-\lambda)(-1) = -2$$

needs to hold. But it holds only for $\lambda \leqslant \frac{1}{2}$. Therefore (2.2) is not satisfied for every $0 \leqslant \lambda \leqslant 1$, and f is not superquadratic according to Definition 1.

B. In order to show that f satisfies (2.1) for all $x, y \in \mathbb{R}^+,\ x \geqslant y$. We choose $a = x - y,\ b = x + y$. It is sufficient to check the following cases for $a, b, \frac{b-a}{2}, \frac{b+a}{2} \geqslant 0$:

(i) $a,\ b,\ \frac{b-a}{2},\ \frac{b+a}{2} \geqslant 1$

(ii) $a \leqslant 1,\quad b,\ \frac{b-a}{2},\ \frac{b+a}{2} \geqslant 1$

(iii) $\frac{b-a}{2} \leqslant 1,\quad a,\ b,\ \frac{b+a}{2} \geqslant 1$

(iv) $a,\ \frac{b-a}{2} \leqslant 1,\quad b,\ \frac{b+a}{2} \geqslant 1$

(v) $a,\ \frac{b-a}{2},\ \frac{b+a}{2} \leqslant 1,\quad b \geqslant 1$

(vi) $a,\ b,\ \frac{b-a}{2},\ \frac{b+a}{2} \leqslant 1.$

By simple calculations that we omit here we get that indeed f satisfies (2.1) – Definition 2 of superquadracity.

This is an example of a continuous function that satisfies only (2.1) but not (2.2). Therefore it is not superquadratic according to Definition 1, but it is superquadratic according to Definition 2.

EXAMPLE 2. Let $g:\mathbb{R}\longrightarrow\mathbb{R}^{-}$ be

$$g(x)=\begin{cases}-3, & x=0\\ -1, & x\neq 0\end{cases}.$$

This is an example of a noncontinuous function satisfying (2.1) and therefore Definition 2 but not (2.2) and therefore not Definition 1. The proof is trivial.

As said before, in [16] superquadracity is defined by (2.1). There it is proved that if a function $f:X\to\mathbb{R}$, where X is a real linear space, satisfies (2.1), and if $f(0)=0$ then f is even.

The following Example 3 shows that a superquadratic function that satisfies (1.4) on $\mathbb{R}^{+}$, when extended as an even function on $\mathbb{R}$ satisfies (2.1) but not necessarily (1.4) on $\mathbb{R}$:

EXAMPLE 3. Let $f(x)=|x|^{3}$, $-\infty<x<\infty$. This is a non-negative superquadratic function for $x\geqslant 0$ according to (1.4) and according to the fact that it satisfies k3 for $x\geqslant 0$ (which guaranties its superquadracity). It also satisfies (2.1) for every x and y in $\mathbb{R}$.

But if we choose $\lambda=\frac{1}{3}$, $x=\frac{1}{2}$, $y=-\frac{1}{3}$ we see that

$$\lambda f(x)+(1-\lambda)f(y)\geqslant f(\lambda x+(1-\lambda)y)+\lambda f((1-\lambda)|x-y|)+(1-\lambda)f(\lambda|x-y|) \tag{2.3}$$

does not hold for $f(x)=|x|^{3}$.

This is an example of an even extension of a superquadratic function that satisfies (2.1) but not (2.3) which is an extension of (1.4) on the whole real line.

For superquadratic functions according to Definition 2, Z. Kominek and K. Troczka proved in [16, Theorem 8], the following theorem:

THEOREM 1. *Let $f:[0,\infty)\to[0,\infty)$ be a non-decreasing and convex function satisfying $f(0)=0$. If $\varphi:\mathbb{R}\to[0,\infty)$ satisfies (2.1) then $f\circ\varphi$ satisfies (2.1) too.*

When we replace everywhere (2.1) with (1.4) in this theorem, we do not succeed in proving or disproving an analogous theorem, except for a subclass of our superquadratic functions as shown here.

As mentioned before, acording to [1], superquadratic functions φ satisfying $\varphi(0)=\varphi'(0)=0$ fall strictly between the class k4 and the class k5.

