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THE GOLDEN–THOMPSON–SEGAL TYPE INEQUALITIES RELATED TO THE WEIGHTED GEOMETRIC MEAN DUE TO LAWSON–LIM

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*Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday*

Abstract. In this paper, by using the weighted geometric mean $G[n,t]$ and the weighted arithmetic one $A[n,t]$ due to Lawson-Lim for each $t \in [0,1]$, we investigate n-variable versions of a complement of the Golden-Thompson-Segal type inequality due to Ando-Hiai: Let $H_1, H_2, \cdots, H_n$ be selfadjoint operators such that $m \leqslant H_i \leqslant M$ for $i = 1, 2, \cdots, n$ and some scalars $m \leqslant M$. Then

$$S(e^{p(M-m)})^{-\frac{2}{p}} \parallel G[n,t](e^{pH_1}, \cdots, e^{pH_n})^{\frac{1}{p}} \parallel \leqslant \parallel e^{A[n,t](H_1,\cdots,H_n)} \parallel \leqslant S(e^{p(M-m)})^{\frac{2}{p}} \parallel G[n,t](e^{pH_1}, \cdots, e^{pH_n})^{\frac{1}{p}} \parallel$$

for all $p > 0$ and the both-hand sides of the inequality above converge to the middle-hand side as $p \downarrow 0$, where $S(\cdot)$ is the Specht ratio and $\parallel \cdot \parallel$ stands for the operator norm.

1. Introduction

A (bounded linear) operator A on a Hilbert space H is said to be positive (in symbol: $A \geqslant 0$) if $(Ax,x) \geqslant 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For some scalars m and M, we write $m \leqslant A \leqslant M$ if $m(x,x) \leqslant (Ax,x) \leqslant M(x,x)$ for all $x \in H$. The order $A \geqslant B$ means that $A - B$ is positive. The symbol $\parallel \cdot \parallel$ stands for the operator norm. Let A and B be two positive operators on a Hilbert space H. For each $t \in [0,1]$, the weighted geometric mean $A \sharp_t B$ of A and B in the sense of Kubo-Ando [10] is defined by

$$A \sharp_t B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t A^{\frac{1}{2}}$$

if A is invertible.

In the commutative case, if H and K are Hermitian matrices, then $e^{H+K} = e^H e^K$. However, in the noncommutative case, it is entirely no relation between e^{H+K} and e^H, e^K under the usual order. For the construction of nonlinear relativistic quantum fields, Segal [13] proved that

$$\parallel e^{H+K} \parallel \leqslant \parallel e^H e^K \parallel .$$

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Also, motivated by quantum statistical mechanics, Golden [8], Symanzik [16] and Thompson [17] independently proved that

$$\operatorname{Tr} e^{H+K} \leqslant \operatorname{Tr} e^H e^K.$$

This inequality is called Golden-Thompson trace inequality.

Throughout this paper, in the setting of Hilbert space operators, we discuss the Golden-Thompson-Segal type inequalities for the operator norm. Ando and Hiai [2] gave a lower bound on $\| e^{H+K} \|$ in terms of the geometric mean: For two selfadjoint operators H and K and $t \in [0,1]$,

$$\| \left(e^{pH} \sharp_t e^{pK}\right)^{\frac{1}{p}} \| \leqslant \| e^{(1-t)H+tK} \| \tag{1.1}$$

for all $p > 0$ and the left-hand side of (1.1) converges to the right-hand side as $p \downarrow 0$.

In [6], we considered a complement of the Golden-Thompson type inequality under the usual order: Let H and K be selfadjoint operators such that $m \leqslant H, K \leqslant M$ for some scalars $m \leqslant M$, and let $t \in [0,1]$. Then

$$\begin{aligned} S(e^{M-m})^{-1} S(e^{p(M-m)})^{-\frac{1}{p}} \left(e^{pH} \sharp_t e^{pK}\right)^{\frac{1}{p}} &\leqslant e^{(1-t)H+tK} \\ &\leqslant S(e^{M-m}) S(e^{p(M-m)})^{\frac{1}{p}} \left(e^{pH} \sharp_t e^{pK}\right)^{\frac{1}{p}} \end{aligned}$$

for all $p > 0$, where $S(\cdot)$ is the Specht ratio. Moreover, in [14], we obtained a reverse of (1.1):

$$\| e^{(1-t)H+tK} \| \leqslant S(e^{p(M-m)})^{\frac{1}{p}} \| \left(e^{pH} \sharp_t e^{pK}\right)^{\frac{1}{p}} \|$$

for all $p > 0$.

