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INEQUALITIES IN SUMMABILITY THEORY OF FOURIER SERIES

FERENC WEISZ

Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. Some recent results on a general summability method, the so-called θ-summability, are summarized for one-dimensional Fourier series. Natural choices of θ are investigated, i.e., if θ is in Wiener amalgam spaces, Feichtinger's algebra or modulation spaces. Sufficient and necessary conditions are given for the uniform and L_1 norm and a.e. convergence of the θ-means $\sigma_n^\theta f$ to the function f. The maximal operator of the θ-means is investigated and it is proved that it is bounded on L_p spaces and on Hardy spaces.

1. Introduction

It was proved by Fejér [7] that the $(C,1)$ or Fejér means of the one-dimensional Fourier series of a continuous function converge uniformly to the function. The same problem for integrable functions was investigated by Lebesgue [10]. He proved that for every integrable function f,

$$\frac{1}{n}\sum_{k=0}^{n-1} s_k f(x) \to f(x) \qquad \text{as} \qquad n \to \infty$$

at each Lebesgue point of f, where $s_k f$ denotes the kth partial sum of the Fourier series of f. Almost every point is a Lebesgue point of f.

In this paper we consider a more general method of summation, the so called θ-summation, which is generated by a single function θ. This method is intensively studied in the literature (see e.g. Butzer and Nessel [2], Trigub and Belinsky [14] and Feichtinger and Weisz [4, 5, 16] and the references therein). A natural choice of θ is a function from the Wiener algebra $W(C,\ell_1)(\mathbb{R})$. All concrete summability methods investigated in the literature satisfy this condition.

We shall investigate some not so standard function spaces in this topic, but known from other parts of analysis, for example Wiener amalgam spaces, Feichtinger's algebra $M_1(\mathbb{R})$, modulation spaces, Hardy and Herz spaces. Feichtinger's algebra and modulation spaces are very intensively investigated in Gabor analysis (see e.g. Feichtinger and Zimmermann [6] and Gröchenig [8]). $M_1(\mathbb{R})$ is the minimal (non-trivial) Banach

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space which is isometrically invariant under translation, modulation and Fourier transform. Hardy spaces extend in a natural way the L_p spaces. Our investigations were made in two directions. On the one hand we give some conditions on the summability function θ, such that the θ-means $\sigma_n^\theta f$ converge to f a.e. and in L_p norm, and such that the maximal operator σ_*^θ of the θ-means is bounded on the L_p spaces. On the other hand, for a fixed θ we extend these convergence and boundedness results to the Hardy spaces.

In Sections 2 and 3 we introduce the function spaces and the basic definitions about summability. In Section 4 we deal with norm convergence of the θ-means $\sigma_n^\theta f$ of the Fourier series of f. We show that $\sigma_n^\theta f \to f$ uniformly (resp. at each point) for all $f \in C(\mathbb{T})$ if and only if $\sigma_n^\theta f \to f$ in L_1 norm for all $f \in L_1(\mathbb{T})$ if and only if $\hat{\theta} \in L_1(\mathbb{R})$. If B is a homogeneous Banach space on $\mathbb{T}$ and $\hat{\theta} \in L_1(\mathbb{R})$ then $\sigma_n^\theta f \to f$ in B norm for all $f \in B$. If θ is in the Feichtinger's algebra $M_1(\mathbb{R})$ or in the Sobolev-type space $V_1^2(\mathbb{R})$, then these convergence results hold.

In Section 5 the a.e. convergence of the θ-means is considered. If $\hat{\theta}$ is in the Herz space $K_1(\mathbb{R})$ then the maximal operator $\sigma_*^\theta f$ of the θ-means of f can be estimated by the Hardy-Littlewood maximal function Mf. Since M is of weak type $(1,1)$ we obtain $\sigma_n^\theta f \to f$ a.e. as $n \to \infty$ for all $f \in L_1(\mathbb{T})$. The set of convergence is also characterized and the condition $\hat{\theta} \in K_1(\mathbb{R})$ is sufficient and necessary. In other words, the convergence holds at every Lebesgue point of $f \in L_1(\mathbb{T})$ if and only if $\hat{\theta} \in K_1(\mathbb{R})$.

In Section 6 we give some sufficient conditions for θ such that $\hat{\theta}$ is in the Herz space. More exactly, if θ is in a weighted modulation space or in the Sobolev-type space then $\hat{\theta} \in K_1(\mathbb{R})$.

