Journal of
Mathematical
Inequalities
Volume 3, Number 3 (2009), 475–483

THE PROBABILISTIC STABILITY FOR A FUNCTIONAL NONLINEAR EQUATION IN A SINGLE VARIABLE

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. We use the fixed point method to prove the probabilistic Hyers–Ulam and generalized Hyers–Ulam–Rassias stability for the nonlinear equation $f(x) = \Phi(x, f(\eta(x)))$ where the unknown is a mapping f from a nonempty set S to a probabilistic metric space (X, F, T_M) and $\Phi : S \times X \to X$, $\eta : S \to X$ are two given functions.

1. Introduction and Preliminaries

The first stability theorem was proved by Hyers in [14], who affirmatively solved a problem of Ulam and proved that the additive Cauchy functional equation is “stable in the sense of Hyers-Ulam”. We quote from the paper by Hyers [14]:

THEOREM 1.1. (Hyers [14], Theorem 1) *Let E and E' be Banach spaces and let $f(x)$ be a δ-linear transformation of E into E', that is, a mapping $f : E \to E'$ such that $\|f(x+y) - f(x) - f(y)\| < \delta$, $\forall x, y \in E$. Then the limit $l(x) = \lim_{n\to\infty} f(2^n x)/2^n$ exists for each x in E, $l(x)$ is a linear transformation, and the inequality $\|f(x) - l(x)\| \leqslant \delta$ is true for all x in E. Moreover $l(x)$ is the only linear transformation satisfying this inequality.*

Subsequently the result of Hyers has been generalized by considering unbounded control functions. Significant results have been obtained by Aoki, Bourgin, Rassias and Găvruţă ([2], [5], [32], [11]). The stability problems of various functional equations have been investigated in last decades by a number of authors, see e.g., [10], [15], [16], [9], [4], [12], [1], [7], [27], [30], [36], [19]. The generalized Hyers-Ulam-Rassias stability of the additive Cauchy functional equation and Jensen functional in probabilistic and fuzzy normed spaces has recently been studied in [28], [23] and [26].

Baker ([3]) used the fixed point method to study the stability of the nonlinear functional equation $f(x) = F(x, f(\eta(x)))$.

Mathematics subject classification (2000): 39B52, 39B82, 47H10, 54E70.
Keywords and phrases: Functional equation, fixed points, Hyers-Ulam stability, probabilistic metric space.

Paper JMI-03-47

THEOREM 1.2. (Baker [3], Theorem 2) *Let S be a nonempty set and (X,d) be a complete metric space. Let $\eta : S \to S, F : S \times X \to X$. Suppose that there is $k \in (0,1)$ such that*

$$d(F(x,u),F(x,v)) \leqslant kd(u,v), \forall x \in S, \forall u,v \in X,$$

and let $f : S \to X, \delta > 0$ be such that

$$d(f(x),F(x,f(\eta(x)))) \leqslant \delta,\ \forall x \in S.$$

Then there is a unique mapping $f_s : S \to X$ with the properties

$$f_s(x) = F(x,f_s(\eta(x))),\ \forall x \in S$$

and

$$d(f(x),f_s(x)) \leqslant \frac{\delta}{1-k},\ \forall x \in S.$$

In this paper we provide some probabilistic versions of the stability result of Baker.

2. Fixed points in generalized metric spaces and in probabilistic metric spaces

The notion of generalized complete metric space has been introduced by Luxemburg in [20].

DEFINITION 2.1. A *generalized metric* on a nonempty set X is a mapping $d : X \times X \to \mathbb{R} \cup \{+\infty\}$ satisfying the following conditions:

i) $d(x,y) \geqslant 0,\ \forall x,y \in X;\ \ d(x,y) = 0$ iff $x = y$

ii) $d(x,y) = d(y,x),\ \forall x,y \in X$

iii) If $d(x,z)$ and $d(y,z)$ are finite, then $d(x,y)$ is finite and

$$d(x,y) \leqslant d(x,z) + d(y,z).$$

If (X,d) is a generalized metric space, then the relation $\sim$, $x \sim y$ if and only if $d(x,y) < +\infty$, is an equivalence relation on X which determines a unique decomposition $X = \cup\{X_\alpha, \alpha \in A\}$ into disjoint equivalence classes, called the canonical decomposition. If $d_\alpha = d\,|_{X_\alpha \times X_\alpha}$, then (X,d) is a complete generalized metric space if and only if (X_α, d_α) is a complete metric space for each $\alpha \in A$.