These properties of superquadratic functions lead to the next partial answer about superquadratic functions φ for which $f\circ\varphi$ is superquadratic too:

THEOREM 2. *Let $\varphi(x)\geqslant 0$ be of class k3 and $\varphi''(x)\geqslant 0$ on $x\geqslant 0$. If $f(x)\geqslant 0$, $f'(x)\geqslant 0$, $f''(x)\geqslant 0$ on $x\geqslant 0$, and $f(0)=0$ then $\psi=f\circ\varphi$ is of class k3 too.*

Proof. Using all the conditions of the theorem and taking into consideration that $\varphi'(x)\geqslant 0$ and $\varphi(0)=0$ as a result from Lemma 2.1 [1]*, we get that $(\psi'(x)/x)'\geqslant 0$ and therefore ψ is of class k3 too.*

COROLLARY 1. *Under the same conditions as of Theorem* 2, $\psi = f \circ \varphi$ *is superquadratic according to our Definition* 1. *This follows as a result of the scale of convexity: As* φ *satisfies k3 it is also superquadratic and as we showed in Theorem* 2 *that* ψ *is of class k3 therefore it is also superquadratric according to our Definition* 1.

THEOREM 3. *Let* $\varphi(x) \geqslant 0$ *be superquadratic and continuously differentiable. If* $f(x) \geqslant 0,\ f'(x) \geqslant 0$, *on* $x \geqslant 0$, *and* $f(0) = 0$ *then* $\psi = f \circ \varphi$ *is of class k5.*

Proof. *The proof is by showing that* $\left(\psi/x^2\right)' \geqslant 0$ *under the conditions of the theorem and by using Lemma 2.1* [1].

Another theorem proved in [16] is Theorem 9 there: *Let* $\varphi : \mathbb{R} \to \mathbb{R}^+$ *satisfy (*2.1*) and assume that* $x \to \varphi \circ \sqrt{x}$ *is a convex function. Then there exists a non-decreasing and convex function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *fulfilling the condition* $f(0) = 0$ *such that* $\varphi(x) = f\left(x^2\right),\quad x \in \mathbb{R}$. *Conversely, if* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *is a non-decreasing and convex function satisfying* $f(0) = 0$, *then the function* $\varphi : \mathbb{R} \to \mathbb{R}^+$ *given by* $\varphi(x) = f\left(x^2\right),\quad x \in \mathbb{R}$ *satisfies (*2.1*).*

However it is easy to see that in order for $\varphi(x) = f\left(x^2\right)$, to be satisfied, when f is convex and $\varphi : \mathbb{R} \to \mathbb{R}^+$ is superquadratic according to Definition 1 and therefore also convex, $\varphi(x)$ has to be of class k3. But we have already shown in the introduction a function which is superquadratic but not of class k3. Therefore, not every positive superquadratic function can be represented by $\varphi(x) = f\left(x^2\right)$, where $f(0) = 0$.

Another partial result obtained by simple computation is the following:

Let $\varphi(x) \geqslant 0$ *be superquadratic and continuously differentiable according to Definition* 1. *Then* $(\varphi(x))^n$ *is superquadratic too when* $n \geqslant 3/2$.

As we have already shown, there are continuous functions that satisfy (2.1) but not (1.4). However we could not prove or disprove yet, a theorem that shows that all non-negative functions φ, $\varphi(0) = 0$ which satisfy (2.1) on $\mathbb{R}$, satisfy (1.4) on $\mathbb{R}^+$ too. Interested researchers are encouraged to continue investigating this subject.

3. A selection of applications

Based on the properties shown above, we present in the sequel few of the many applications obtained in the last several years for superquadratic functions as defined in (1.4). The results are either analogies or refinements of results related to convex functions.