In this paper, by using the weighted geometric mean $G[n,t]$ and the weighted arithmetic one $A[n,t]$ due to Lawson-Lim for each $t \in [0,1]$, we investigate n-variable versions of a complement of the Golden-Thompson-Segal type inequality due to Ando-Hiai: Let $H_1, H_2, \cdots, H_n$ be selfadjoint operators such that $m \leqslant H_i \leqslant M$ for $i = 1, 2, \cdots, n$ and some scalars $m \leqslant M$. Then

$$\begin{aligned} S(e^{p(M-m)})^{-\frac{2}{p}} & \| G[n,t](e^{pH_1}, \cdots, e^{pH_n})^{\frac{1}{p}} \| \\ &\leqslant \| e^{A[n,t](H_1, \cdots, H_n)} \| \leqslant S(e^{p(M-m)})^{\frac{2}{p}} \| G[n,t](e^{pH_1}, \cdots, e^{pH_n})^{\frac{1}{p}} \| \end{aligned} \tag{1.2}$$

for all $p > 0$ and the both-hand sides of (1.2) converge to the middle-hand side as $p \downarrow 0$.

2. Preliminary

In [3], Ando, Li and Mathias proposed a definition of the geometric mean for an n-tuple of positive operators and showed that it has many required properties on the geometric mean. Afterward, by virtue of geometry, Lawson and Lim [11, 12] established a definition of the weighted geometric mean for an n-tuple of positive operators. In

[4], we considered it in the framework of operator theory. Following [11, 4], we recall the definition of the weighted geometric mean $G[n,t]$ with $t \in [0,1]$ for an n-tuple of positive invertible operators $A_1, A_2, \cdots, A_n$. Let $G[2,t](A_1,A_2) = A_1 \sharp_t A_2$. For $n \geqslant 3$, $G[n,t]$ is defined inductively as follows: Put $A_i^{(1)} = A_i$ for all $i = 1,2,\cdots,n$ and

$$A_i^{(r)} = G[n-1,t]((A_j^{(r-1)})_{j\neq i}) = G[n-1,t](A_1^{(r-1)},\cdots,A_{i-1}^{(r-1)},A_{i+1}^{(r-1)},\cdots,A_n^{(r-1)})$$

inductively for r. Then sequences $\{A_i^{(r)}\}$ have the same limit for all $i = 1,2,\cdots,n$ in the Thompson metric. So we can define

$$G[n,t](A_1,A_2,\cdots,A_n) = \lim_{r\to\infty} A_i^{(r)}.$$

Similarly, we can define the weighted arithmetic mean as follows: Let $A[2,t](A_1,A_2) = (1-t)A_1 + tA_2$. For $n \geqslant 3$, $A[n,t]$ is defined inductively as follows: Put $\widetilde{A_i^{(1)}} = A_i$ for all $i = 1,2,\cdots,n$ and

$$\widetilde{A_i^{(r)}} = A[n-1,t]((\widetilde{A_j^{(r-1)}})_{j\neq i}) = A[n-1,t](\widetilde{A_1^{(r-1)}},\cdots,\widetilde{A_{i-1}^{(r-1)}},\widetilde{A_{i+1}^{(r-1)}},\cdots,\widetilde{A_n^{(r-1)}})$$

inductively for r. Then we see that sequences $\{\widetilde{A_i^{(r)}}\}$ have the same limit for all $i = 1,2,\cdots,n$ because it is just the problems on weights. If we put