Recently, Radu [31] pointed out that many theorems concerning the stability of functional equations are consequences of the fixed point alternative of Margolis and Diaz [21]. On the other hand, as it is shown in [18], the fixed point theorems of the alternative on generalized metric spaces can be obtained from the corresponding fixed point theorems on appropriate metric spaces: if (X,d) is a generalized metric space, $X = \cup\{X_\alpha, \alpha \in A\}$ is its canonical decomposition and $T : X \to X$ is a mapping such that

$$d(T(x),T(y)) < +\infty \text{ whenever } d(x,y) < +\infty,$$

then T has a fixed point if and only if $T_\alpha = T\,|_{X_\alpha} : X_\alpha \to X_\alpha$ has a fixed point for some $\alpha \in A$ ([18], Theorem 3.1). Accordingly, we obtain:

THEOREM 2.1. [18] *Let* (X,d) *be a complete generalized metric space and* $T: X \to X$ *be a strict contraction with the Lipschitz constant* k, *such that* $d(x_0,T(x_0)) < +\infty$ *for some* $x_0 \in X$. *Then* T *has a unique fixed point in the set* $Y := \{y \in X, d(x_0,y) < \infty\}$ *and the sequence* $(T^n(x))_{n\in N}$ *converges to the fixed point* x^* *for every* $x \in Y$. *Moreover,* $d(x_0,T(x_0)) \leqslant \delta$ *implies* $d(x^*,x_0) \leqslant \frac{\delta}{1-k}$.

The above theorem can immediately be extended to uniform spaces with the uniformity generated by the separated family $D = \{d_\alpha\}_{\alpha\in I}$ of generalized pseudometrics on X (see e.g., [29], [6]):

THEOREM 2.2. *Let* X *be a sequentially complete generalized uniform space and* f *be a mapping from* X *into* X, *with the property that for every* $\alpha \in I$ *there is* $k_\alpha \in (0,1)$ *such that* $d_\alpha(f(x),f(y)) \leqslant k_\alpha d_\alpha(x,y)$ *for all* $x,y \in X$. *Suppose that there exist* $x \in X$ *and* $\delta_\alpha > 0$ *such that* $d_\alpha(x,f(x)) < \delta_\alpha\ \forall \alpha \in I$. *Then the sequence* $\{x_n\}_{n>0}$, $x_n = f^n(x)$ *is convergent in* X *and* $u := \lim_{n\to\infty} f^n(x)$ *is the fixed point of* f. *Moreover,* $d_\alpha(u,x) \leqslant \frac{\delta_\alpha}{1-k_\alpha}\ \forall \alpha \in I$.

Probabilistic contractions were first defined and studied by V.M. Sehgal in his doctoral dissertation at Wayne State University. In the following we recall some useful facts from fixed point theory of probabilistic metric spaces. For more details the reader is referred to the books [13] and [33].

A *triangular norm* (shorter *t-norm*) is a binary operation $T : [0,1]\times[0,1] \to [0,1]$ which is commutative, associative, nondecreasing in each variable and has 1 as the unit element. Basic examples are: $T_L(a,b) = Max(a+b-1,0)$ (*Łukasiewicz* t-norm), $T_P(a,b) = ab$ and $T_M(a,b) = Min\{a,b\}$.

Let (b_n) be a strictly increasing sequence of positive numbers with $\lim_{n\to\infty} b_n = 1$. By a (b_n) *t-norm of Hadžić type* we understand a t-norm T with $T(b_n,b_n) = b_n\ \forall n$.