3.1. Inequalities for averages

In [2] it was proved: If f is superquadratic and non-negative, then for $n \geqslant 3$:

$$\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n+1}\right) - \frac{1}{n-1}\sum_{r=1}^{n-1} f\left(\frac{r}{n}\right) \geqslant f\left(\frac{1}{3n}\right) + f\left(\frac{16}{81(n+3)}\right),$$

and for $n \geqslant 2$:

$$\frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n-1}\right) - \frac{1}{n+1}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) \geqslant f\left(\frac{1}{3n}\right) + f\left(\frac{16}{81n}\right).$$

There, many more results concerning generalized versions of averages were obtained for a superquadratic function.

3.2. Fejer and Hermite-Hadamard type inequalities

In [4] Fejer and Hermite-Hadamard type inequalities for superquadratic functions were discussed.

Here are two results presented there: Let f be a superquadratic integrable function on $[a,b]$ and let p be non-negative integrable and symmetric about $x = \frac{a+b}{2}$, $0 \leqslant a < b$. Let $P(t)$ be

$$P(t) = \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) p(x)\, dx, \quad t \in [0,1],$$

and let $Q(t)$ be

$$\begin{aligned} Q(t) = \int_a^b \frac{1}{2}[f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right) p\left(\frac{x+a}{2}\right) \\ + f\left(\frac{1+t}{2}b + \frac{1-t}{2}x\right) p\left(\frac{x+b}{2}\right)]dx, \quad t \in [0,1], \end{aligned}$$

then for $0 \leqslant s \leqslant t \leqslant 1,\ t > 0$

$$\begin{aligned} P(s) \leqslant P(t) - \int_a^b \frac{t+s}{2t} f\left((t-s)\left(\left|\frac{a+b}{2} - x\right|\right)\right) p(x)\, dx \\ - \int_a^b \frac{t-s}{2t} f\left((t+s)\left(\left|\frac{a+b}{2} - x\right|\right)\right) p(x)\, dx. \end{aligned}$$

And, if $0 \leqslant s \leqslant t \leqslant 1$, we get that

$$\begin{aligned} Q(s) - Q(t) \leqslant -\int_a^b \frac{\left(1 - \frac{t+s}{2}\right)|2x-a-b| + \frac{t+s}{2}(b-a)}{(1-t)|2x-a-b| + t(b-a)} \\ \times f\left(\frac{t-s}{2}(b-a-|a+b-2x|)\right) p(x)\, dx \\ - \int_a^b \frac{\frac{t-s}{2}(b-a-|a+b-2x|)}{(1-t)|2x-a-b| + t(b-a)} \\ \times f\left(\left(1 - \frac{t+s}{2}\right)|2x-a-b| + \frac{t+s}{2}(b-a)\right) p(x)\, dx. \end{aligned}$$

3.3. Refinements of some classical Inequalities

In [10] the authors obtained a sequence of inequalities for superquadratic functions. Especially, when the superquadratic function is convex too, then refinements of classical known results are obtained.

Here we demonstrate two of their results:

In Theorem 1 there, a converse of Jensen's inequality for superquadratic functions is proved: Let (Ω, A, μ) be a measure space with $0 < \mu(\Omega) < \infty$ and let φ be a superquadratic function. If $f : \Omega \subseteq [m, \mu] \to [0, \infty)$ is such that $f, \varphi \circ f \in L_1(\mu)$, then we have

$$\frac{1}{\mu(\Omega)} \int_r (\varphi \circ f) \, d\mu + \Delta_c \leqslant \frac{M - \overline{f}}{M - m} \varphi(m) + \frac{\overline{f} - m}{M - m} \varphi(\mu)$$

where $\overline{f} = \frac{1}{\mu(\Omega)} \int_\Omega f d\mu$ and

$$\Delta_c = \frac{1}{\mu(\Omega)} \frac{1}{M - m} \int_\Omega [(M - f) \varphi(f - m) + (f - m) \varphi(M - f)] \, d\mu.$$