In Section 7 our results are extended to Hardy spaces. Under some conditions on θ the boundedness of σ_*^θ is proved from the Hardy space $H_p(\mathbb{T})$ to $L_p(\mathbb{T})$, when $(1>)p_0 < p \leqslant 1$. Moreover, $\sigma_n^\theta f$ converge to f in H_p norm. In the last section some well known summability methods are listed as special cases of the θ-summability.

Most of the proofs of the results of this paper can be found in Feichtinger and Weisz [4, 5, 17]. This paper was the base of my talk given at the conference "Mathematical Inequalities and Applications", Trogir - Split, Croatia, 2008.

2. Wiener amalgams and Feichtinger's algebra

We briefly write L_p instead of $L_p(\mathbb{T},\lambda)$ space equipped with the norm (or quasi-norm) $\|f\|_p := (\int_\mathbb{T} |f|^p \, d\lambda)^{1/p}$ $(0 < p \leqslant \infty)$, where $\mathbb{T}$ is the torus and λ is the Lebesgue measure. We use the notation $|I|$ for the Lebesgue measure of the set I.

The *weak* L_p space, $L_{p,\infty}(\mathbb{T})$ $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{L_{p,\infty}} := \sup_{\rho>0} \rho\lambda(|f| > \rho)^{1/p} < \infty.$$

Note that $L_{p,\infty}(\mathbb{T})$ is a quasi-normed space (see Bergh and Löfström [1]). It is easy to see that for each $0 < p < \infty$, $L_p(\mathbb{T}) \subset L_{p,\infty}(\mathbb{T})$ and $\|\cdot\|_{L_{p,\infty}} \leqslant \|\cdot\|_p$.

A measurable function f belongs to the *Wiener amalgam space* $W(L_p, \ell_q^{v_s})(\mathbb{R})$

$(1 \leqslant p, q \leqslant \infty)$ if

$$\|f\|_{W(L_p,\ell_q^{v_s})} := \Big(\sum_{k\in\mathbb{Z}} \|f(\cdot+k)\|_{L_p[0,1)}^q v_s(k)^q \Big)^{1/q} < \infty$$

with the obvious modification for $q = \infty$, where the weight function v_s is defined by $v_s(\omega) := (1+|\omega|)^s$ $(\omega \in \mathbb{R})$. If $s = 0$ then we write simply $W(L_p, \ell_q)(\mathbb{R})$. The closed subspace of $W(L_\infty, \ell_q)(\mathbb{R})$ containing continuous functions is denoted by $W(C, \ell_q)(\mathbb{R})$ $(1 \leqslant q \leqslant \infty)$. The space $W(C, \ell_1)(\mathbb{R})$ is called *Wiener algebra*. It is used quite often in Gabor analysis, because it provides a convenient and general class of windows. It plays an important rule in summability theory, too (see Feichtinger and Weisz [4, 5]).

A Banach space B consisting of Lebesgue measurable functions on $\mathbb{T}$ is called a *homogeneous Banach space*, if

(a) for all $f \in B$ and $x \in \mathbb{T}$, $T_x f \in B$ and $\|T_x f\|_B = \|f\|_B$,

(b) the function $x \mapsto T_x f$ from $\mathbb{T}$ to B is continuous for all $f \in B$,

(c) $\|f\|_1 \leqslant C\|f\|_B \qquad (f \in B)$.

For an introduction to homogeneous Banach spaces see Katznelson [9].

The *Fourier transform* and the *short-time Fourier transform* (STFT) with respect to a window function g are defined by

$$\hat f(\omega) := \int_{\mathbb{R}} f(t) e^{-2\pi\imath\omega t}\, dt, \qquad S_g f(x,\omega) := \int_{\mathbb{R}} f(t)\overline{g(t-x)} e^{-2\pi\imath\omega t}\, dt,$$

$(x, \omega \in \mathbb{R})$. Let $g_0(x) := e^{-\pi|x|^2}$, $v_s(x,\omega) := (1+|\omega|)^s$. The *modulation spaces* $M_1^{v_s}(\mathbb{R})$ consists of all $f \in L^2(\mathbb{R})$ functions for which

$$\|f\|_{M_1^{v_s}} := \|S_{g_0} f \cdot v_s\|_{L_1(\mathbb{R}^2)} < \infty.$$

If $s = 0$ then $M_1(\mathbb{R})$ is called *Feichtinger's algebra*. Any other non-zero Schwartz function defines the same space and an equivalent norm. It is known that $M_1(\mathbb{R})$ is isometrically invariant under translation, modulation and Fourier transform. Furthermore, the embedding $M_1(\mathbb{R}) \hookrightarrow W(C, \ell_1)(\mathbb{R})$ is dense and continuous (see Feichtinger and Zimmermann [6] and Gröchenig [8]).