$$A[n,t](A_1,\cdots,A_n) = \lim_{r\to\infty} \widetilde{A_i^{(r)}},$$

then it is expressed by

$$A[n,t](A_1,\cdots,A_n) = t[n]_1 A_1 + \cdots + t[n]_n A_n$$

where $t[n]_i \geqslant 0$ for $i = 1,2,\cdots,n$ and $\sum_{i=1}^n t[n]_i = 1$. Also, the weighted harmonic mean $H[n,t](A_1,\cdots,A_n)$ is defined as

$$H[n,t](A_1,\cdots,A_n) = \left(t[n]_1 A_1^{-1} + \cdots + t[n]_n A_n^{-1}\right)^{-1}.$$

We remark that the coefficient $\{t[n]_i\}$ depends on n only. For example, in the case of $n = 2,3$, it follows from [11] that

$$A[2,t](A_1,A_2) = t[2]_1 A_1 + t[2]_2 A_2 = (1-t)A_1 + tA_2,$$
$$A[3,t](A_1,A_2,A_3) = t[3]_1 A_1 + t[3]_2 A_2 + t[3]_3 A_3 = \frac{1-t}{2-t}A_1 + \frac{1-t+t^2}{2+t-t^2}A_2 + \frac{t}{1+t}A_3.$$

For the sake of convenience, we show the general term of the coefficient $\{t[n]_i\}$ in [4]: For any positive integer $n \geqslant 2$

$$t[n]_{n-m} = \frac{m(m+1) + 2m(n-2m-2)t + (n^2 - (4m+1)n + 4m(m+1))t^2}{(n-1)(m+(n-2m)t)(m+1+(n-2(m+1))t)}$$

for $m = 0, 1, \cdots, n-1$.

Moreover, the arithmetic-geometric-harmonic mean inequality holds:

$$H[n,t](A_1,\cdots,A_n) \leqslant G[n,t](A_1,\cdots,A_n) \leqslant A[n,t](A_1,\cdots,A_n). \qquad \text{(AGH)}$$

As a converse of the arithmetic-geometric mean inequality, Specht [15] estimated the upper bound of the arithmetic mean by the geometric one for positive numbers: For $x_1,\cdots,x_n \in [m,M]$ with $0 < m \leqslant M$,

$$\frac{x_1 + \cdots x_n}{n} \leqslant S(h)\sqrt[n]{x_1\cdots x_n}, \tag{2.1}$$

where $h = \frac{M}{m}(\geqslant 1)$ is a generalized condition number in the sense of Turing [19] and the Specht ratio is defined for $h > 0$ as

$$S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e\log h} \ (h \neq 1) \quad \text{and} \quad S(1) = 1. \tag{2.2}$$

Alić, Bullen, Pečarić and Volenec in [1] showed noncommutative version of Specht inequality (2.1) in the case of $n = 2$ as follows:

$$(1-t)A_1 + tA_2 \leqslant S(h)A_1 \sharp_t A_2,$$

also see [18].

Moreover, we showed n-variable noncommutative operator version of (2.1) in [4]: For any positive integer $n \geqslant 3$, let $A_1, A_2, \cdots, A_n$ be positive invertible operators such that $m \leqslant A_i \leqslant M$ for $i = 1,2,\cdots,n$ and some scalars $0 < m \leqslant M$. Put $h = \frac{M}{m}$. Then for each $t \in [0,1]$

$$A[n,t](A_1,\cdots,A_n) \leqslant S(h)^2\, G[n,t](A_1,\cdots,A_n). \tag{2.3}$$

We collect basic properties of the Specht ratio ([7, Lemma 2.47], [20]):

LEMMA 2.1. *Let $h > 0$ be given. Then the Specht ratio has the following properties:*

(1) $S(h^{-1}) = S(h)$.

(2) *A function $S(h)$ is strictly decreasing for $0 < h < 1$ and strictly increasing for $h > 1$.*

(3) $\lim_{p\to 0} S(h^p)^{\frac{1}{p}} = 1$.