In Theorem 4 there, the integral version of the Reversal of Jensen's inequality is proved : Let (Ω, A, μ) be a measure space with $0 < \mu(\Omega) < \infty$ and let φ be a superquadratic function. If $p, g : \Omega \to [0, \infty)$ are functions and $a, u \in [0, \infty)$ are real numbers such that

$$p,\ pg,\ p\varphi(g),\ p\varphi\left(\left|\frac{\int_\Omega pg d\mu}{\int_\Omega p d\mu} - g\right|\right) \in L_1(\mu), \quad 0 < \int_\Omega p d\mu < u$$

and $ua - \int_\Omega pg d\mu \geqslant 0$, then

$$\varphi\left(\frac{ua - \int_\Omega pg d\mu}{u - \int_\Omega p d\mu}\right) \geqslant \frac{u\varphi(a) - \int_\Omega p\varphi(g) \, d\mu}{u - \int_\Omega p d\mu} + \Delta_{RJ},$$

where

$$\Delta_{RJ} = \frac{1}{u - \int_\Omega p d\mu} \left[\int_\Omega p\varphi\left(\left|\frac{\int_\Omega pg d\mu}{\int_\Omega p d\mu} - g\right|\right) d\mu + \left(\int_\Omega p d\mu\right) \varphi\left(\left|\frac{\int_\Omega pg d\mu}{\int_\Omega p d\mu} - a\right|\right) \right.$$
$$\left. + \left(u - \int_\Omega p d\mu\right) \varphi\left(\frac{\int_\Omega p d\mu}{u - \int_\Omega p d\mu} \left|\frac{\int_\Omega pg d\mu}{\int_\Omega p d\mu} - a\right|\right)\right].$$

3.4. Jensen-Steffensen's and Slater-Pečarić inequalities

In [6] the authors dealt with refinements of Jensen-Steffensen's inequality and Slater-Pečarić inequality for superquadratic functions. Two of the results are as follows:

Let $\varphi : [0, \infty) \to \mathbb{R}$ be differentiable superquadratic and nonnegative, let ζ be a nonnegative monotonic n-tuple in $\mathbb{R}^n$, and ρ a real n-tuple satisfying Steffensen's coefficients, that is

$$0 \leqslant P_j \leqslant P_n, \quad j = 1, \ldots, n, \quad P_n > 0, \tag{3.1}$$

$$P_j = \sum_{i=1}^{j} \rho_i, \quad \overline{P}_j = \sum_{i=j}^{n} \rho_i, \quad j = 1, \ldots, n.$$

Let $\overline{\zeta}$ be defined by

$$\overline{\zeta} = \frac{1}{P_n}\sum_{i=1}^{n}\rho_i\zeta_i\ . \tag{3.2}$$

Then

$$\begin{aligned}
\sum_{i=1}^{n}\rho_i\varphi(\zeta_i) - P_n\varphi\left(\overline{\zeta}\right) &\geqslant \sum_{j=1}^{k-1}P_j\varphi\left(\zeta_{j+1}-\zeta_j\right) + P_k\varphi\left(\overline{\zeta}-\zeta_k\right)\\
&\quad + \overline{P}_{k+1}\varphi\left(\zeta_{k+1}-\overline{\zeta}\right) + \sum_{j=k+2}^{n}\overline{P}_j\varphi\left(\zeta_j-\zeta_{j-1}\right)\\
&\geqslant \left(\sum_{i=1}^{k}P_i + \sum_{i=k+1}^{n}\overline{P_i}\right)\varphi\left(\frac{\sum_{i=1}^{n}\rho_i\left(\left|\zeta_i-\overline{\zeta}\right|\right)}{\sum_{i=1}^{k}P_i+\sum_{i=k+1}^{n}\overline{P_i}}\right)\\
&\geqslant \left((n-1)P_n\right)\varphi\left(\frac{\sum_{i=1}^{n}\rho_i\left(\left|\zeta_i-\overline{\zeta}\right|\right)}{(n-1)P_n}\right),
\end{aligned}$$

where $k\in\{1,\dots,n-1\}$ satisfies $\zeta_k\leqslant\overline{\zeta}\leqslant\zeta_{k+1}$.