3. θ-summability of Fourier series

The θ-summation was considered in a great number of papers and books, such as Butzer and Nessel [2], Trigub and Belinsky [14], Natanson and Zuk [12] and Feichtinger and Weisz [4, 5, 15, 16]. In these investigations usually it was supposed that $\theta \in L_1(\mathbb{R})$ is an even continuous function satisfying

$$\sum_{k=-\infty}^{\infty} \Big|\theta\Big(\frac{k}{n+1}\Big)\Big| < \infty, \qquad (n \in \mathbb{N}). \tag{1}$$

Now we omit the condition that θ is even and, on the other hand, we require a little bit more on θ, to be more precise we assume that the function θ is from the Wiener algebra $W(C,\ell_1)(\mathbb{R})$. We have seen in Feichtinger and Weisz [4, 5] that this is a natural choice of θ and all summability methods considered in Butzer and Nessel [2] and Weisz [16] satisfy this condition. It is easy to see that

$$\sum_{k=-\infty}^{\infty}\left|\theta\left(\frac{k}{n+1}\right)\right| \leqslant \sum_{l\in\mathbb{Z}}(n+1)\sup_{x\in[0,1)}|\theta(x+l)| = (n+1)\|\theta\|_{W(C,\ell_1)} < \infty, \qquad (2)$$

which shows (1).

The nth partial sum of a distribution f over $\mathbb{T}$ is denoted by

$$s_n f(x) := \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi\imath kx},$$

where $\hat{f}(k) := \int_{\mathbb{T}} f(t) e^{-2\pi\imath kt}\,dt \ \ (k\in\mathbb{Z})$ are the Fourier coefficients. One of the deepest results in harmonic analysis (see Carleson [3]) says that for all $f\in L_p(\mathbb{T})\ \ (1<p<\infty)$

$$s_n f \to f \qquad \text{a.e. and in } L_p\text{-norm as } n\to\infty$$

and

$$\|s_* f\|_p \leqslant C_p \|f\|_p \qquad \text{where} \qquad s_* f := \sup_{n\in\mathbb{N}} |s_n f|.$$

These results are not true for $p=1$. However, considering a suitable summability method, we can extend the results. The Fejér means are given with

$$\sigma_n f(x) := \frac{1}{n+1}\sum_{k=0}^{n} s_k f(x) = \sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)\hat{f}(k) e^{2\pi\imath kx}.$$

We generalize the Fejér means and introduce the θ-*means* of a distribution f by

$$\sigma_n^{\theta} f(x) := \sum_{k=-\infty}^{\infty}\theta\left(\frac{-k}{n+1}\right)\hat{f}(k)e^{2\pi\imath kx} = \int_{\mathbb{T}} f(x-t) K_n^{\theta}(t)\,dt,$$

where $\theta\in W(C,\ell_1)(\mathbb{R})$ is a fixed function and the θ-*kernels* K_n^{θ} are given by

$$K_n^{\theta}(t) = \sum_{k=-\infty}^{\infty}\theta\left(\frac{-k}{n+1}\right)e^{2\pi\imath kt}.$$

This function is well defined and integrable because of (2). If $\theta(x) := (1-|x|)\vee 0$ then we get back the Fejér means. Another well known summability method is the Riesz method with $\theta(x) := (1-|x|^{\gamma})^{\alpha}\vee 0 \ \ (0<\alpha\leqslant 1\leqslant\gamma)$.

4. Norm convergence of θ-means

In this section we present some results about the norm convergence of $\sigma_n^\theta f$.

THEOREM 1. *If* $\theta \in W(C,\ell_1)(\mathbb{R})$ *and* $\theta(0)=1$ *then* $\sigma_n^\theta f \to f$ *in* $L_2(\mathbb{T})$ *norm for all* $f \in L_2(\mathbb{T})$ *as* $n \to \infty$.

If the Fourier transform of θ is integrable then the θ-means can be written as a singular integral of f and the Fourier transform of θ in the following way.