A *distribution function* is a mapping $F : [0,\infty) \to [0,1]$ with the properties: $F(0) = 0$, F is increasing and F is left continuous on $(0,\infty)$. The class of all distribution functions is denoted by Δ_+ and D_+ is the subset of Δ_+ containing the functions F which also satisfy the condition $\lim\limits_{x\to\infty} F(x) = 1$.

A special element in D_+ is ε_0,

$$\varepsilon_0(t) = \begin{cases} 0, \text{ if } t = 0 \\ 1, \text{ if } t > 0 \end{cases}.$$

A *probabilistic distance* on a nonempty set X is a a mapping F from $X\times X$ to Δ_+ ($F(x,y)$ will be denoted by F_{xy}). A triple (X,F,T), where X is a nonempty set, F is a probabilistic distance on X and T is a t-norm, is called *a generalized Menger space* or a *probabilistic metric space in the sense of Schweizer and Sklar* if the following conditions are satisfied:

$$(PM0) \qquad F_{xy} = \varepsilon_0 \text{ if and only if } x = y$$

$$(PM1) \qquad F_{xy} = F_{yx}\ \forall x,y \in X$$

$$(PM2_M) \qquad F_{xy}(t+s) \geqslant T(F_{xz}(t),F_{zy}(s)),\ \forall x,y,z \in X,\ \forall t,s > 0.$$

If (X,F,T) is a generalized Menger space with $\sup_{t<1} T(t,t) = 1$, then the family $\{U_{\varepsilon,\lambda}\}_{\varepsilon>0,\lambda\in(0,1)}$, where $U_{\varepsilon,\lambda} := \{(x,y) \in X \times X : F_{xy}(\varepsilon) > 1-\lambda\}$ is a base for a metrizable uniformity on X, named the F-*uniformity* and denoted by $\mathscr{U}_F$ ([33]).

DEFINITION 2.2. ([22]) Let b be the collection of all strictly increasing sequences $(b_n)_{n\in\mathbf{N}}$, with $\lim_{n\to\infty} b_n = 1$ and let $(b_n) \in b$. A pair (S,F), where F is a probabilistic distance on S satisfying (PM0) and (PM1) is called a *probabilistic* (b_n)-*structure* if the following triangle inequality takes place:

$$(PM3_b)\ \ (F_{pq}(s) \geqslant b_n, F_{qr}(t) \geqslant b_n) \Rightarrow F_{pr}(s+t) \geqslant b_n.$$

It is easy to see that every generalized Menger space under a $(b_n)t$-norm T of Hadžić- type is a probabilistic (b_n)-structure.

DEFINITION 2.3. ([34]) Let S be a nonempty set and F be a probabilistic distance on S. A mapping $f : S \to S$ is called a *probabilistic contraction* (or B-*contraction*) if there exists $k \in (0,1)$ such that

$$F_{f(p)f(q)}(kt) \geqslant F_{pq}(t), \forall p,q \in S, \forall t > 0.$$

The following theorem is a probabilistic analogous of the fixed point theorem of Luxemburg-Yung.

THEOREM 2.3. (see [17], [35]) *Let (S,F,T) be a complete generalized Menger space with T of Hadžić-type and $f : S \to S$ be a B-contraction. Then f has a fixed point iff there is $p \in S$ such that $F_{pf(p)} \in D_+$. If $F_{pf(p)} \in D_+$, then $u := \lim_{n\to\infty} f^n(p)$ is the unique fixed point of f in the set $Y = \{q \in S : F_{pq} \in D_+\}$.*

3. Probabilistic stability of the equation $f(x) = \Phi(x, f(\eta(x)))$

In this section we study the probabilistic Hyers–Ulam and generalized Hyers–Ulam–Rassias stability for the nonlinear equation $f(x) = \Phi(x, f(\eta(x)))$ where the unknown is a mapping f from a nonempty set S to a probabilistic metric space (X,F,T_M) and $\Phi : S \times X \to X$, $\eta : S \to X$ are two given functions.