If also $\sum_{i=1}^{n}\rho_i\varphi'(\zeta_i)\neq 0$, and we define $M=\frac{\sum_{i=1}^{n}\rho_i\zeta_i\varphi'(\zeta_i)}{\sum_{i=1}^{n}\rho_i\varphi'(\zeta_i)}$, then, for s satisfying $\zeta_s\leqslant M\leqslant\zeta_{s+1},\ \ s+1\leqslant n$,

$$\begin{aligned}
\sum_{i=1}^{n}\rho_i\varphi(\zeta_i) &\leqslant P_n\varphi(M) - \left(\sum_{j=1}^{s-1}P_j\varphi\left(\zeta_{j+1}-\zeta_j\right) + P_s\varphi(M-\zeta_s)\right.\\
&\quad \left. + \overline{P}_{s+1}\rho\left(\zeta_{s+1}-M\right) + \sum_{j=s+2}^{n}\overline{P}_j\varphi\left(\zeta_j-\zeta_{j-1}\right)\right)\\
&\leqslant P_n\varphi(M) - \left(\sum_{j=1}^{s}P_j + \sum_{j=s+1}^{n}\overline{P}_j\right)\varphi\left(\frac{\sum_{i=1}^{n}\rho_i\left|\zeta_i-M\right|}{\sum_{j=1}^{s}P_j+\sum_{j=s+1}^{n}\overline{P}_j}\right)\\
&\leqslant P_n\varphi(M) - \left((n-1)P_n\right)\varphi\left(\frac{\sum_{i=1}^{n}\rho_i\left|\zeta_i-M\right|}{(n-1)P_n}\right).
\end{aligned}$$

3.5. Normalized Jensen functional

In [7] the authors consider the normalized Jensen functional

$$J_n(f,\mathbf{x},\mathbf{p}) = \sum_{i=1}^{n}p_if(x_i) - f\left(\sum_{i=1}^{n}p_ix_i\right),$$

where $\sum_{i=1}^{n}p_i=1,\ \ f:I\longrightarrow\mathbb{R}$, and I is an interval in $\mathbb{R}$.

We quote here only one of the theorems there: Let $x=(x_1,\dots,x_n)\in I^n,\quad p=(p_1,\dots,p_n),\ \ q=(q_1,\dots,q_n)$ be nonnegative n-tuples satisfying $\sum_{i=1}^{n}p_i=1,\ \ \sum_{i=1}^{n}q_i=$

$1,\ \ q_i > 0,\ \ i = 1,...,n$. Let

$$m = \min_{1 \leqslant i \leqslant n} \left(\frac{p_i}{q_i} \right), \quad M = \max_{1 \leqslant i \leqslant n} \left(\frac{p_i}{q_i} \right)$$

If I is $[0,a)$ or $[0,\infty)$ and f is a superquadratic function on I, then

$$J_n(f,\mathbf{x},\mathbf{p}) - mJ_n(f,\mathbf{x},\mathbf{q}) \geqslant mf\left(\left| \sum_{i=1}^{n} (q_i - p_i) x_i \right| \right) + \sum_{i=1}^{n} (p_i - mq_i) f\left(\left| x_i - \sum_{j=1}^{n} p_j x_j \right| \right)$$

and

$$J_n(f,\mathbf{x},\mathbf{p}) - MJ_n(f,\mathbf{x},\mathbf{q}) \leqslant -\sum_{i=1}^{n} (Mq_i - p_i) f\left(\left| x_i - \sum_{j=1}^{n} q_j x_j \right| \right) - f\left(\left| \sum_{i=1}^{n} (p_i - q_i) x_i \right| \right).$$

3.6. Refinement of Hardy's inequality

I finish this paper with a very interesting applications of the properties of superquadratic functions dealt by Oguntuase and Persson in [14]. They considered several Hardy's inequalities, for instance:

$$\int_0^\infty x^{-k} \left(\int_0^x f(t)\,dt \right)^p dx \leqslant \left(\frac{p}{|k-1|} \right)^p \int_0^\infty x^{p-k} f^p(x)\,dx, \quad k > 1, \quad p \geqslant 1.$$

One of their refinements is as follows: Let $p > 1,\ \ k > 1,\ 0 < b \leqslant \infty$, and let the function f be locally integrable on $(0,b)$ such that $0 < \int_0^b x^{p-k} f^p(x)\,dx < \infty$.