THEOREM 2. *If* $\theta \in W(C,\ell_1)(\mathbb{R})$ *and* $\hat\theta \in L_1(\mathbb{R})$ *then*

$$\sigma_n^\theta f(x) = (n+1)\int_{\mathbb{R}} f(x-t)\hat\theta\Big((n+1)t\Big)\,dt$$

for all $x \in \mathbb{T}$, $n \in \mathbb{N}$ *and* $f \in L_1(\mathbb{T})$.

For the uniform and L_1 norm convergence of the θ-means a sufficient and necessary condition can be given.

THEOREM 3. *If* $\theta \in W(C,\ell_1)(\mathbb{R})$ *and* $\theta(0)=1$ *then the following conditions are equivalent:*

(i) $\hat\theta \in L_1(\mathbb{R})$,

(ii) $\sigma_n^\theta f \to f$ *uniformly for all* $f \in C(\mathbb{T})$ *as* $n \to \infty$,

(iii) $\sigma_n^\theta f(x) \to f(x)$ *for all* $x \in \mathbb{T}$ *and* $f \in C(\mathbb{T})$ *as* $n \to \infty$,

(iv) $\sigma_n^\theta f \to f$ *in* $L_1(\mathbb{T})$ *norm for all* $f \in L_1(\mathbb{T})$ *as* $n \to \infty$.

One part of the preceding result is generalized for homogeneous Banach spaces.

THEOREM 4. *Assume that* B *is a homogeneous Banach space on* $\mathbb{T}$. *If* $\theta \in W(C,\ell_1)(\mathbb{R})$, $\hat\theta \in L_1(\mathbb{R})$ *and* $\theta(0)=1$ *then* $\sigma_n^\theta f \to f$ *in* B *norm for all* $f \in B$ *as* $n \to \infty$.

Note that $L_p(\mathbb{T})$ $(1 \leqslant p < \infty)$, $C(\mathbb{T})$, Lorentz spaces $L_{p,q}(\mathbb{T})$ $(1 < p < \infty, 1 \leqslant q < \infty)$ and Hardy space $H_1(\mathbb{T})$ are all homogeneous Banach spaces.

Since $\theta \in M_1(\mathbb{R})$ implies $\theta \in W(C,\ell_1)(\mathbb{R})$ and $\hat\theta \in M_1(\mathbb{R}) \subset L_1(\mathbb{R})$, the next corollary follows from Theorems 3 and 4.

COROLLARY 1. *If* $\theta \in M_1(\mathbb{R})$ *and* $\theta(0)=1$ *then*

(i) $\sigma_n^\theta f \to f$ *uniformly for all* $f \in C(\mathbb{T})$ *as* $n \to \infty$,

(ii) $\sigma_n^\theta f \to f$ *in* $L_1(\mathbb{T})$ *norm for all* $f \in L_1(\mathbb{T})$ *as* $n \to \infty$,

(iii) $\sigma_n^\theta f \to f$ *in* B *norm for all* $f \in B$ *as* $n \to \infty$ *if* B *is a homogeneous Banach space.*

COROLLARY 2. *If $\theta \in C(\mathbb{R})$ has compact support and $\theta(0)=1$ then the following conditions are equivalent:*

(i) $\theta \in M_1(\mathbb{R})$,

(ii) $\sigma_n^\theta f \to f$ *uniformly for all* $f \in C(\mathbb{T})$ *as* $n \to \infty$,

(iii) $\sigma_n^\theta f(x) \to f(x)$ *for all* $x \in \mathbb{T}$ *and* $f \in C(\mathbb{T})$ *as* $n \to \infty$,

(iv) $\sigma_n^\theta f \to f$ *in* $L_1(\mathbb{T})$ *norm for all* $f \in L_1(\mathbb{T})$ *as* $n \to \infty$.

Next we give a sufficient result for θ to be in $M_1(\mathbb{R})$. A function θ is in the Sobolev-type space $V_1^k(\mathbb{R})$, if there are numbers $-\infty = a_0 < a_1 < \ldots < a_n < a_{n+1} = \infty$ such that $n = n(\theta)$ is depending on θ and

$$\theta \in C^{k-2}(\mathbb{R}), \quad \theta \in C^k(a_i, a_{i+1}), \quad \theta^{(j)} \in L_1(\mathbb{R})$$

for all $i = 0, \ldots, n$ and $j = 0, \ldots, k$. Here C^k denotes the set of k-times continuously differentiable functions. The norm of this space is introduced by

$$\|\theta\|_{V_1^k} := \sum_{j=0}^{k} \|\theta^{(j)}\|_1 + \sum_{i=1}^{n} |\theta^{(k-1)}(a_i + 0) - \theta^{(k-1)}(a_i - 0)|,$$

where $\theta^{(k-1)}(a_i \pm 0)$ denote the right and left limits of $\theta^{(k-1)}$. It is easy to see that these limits do exist and are finite.