First we define the notion of probabilistic approximate solution of the given equation.

DEFINITION 3.1. A probabilistic uniform approximate solution of the equation $f(x) = \Phi(x, f(\eta(x)))$ is any function $f : S \to X$ such that

$$\lim_{t\to\infty} F_{f(u)\Phi(u,f(\eta(u)))}(t) = 1$$

uniformly on S.

The proof of our first theorem makes essential use of the following lemma, which completes Theorem 2.3 with an estimation relation in the case $T = T_M$:

LEMMA 3.1. ([24]) *Let* (S,F,T_M) *be a complete generalized Menger space and* $f:S\to S$ *be a* $k-B$ *contraction. Suppose that* $F_{pf(p)}\in D_+$ *and let* $u=\lim_{n\to\infty}f^n(p)$. *Then*

$$F_{up}(t+0)\geqslant F_{pf(p)}((1-k)t),\ \forall t>0.$$

THEOREM 3.1. *Let* S *be a nonempty set and* (X,F,T_M) *be a complete generalized Menger space. Suppose that the mapping* $\Phi:S\times X\to X$ *satisfies the condition: there exists* $k\in(0,1)$ *such that*

$$F_{\Phi(u,x)\Phi(u,y)}(kt)\geqslant F_{xy}(t)$$

for all $u\in S,\ x\in X,\ t>0$ *and let* $f:S\to X$ *be a probabilistic uniform approximate solution of the given equation. Then there is a function* $f_s:S\to X$ *with the properties:*

$$f_s(u)=\Phi(u,f_s(\eta(u))),\ \forall u\in S$$

and, if for some $\varepsilon\in(0,1)$ *and* $\delta>0$

$$F_{f(u)\Phi(u,f(\eta(u)))}(\delta)>1-\varepsilon,\ \forall u\in S$$

then

$$F_{f(u)f_s(u)}\left(\frac{\delta}{1-k}+0\right)\geqslant 1-\varepsilon,\ \forall u\in S.$$

Proof. Let Y be the set of all mappings a from S to X and the operator of Baker $T:Y\to Y, T(a)(x)=\Phi(x,a((\eta(x)))$ for all $a\in Y$ and $x\in S$. For every $a,b\in Y$ define the distribution function $\mathbf{F}_{ab}$ by the formula

$$\mathbf{F}_{ab}(t)=\sup_{s<t}\inf_{x\in S}F_{a(x)b(x)}(s).$$

It can immediately be proved that $(Y,\mathbf{F},T_M)$ is a complete generalized probabilistic metric space. We also have

$$\begin{aligned}\mathbf{F}_{T(a)T(b)}(kt)&=\sup_{s<kt}\inf_{y\in S}F_{T(a)(y)T(b)(y)}(s)\\&=\sup_{s<t}\inf_{y\in S}F_{T(a)(y)T(b)(y)}(ks)\geqslant\sup_{s<t}\inf_{y\in S}F_{a(\eta(y))b(\eta((y))}(s)\geqslant\mathbf{F}_{ab}(t),\end{aligned}$$

that is, T is a $B-$ contraction on $(Y,\mathbf{F},T_M)$.

As the condition $\lim_{t\to\infty}F_{f(u)\Phi(u,f(\eta(u))}(t)=1$, uniformly on X means that $\mathbf{F}_{fT(f)}\in D_+$, from Theorem 2.3 it follows that T has a fixed point, that is, there is a selfmapping f_s of Y such that $f_s(u)=\Phi(u,f_s(\eta(u)))$, for all $u\in S$.

If $F_{f(u)\Phi(u,f(\eta(u)))}(\delta)>1-\varepsilon,\ \forall u\in S$ then $\mathbf{F}_{fT(f)}(\delta)\geqslant 1-\varepsilon$. It follows that $\mathbf{F}_{ff_s}(\frac{\delta}{1-k}+0)\geqslant 1-\varepsilon$, that is, $\inf_{u\in S}F_{f(u)f_s(u)}(\frac{\delta}{1-k}+0)\geqslant 1-\varepsilon$, concluding that $F_{f(u)f_s(u)}(\frac{\delta}{1-k}+0)\geqslant 1-\varepsilon,\ \forall u\in S.$ □

The next stability result refers to mappings on probabilistic (b_n)-structures. In the proof of Theorem 3.2 we use the fixed point theorem of Monna.