(i) If $p \geqslant 2$ then,

$$\int_0^b x^{-k} \left(\int_0^x f(t)\,dt \right)^p dx + \frac{k-1}{p} \int_0^b \int_t^b \left| \frac{p}{k-1} \left(\frac{t}{x} \right)^{1-\frac{k-1}{p}} f(t) - \frac{1}{x} \int_0^x f(t)\,dt \right|^p$$
$$\times x^{p-k-\frac{k-1}{p}} dx t^{\frac{k-1}{p}-1} dt$$
$$\leqslant \left(\frac{p}{k-1} \right)^p \int_0^b \left(1 - \left(\frac{x}{b} \right)^{\frac{k-1}{p}} \right) x^{p-k} f^p(x)\,dx.$$

(ii) If $1 < p \leqslant 2$, then the inequality holds in the reversed direction.

REFERENCES

[1] S. ABRAMOVICH, G. JAMESON AND G. SINNAMON, *Refining Jensen's inequality*, Bull. Sci. Math. Roum, **47**, 95 (2004), 3–14.

[2] S. ABRAMOVICH, G. JAMESON AND G. SINNAMON, *Inequalities for averages of convex and superquadratic functions*, JIPAM, **5** (2004), article 91.

[3] S. ABRAMOVICH, S. BANIĆ, M. KLARIČIĆ BAKULA, *A variant of Jensen-Steffensen's inequality for convex and for superquadratic functions*, JIPAM, **7**, 2 (2006), article 70.

[4] S. ABRAMOVICH, J. BARIĆ AND J. PEČARIĆ, *Fejer and Hermite-Hadamard type inequalities for superquadratic functions*, JMAA, (2008).

[5] S. ABRAMOVICH, S. BANIĆ AND M. MATIĆ, *Superquadratic functions in several variables*, JMAA, **327** (2007), 1444–1460.
[6] S. ABRAMOVICH, S. BANIĆ, M. MATIĆ AND J. PEČARIĆ, *Jensen-Steffensen's and related inequalities for superquadratic functions*, MIA (2008).
[7] S. ABRAMOVICH AND S. S. DRAGOMIR, *Normalized Jensen functional, superquadracity and related inequalities*, Proceedings of the CIA07 – Conference of Inequalities and Applications – Hungary.
[8] S. ABRAMOVICH, *Superquadracity of functions and rearrangements of sets*, JIPAM, **8**, 2 (2007), Article 46.
[9] S. BANIĆ, *Superquadratic functions*, PhD Dissertation 2007, Zagreb, Croatia.
[10] S. BANIĆ, J. PEČARIĆ AND S. VAROŠANEC, *Superquadratic functions and refinements of some classical inequalities*, J. Korean Math. Soc. (to appear).
[11] E. F. BECKENBACH, *Superadditivity inequalities*, Pac. J. Math., **14** (1964), 421–438.
[12] A. M. BRUCKNER AND E. OSTROW, *Some function classes related to the class of convex functions*, Pac. J. Math., **12** (1962), 1203–1215.
[13] A. GILÁNYI, *Remark on subquadratic functions*, Presented at the conference on Inequality and applications 2007, Noszvaj, Hungary, September 2007.
[14] J. A. OGUNTUASE AND L.E. PERSSON, *Refinement of Hardy's inequalities via superquadratic and subquadratic functions*, JMAA.
[15] W. SMAJDOR, *Subadditive and subquadratic set-valuedfunctions*, Scientific Publications of the University of Silesia, 889, Katowice, 1987.
[16] Z. KOMINEK AND K. TROCZKA, *Some remarks on subquadratic function*, Demonstratio Mathematica, **39** (2006), 751–758.

(Received September 30, 2008)

S. Abramovich
Department of Mathematics
Faculty of Science and science Education
University of Haifa
Haifa
Israel
e-mail: abramos@math.haifa.ac.il