THEOREM 5. *If $\theta \in V_1^2(\mathbb{R})$ then $\theta \in M_1(\mathbb{R})$,*

$$\|\theta\|_{M_1} \leqslant \|\theta\|_{V_1^2} \qquad (f \in V_1^2(\mathbb{R}))$$

and Corollary 1 *holds.*

5. A.e. convergence of θ-means

First we define the *Hardy-Littlewood maximal function* by

$$Mf(x) := \sup_{x \in I} \frac{1}{|I|} \int_I |f| \, d\lambda,$$

where the supremum is taken over all intervals. It is known (see Stein [13] or Weisz [16]) that the maximal function M is of *weak type* $(1,1)$ and bounded on $L_p(\mathbb{T})$ $(1 < p \leqslant \infty)$, i.e.,

$$\|Mf\|_{L_{1,\infty}} = \sup_{\rho > 0} \rho \lambda(Mf > \rho) \leqslant C\|f\|_1, \qquad (f \in L_1(\mathbb{T})) \tag{3}$$

and, for all $1 < p \leqslant \infty$

$$\|Mf\|_p \leqslant C_p \|f\|_p, \qquad (f \in L_p(\mathbb{T}), 1 < p \leqslant \infty). \tag{4}$$

Next we introduce the special *Herz spaces* $K_p(\mathbb{R})$ with the norm

$$\|f\|_{K_p} := \Big(\sum_{k=-\infty}^{\infty} 2^k \|f\mathbf{1}_{P_k}\|_\infty^p\Big)^{1/p} < \infty \qquad (0 < p \leqslant \infty),$$

where $P_k := (-2^k, 2^k) \setminus (-2^{k-1}, 2^{k-1})$ $(k \in \mathbb{Z})$. Of course $K_1(\mathbb{R}) \subset L_1(\mathbb{R})$, since

$$\|f\|_1 = \sum_{k=-\infty}^{\infty} \|f\mathbf{1}_{P_k}\|_1 \leqslant \sum_{k=-\infty}^{\infty} 2^k \|f\mathbf{1}_{P_k}\|_\infty = \|f\|_{K_1}.$$

In the next theorem we give an equivalent norm on the Herz spaces.

THEOREM 6. *For $\theta \in L_1(\mathbb{R})$ let $\eta(x) := \sup_{|t|\geqslant|x|} |\hat{\theta}(t)|$. Then $\hat{\theta} \in K_p(\mathbb{R})$ if and only if $\eta \in L_p(\mathbb{R})$ and*

$$C_p^{-1}\|\eta\|_p \leqslant \|\hat{\theta}\|_{K_p} \leqslant C_p\|\eta\|_p \qquad (0 < p < \infty).$$

To prove pointwise convergence of the θ-means we will investigate the *maximal operator* σ_*^θ given by

$$\sigma_*^\theta f := \sup_{n\in\mathbb{N}} |\sigma_n^\theta f|.$$

It is easy to see that if $\hat{\theta} \in L_1(\mathbb{R})$ then

$$\|\sigma_*^\theta f\|_\infty \leqslant \|\hat{\theta}\|_1 \|f\|_\infty, \qquad (f \in L_\infty(\mathbb{T})).$$

THEOREM 7. *If $\theta \in W(C,\ell_1)(\mathbb{R})$ and $\hat{\theta} \in K_1(\mathbb{R})$ then*

$$\sigma_*^\theta f(x) \leqslant C\|\hat{\theta}\|_{K_1} Mf(x) \qquad a.e.$$

The inequalities in (3) and (4) imply

THEOREM 8. *If $\theta \in W(C,\ell_1)(\mathbb{R})$ and $\hat{\theta} \in K_1(\mathbb{R})$ then*

$$\|\sigma_*^\theta f\|_{L_{1,\infty}} \leqslant C\|\hat{\theta}\|_{K_1} \|f\|_1 \qquad (f \in L_1(\mathbb{T})).$$

Moreover, for every $1 < p \leqslant \infty$,

$$\|\sigma_*^\theta f\|_p \leqslant C_p\|\hat{\theta}\|_{K_1} \|f\|_p \qquad (f \in L_p(\mathbb{T})).$$

The next result follows from the usual density argument due to Marcinkiewicz and Zygmund [11].