LEMMA 3.1. ([25]) *If* (S,F) *is a* (b_n)*-probabilistic structure then, for every* $n \in N$*, the mapping*

$$r_n : S \times S \to [0,\infty],\ r_n(p,q) = \inf\{t > 0 |\ F_{pq}(t) \geqslant b_n\}$$

is a generalized pseudometric on S *and the family* $\{r_n\}_{n\in\mathbf{N}}$ *generates the uniformity* $\mathscr{U}_F$*. Moreover,* $r_n(p,q) < \varepsilon \Rightarrow F_{pq}(\varepsilon) \geqslant b_n$ *and* $F_{pq}(\varepsilon) \geqslant b_n \Rightarrow r_n(p,q) \leqslant \varepsilon$*.*

THEOREM 3.2. *Let* S *be a nonempty set and* (X,F) *be a complete probabilistic* (b_n)*-structure. Suppose that the mapping* $\Phi : S \times X \to X$ *satisfies the condition:*

$$\forall n \in N\ \exists k_n \in (0,1) : F_{xy}(t) \geqslant b_n \Rightarrow F_{\Phi(u,x)\Phi(u,y)}(k_n t) \geqslant b_n.$$

Let $f : S \to X$ *be a mapping with the property:*

$$\lim_{t\to\infty} F_{f(u)\Phi(u,f(u))}(t) = 1,$$

uniformly on S*.*

Then there is a function $f_s : S \to X$ *such that*

$$f_s(u) = \Phi(u, f_s(u)),\ \forall u \in S$$

and if

$$F_{f(u)\Phi(u,f(u))}(\delta_n) \geqslant b_n,\ \forall u \in S$$

then

$$F_{f(u)f_s(u)}\left(\frac{\delta_n}{1-k_n}\right) \geqslant b_n, \forall u \in S.$$

Proof. Let again Y be the set of all mappings a from S to X. Consider the family of pseudometrics $\{r_n\}_{n\in N}$ defined in the above lemma and, for every $n \in N$ and all $a,b \in Y$,

$$d_n(a,b) = \sup\{r_n(a(x),b(x)), x \in S\}.$$

Then $\{d_n\}_{n\in N}$ is a complete family of generalized pseudometrics on Y. Consider the mapping $T : Y \to Y$, $T(a)(x) = \Phi(x,a(x))$ for all $a,b \in Y$ and $x \in S$.

Let $a,b \in Y, \varepsilon > 0$ and $x \in S$ be given. Then, for all $a,b \in Y$,

$$k_n r_n(a(x),b(x)) < \varepsilon \Rightarrow r_n(a(x),b(x)) < \frac{\varepsilon}{k_n}$$

$$\Rightarrow F_{a(x)b(x)}\left(\frac{\varepsilon}{k_n}\right) \geqslant b_n \Rightarrow F_{T(a)(x))T(b)(x)}(\varepsilon) \geqslant b_n$$

$$\Rightarrow r_n(T(a)(x), T(b)(x)) \leqslant \varepsilon.$$

Therefore, $r_n(T(a)(x),T(b)(x)) \leqslant k_n r_n(a(x),b(x)) \leqslant k_n d_n(a,b)$. This means that $d_n(T(a),T(b)) \leqslant k_n d_n(a,b)$ for all $a,b \in Y$, and now we can apply Theorem 2.2 to conclude the proof. □

The generalized Hyers-Ulam-Rassias stability of the equation $f(x) = \Phi(x, f(\eta(x)))$ has been studied in [8]. In order to establish a similar result in probabilistic setting, we need the following lemma (cf. [26]):

LEMMA 3.2. *Let X be a nonempty set, (Y,F,T_M) be a complete probabilistic metric space and G be a mapping from $X\times R$ into [0,1], such that $G(x,\cdot)\in D_+$ for all x. Consider the set $E:=\{g:X\to Y, g(0)=0\}$ and the mapping d_G defined on $E\times E$ by*

$$d_G(g,h)=\inf\{a\in R_+, F_{g(x)h(x)}(at)\geqslant G(x,t) \text{ for all } x\in X \text{ and } t>0\}$$

($\inf\emptyset=+\infty$). Then d_G is a complete generalized metric on E.