COROLLARY 3. *If $\theta \in W(C,\ell_1)(\mathbb{R})$, $\theta(0) = 1$, $\hat{\theta} \in K_1(\mathbb{R})$ and $f \in L_1(\mathbb{T})$ then*

$$\lim_{n\to\infty} \sigma_n^\theta f = f \qquad a.e.$$

We will characterize the points of convergence. To this end we generalize the concept of Lebesgue points. Lebesgue differentiation theorem says that

$$\lim_{h\to 0}\frac{1}{h}\int_0^h f(x+u)\,du = f(x)$$

for a.e. $x\in\mathbb{T}$, where $f\in L_1(\mathbb{T})$. A point $x\in\mathbb{T}$ is called a *Lebesgue point* of $f\in L_1(\mathbb{T})$ if

$$\lim_{h\to 0}\frac{1}{h}\int_0^h |f(x+u)-f(x)|\,du = 0.$$

It is known that almost every point is a Lebesgue point of $f\in L_1(\mathbb{T})$. We can prove the next sufficient and necessary condition for the convergence of the θ-means at every Lebesgue point.

THEOREM 9. *Suppose that $\theta\in W(C,\ell_1)(\mathbb{R})$, $\theta(0)=1$ and $\hat\theta\in L_1(\mathbb{R})$. Then*

$$\lim_{n\to\infty}\sigma_n^\theta f(x) = f(x)$$

for all Lebesgue points of $f\in L_1(\mathbb{T})$ if and only if $\hat\theta\in K_1(\mathbb{R})$.

If f is continuous at a point x then x is a Lebesgue point of f.

COROLLARY 4. *Let $\theta\in W(C,\ell_1)(\mathbb{R})$, $\theta(0)=1$ and $\hat\theta\in K_1(\mathbb{R})$. If $f\in L_1(\mathbb{T})$ is continuous at a point x then*

$$\lim_{n\to\infty}\sigma_n^\theta f(x) = f(x).$$

6. Modulation and Sobolev-type spaces

In this section we present some sufficient conditions for θ such that $\hat\theta\in K_1(\mathbb{R})$.

THEOREM 10. *If $\theta\in M_1^{v_1}(\mathbb{R})$ then $\hat\theta\in K_1(\mathbb{R})$ and*

$$\|\hat\theta\|_{K_1}\leqslant C\|\theta\|_{M_1^{v_1}} \qquad (f\in M_1^{v_1}(\mathbb{R})).$$

The next result is an easy consequence of Theorems 8 and 9.

THEOREM 11. *If $\theta\in M_1^{v_1}(\mathbb{R})$ then*

$$\lim_{n\to\infty}\sigma_n^\theta f(x) = f(x)$$

for all Lebesgue points of $f\in L_1(\mathbb{T})$. Moreover,

$$\sup_{\rho>0}\rho\,\lambda(\sigma_*^\theta f>\rho)\leqslant C\|\theta\|_{M_1^{v_1}}\|f\|_1 \qquad (f\in L_1(\mathbb{T})),$$

and, for every $1<p\leqslant\infty$,

$$\|\sigma_*^\theta f\|_p\leqslant C_p\|\theta\|_{M_1^{v_1}}\|f\|_p \qquad (f\in L_p(\mathbb{T})).$$

THEOREM 12. *If* $\theta \in V_1^k(\mathbb{R})$ *for some* $k > 2$ *then* $\theta \in M_1^{v_1}(\mathbb{R})$*,*

$$\|\theta\|_{M_1^{v_1}} \leqslant C\|\theta\|_{V_1^k} \qquad (f \in V_1^k(\mathbb{R}))$$

and Theorem 11 *holds.*

This result is not true for $k = 2$, however, we have

THEOREM 13. *If* $\theta \in V_1^2(\mathbb{R})$ *then* $\hat{\theta} \in K_1(\mathbb{R})$ *and*

$$\lim_{n\to\infty} \sigma_n^\theta f(x) = f(x)$$

for all Lebesgue points of $f \in L_1(\mathbb{T})$*. Moreover,*

$$\sup_{\rho>0} \rho\,\lambda(\sigma_*^\theta f > \rho) \leqslant C\|\hat{\theta}\|_{K_1}\|f\|_1 \qquad (f \in L_1(\mathbb{T})),$$

and, for every $1 < p \leqslant \infty$*,*

$$\|\sigma_*^\theta f\|_p \leqslant C_p\|\hat{\theta}\|_{K_1}\|f\|_p \qquad (f \in L_p(\mathbb{T})).$$