THEOREM 3.3. *Let S be a nonempty set, (X,F,T_M) be a complete generalized Menger space and $\Phi:S\times X\to X$ be a mapping with the property that there is $k>0$ such that*

$$F_{\Phi(u,x)\Phi(u,y)}(kt)\geqslant F_{xy}(t)$$

for all $u\in S, x\in X, t>0$. If

$$F_{f(u)\Phi(u,f(\eta(u)))}\geqslant\Psi_u,\ \forall u\in S$$

where $\Psi:S\to D_+$ ($\Psi(u)$ is denoted by Ψ_u) is a mapping with the property:

$$\exists L\in(0,1):\Psi_{\eta(u)}(Lt)\geqslant\Psi_u(kt),\forall\ u\in S,\forall t>0$$

then there is a unique function $f_s:S\to X$ such that

$$f_s(u)=\Phi(u,f_s(\eta(u))),\ \forall u\in S$$

and $F_{f(u)f_s(u)}(t)\geqslant\Psi_u((1-L)t),\forall u\in S,\ \forall t>0$.

Proof. Consider the set $E:=\{g:S\to X\}$ equipped with the complete generalized metric D defined by

$$D(g,h)=\inf\left\{K\in\mathbb{R}_+, F_{g(u)h(u)}(Kt)\geqslant\Psi_u(t),\forall u\in S,\forall t>0\right\}.$$

Let $T:E\to E, T(a)(u)=\Phi(u,a(\eta(u)))$ for all $a,b\in E$ and $u\in S$ be the Baker operator.

If we suppose that $a,b\in E$ are such that $D(a,b)<\varepsilon$ then

$$F_{a(u)b(u)}(\varepsilon t)\geqslant\Psi_u(t),\forall u\in S,\forall t>0$$

and

$$\begin{aligned}F_{T(a)(u)T(b)(u)}(L\varepsilon t)&=F_{\Phi(u,a(\eta(u)))\Phi(u,b(\eta(u)))}(L\varepsilon t)\\&\geqslant F_{a(\eta(u))b(\eta(u))}\left(\frac{L}{k}\varepsilon t\right)\geqslant\Psi_{\eta(u)}(\frac{L}{k}t)\geqslant\Psi_u(t)\end{aligned}$$

for every $u\in S$ and $t>0$. Therefore, $D(T(a),T(b))\leqslant L\varepsilon$, which implies

$$D(T(a),T(b))\leqslant LD(a,b),\forall a,b\in E.$$

Next, from $F_{f(u)\Phi(u,f(\eta(u)))}\geqslant\Psi_u,\ \forall u\in S$ it follows $D(f,Tf)\leqslant1$. From Theorem 2.1 we obtain a fixed point of T, that is the existence of a mapping $f_s:S\to X$ such that $f_s(u)=\Phi(u,f_s(\eta(u))),\ \forall u\in S$. Moreover, $D(f,f_s)\leqslant\frac{1}{1-L}$ that is, the estimation $F_{f(u)f_s(u)}(\frac{1}{1-L}t)\geqslant\Psi_u(t),\forall u\in S,\ \forall t>0$ holds and, as f_s is the unique fixed point of T in the set $\{g\in E,\ D(f,g)<\infty\}$, f_s is the unique mapping verifying both $f_s(u)=\Phi(u,f_s(\eta(u))),\ \forall u\in S$ and the above estimation. $\square$

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(Received September 22, 2008)

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