7. Summability and Hardy spaces

If we suppose a little bit more on θ then we can obtain the extensions of the previous results to Hardy spaces. A distribution f is in the *Hardy space* $H_p(\mathbb{T})$ if

$$\|f\|_{H_p} := \|\sup_{t>0}|f * P_t|\|_p < \infty,$$

where $*$ denotes the convolution and

$$P_t(x) := \sum_{m\in\mathbb{Z}} e^{-t|m|}e^{2\pi \imath mx} = \frac{1-r^2}{1+r^2-2r\cos(2\pi x)}$$

is the usual *Poisson kernel*. It is known that $H_p(\mathbb{T})$ is equivalent to $L_p(\mathbb{T})$ if $1 < p < \infty$ and $H_1(\mathbb{T}) \subset L_1(\mathbb{T})$ (see e.g. Stein [13] or Weisz [16]).

To prove the boundedness of σ_*^θ on Hardy spaces we will estimate $\hat{\theta}$ by an even non-increasing function η, i.e.

$$\left.\begin{array}{l} |\hat{\theta}| \leqslant \eta, \\ \eta \text{ is even and non-increasing on } \mathbb{R}_+. \end{array}\right\} \tag{5}$$

Of course the smallest such function is defined by $\eta(x) := \sup_{|t|\geqslant|x|}|\hat{\theta}(t)|$.

THEOREM 14. *Suppose that* $\theta \in W(C, \ell_1^{v_1})(\mathbb{R})$ *and* $\eta \in L_1(\mathbb{R}_+ \setminus (0, 1/4))$ *satisfies (*5*). If the function* $s \mapsto s\eta(s)$ *is non-increasing on* $\mathbb{R}_+$ *then*

$$\|\sigma_*^\theta f\|_1 \leqslant C\|f\|_{H_1} \qquad (f \in H_1(\mathbb{T})).$$

Note that $\eta \in L_p(\mathbb{R}_+ \setminus (0,1/4))$ implies $\hat{\theta} \in K_p(\mathbb{R})$. Theorem 11 says that if $\theta \in M_1^{v_1}(\mathbb{R}) \subset W(C,\ell_1)(\mathbb{R})$ then the maximal θ-operator is of weak type $(1,1)$ and of type (p,p) $(1 < p \leqslant \infty)$. As we will see in the next theorem, if $\theta \in W(C,\ell_1^{v_1})(\mathbb{R}) \cap M_1^{v_s}(\mathbb{R})$ for some $s > 1$, then it is bounded from $H_1(\mathbb{T})$ to $L_1(\mathbb{R})$.

THEOREM 15. *If $\theta \in W(C,\ell_1^{v_1})(\mathbb{R}) \cap M_1^{v_s}(\mathbb{R})$ for some $s > 1$, then*

$$\|\sigma_*^\theta f\|_1 \leqslant C\|f\|_{H_1} \qquad (f \in H_1(\mathbb{T})).$$

THEOREM 16. *If $\theta \in V_1^k(\mathbb{R})$ then $\theta \in M_1^{v_s}(\mathbb{R})$ for all $0 \leqslant s < k-1$ and*

$$\|\theta\|_{M_1^{v_1}} \leqslant C\|\theta\|_{V_1^k} \qquad (f \in V_1^k(\mathbb{R})).$$

The next result extends Theorem 13.

THEOREM 17. *If $\theta \in W(C,\ell_1^{v_1})(\mathbb{R}) \cap V_1^2(\mathbb{R})$, then*

$$\|\sigma_*^\theta f\|_1 \leqslant C\|f\|_{H_1} \qquad (f \in H_1(\mathbb{T})).$$

If we know some information about the derivatives of $\hat{\theta}$ then we can state a stronger result. Similarly to (5) assume that $\hat{\theta}^{(m)}$ can be estimated by a non-increasing even function η_m, i.e.

$$\left.\begin{array}{l} |\hat{\theta}^{(m)}| \leqslant \eta_m, \\ \eta_m \text{ is even and non-increasing on } \mathbb{R}_+. \end{array}\right\} \tag{6}$$

THEOREM 18. *Let $0 < p \leqslant 1$, $\theta \in W(C,\ell_1)(\mathbb{R})$ and $\hat{\theta} \in L_1(\mathbb{R})$ be $(N+1)$-times differentiable $(N \in \mathbb{N})$. Suppose that $\eta_m \in L_p(\mathbb{R}_+ \setminus (0,1))$ satisfies (6) for $m = N, N+1$. If $s \mapsto s^{N+1}\eta_N(s)$ is non-increasing and $s \mapsto s^{N+2}\eta_{N+1}(s)$ is non-decreasing on $\mathbb{R}_+$ then*

$$\|\sigma_*^\theta f\|_p \leqslant C_p\|f\|_{H_p} \qquad (f \in H_p(\mathbb{T})).$$

Moreover, for all $p < r < \infty$,

$$\|\sigma_*^\theta f\|_r \leqslant C_{r,\theta}\|f\|_{H_r} \qquad (f \in H_r(\mathbb{T})).$$

If $p < 1$ then

$$\|\sigma_*^\theta f\|_{L_{1,\infty}} \leqslant C_\theta\|f\|_1 \qquad (f \in L_1(\mathbb{T})).$$

If $\eta_m \in L_{p,\infty}(\mathbb{R}) \setminus L_p(\mathbb{R}_+ \setminus (0,1))$ $(m = N, N+1, p \neq 1)$ then

$$\|\sigma_*^\theta f\|_{L_{p,\infty}} \leqslant C_p\|f\|_{H_p} \qquad (f \in H_p(\mathbb{T})).$$

COROLLARY 5. *Let* $\theta \in W(C,\ell_1)(\mathbb{R})$ *and* $\hat{\theta} \in L_1(\mathbb{R})$ *be* $(N+1)$*-times differentiable* $(N \in \mathbb{N})$. *Assume that there exists* $N+1 < \beta \leqslant N+2$ *such that*

$$|\hat{\theta}^{(m)}(x)| \leqslant C|x|^{-\beta} \qquad (x \neq 0)$$

whenever $m = N$ *or* $m = N+1$. *Then*

$$\|\sigma_*^\theta f\|_r \leqslant C_{r,\theta}\|f\|_{H_r} \qquad (f \in H_r(\mathbb{T})),$$

$$\|\sigma_*^\theta f\|_{L_{1,\infty}} \leqslant C_\theta \|f\|_1 \qquad (f \in L_1(\mathbb{T}))$$

hold for all $1/\beta < r < \infty$ *and*

$$\|\sigma_*^\theta f\|_{L_{1/\beta,\infty}} \leqslant C\|f\|_{H_{1/\beta}} \qquad (f \in H_{1/\beta}(\mathbb{T})).$$

COROLLARY 6. *Under the conditions of Theorem* 18 *or Corollary* 5

$$\lim_{n\to\infty} \sigma_n^\theta f = f \quad \text{ in } H_p(\mathbb{T}) \text{ norm}$$

for all $f \in H_p(\mathbb{T})$.

8. Some summability methods

In this section we consider some well-known summability methods as special cases of the θ-summation. All theorems above hold for the next summability methods. For more examples see [5] or [17].

EXAMPLE 1. (Fejér summation) Let $\theta(x) = (1-|x|) \vee 0$. Then $\hat{\theta}(x) = (\frac{\sin x/2}{x/2})^2$.

EXAMPLE 2. (Riesz summation) For $0 < \alpha < \infty, 1 \leqslant \gamma < \infty$ let $\theta(x) = (1-|x|^\gamma)^\alpha \vee 0$. It is known that

$$|\hat{\theta}^{(m)}(x)| \leqslant C|x|^{-1-\alpha} \qquad (x \neq 0)$$

for all $m \in \mathbb{N}$. Then Corollary 5 holds with $1/(1+\alpha) < r < \infty$.

EXAMPLE 3. (Weierstrass summation) $\theta(x) = e^{-|x|^\gamma}$ $(1 \leqslant \gamma < \infty)$.

EXAMPLE 4. (Picard and Bessel summations) $\theta(x) = (1+|x|^\gamma)^{-\alpha}$ $(0 < \alpha < \infty$, $1 \leqslant \gamma < \infty$, $\alpha\gamma > 1)$.

EXAMPLE 5. (de La Vallée-Poussin summation) Let

$$\theta(x) = \begin{cases} 1, & \text{if } |x| \leqslant 1/2 \\ -2|x|+2, & \text{if } 1/2 < |x| \leqslant 1 \\ 0, & \text{if } |x| > 1. \end{cases}$$

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Ferenc Weisz
Department of Numerical Analysis
Eötvös L. University
H-1117 Budapest
Pázmány P. sétány 1/C.
Hungary
e-mail: weisz@numanal.inf.elte.hu