3. Results

In [9], Hiai and Petz showed the following geometric mean version of the Lie-Trotter formula: If A and B are positive invertible and $t \in [0,1]$, then

$$\lim_{p\to 0} (A^p \sharp_t B^p)^{\frac{1}{p}} = e^{(1-t)\log A + t\log B}.$$

One of the authors and Nakamoto [5] defined the chaotically t-geometric mean $A \diamondsuit_t B$ which is different from the usual t-geometric mean $A \sharp_t B$:

$$A \diamondsuit_t B = e^{(1-t)\log A + t\log B}.$$

We firstly show an n-variable version of the Lie-Trotter formula for the weighted geometric mean due to Lawson-Lim:

LEMMA 3.1. *Let $A_1, A_2, \cdots, A_n$ be positive invertible operators such that $m \leqslant A_i \leqslant M$ for $i = 1, 2, \cdots, n$ and some scalars $0 < m \leqslant M$, and let $p > 0$. Then $G[n,t](A_1^p, \cdots, A_n^p)^{\frac{1}{p}}$ uniformly converges to the chaotically geometric mean $e^{A[n,t](\log A_1, \cdots, \log A_n)}$ as $p \downarrow 0$.*

Proof. It follows from [6, Lemma 3.5] that for all $\lambda_i \in [0,1]$, $i = 1, 2, \cdots, n$ such that $\sum_{i=1}^n \lambda_i = 1$,

$$0 \leqslant \log \sum_{i=1}^n \lambda_i A_i - \sum_{i=1}^n \lambda_i \log A_i \leqslant \log S(h).$$

In particular, we have

$$0 \leqslant \log A[n,t](A_1, \cdots, A_n) - A[n,t](\log A_1, \cdots, \log A_n) \leqslant \log S(h).$$

Replacing A_i by A_i^p for $p > 0$,

$$0 \leqslant \log A[n,t](A_1^p, \cdots, A_n^p) - A[n,t](\log A_1^p, \cdots, \log A_n^p) \leqslant \log S(h^p)$$

and hence

$$0 \leqslant \log A[n,t](A_1^p, \cdots, A_n^p)^{\frac{1}{p}} - A[n,t](\log A_1, \cdots, \log A_n) \leqslant \log S(h^p)^{\frac{1}{p}}.$$

Since $S(h^p)^{\frac{1}{p}} \to 1$ as $p \downarrow 0$, it follows that $A[n,t](A_1^p, \cdots, A_n^p)^{\frac{1}{p}}$ uniformly converges to the chaotically geometric mean $e^{A[n,t](\log A_1, \cdots, \log A_n)}$ as $p \downarrow 0$.

On the other hand, since

$$0 \leqslant \log A[n,t](A_1^{-1}, \cdots, A_n^{-1}) - A[n,t](\log A_1^{-1}, \cdots, \log A_n^{-1}) \leqslant \log S(h^{-1}),$$

it follows from $S(h^{-1}) = S(h)$ by Lemma 2.1 that

$$0 \geqslant \log H[n,t](A_1, \cdots, A_n) - A[n,t](\log A_1, \cdots, \log A_n) \geqslant -\log S(h)$$

and this implies

$$0 \geqslant \log H[n,t](A_1^p, \cdots, A_n^p)^{\frac{1}{p}} - A[n,t](\log A_1, \cdots, \log A_n) \geqslant -\log S(h^p)^{\frac{1}{p}}$$

for all $p > 0$.

Hence $H[n,t](A_1^p, \cdots, A_n^p)^{\frac{1}{p}}$ uniformly converges to the chaotically geometric mean $e^{A[n,t](\log A_1, \cdots, \log A_n)}$ as $p \downarrow 0$.

By (AGH), we have

$$\log H[n,t](A_1^p,\cdots,A_n^p)^{\frac{1}{p}} \leqslant \log G[n,t](A_1^p,\cdots,A_n^p)^{\frac{1}{p}} \leqslant \log A[n,t](A_1^p,\cdots,A_n^p)^{\frac{1}{p}}$$

for all $p>0$ and hence we have this lemma. □

For the case of $n=2$, Ando-Hiai showed that the norm $\| (A_1^p \sharp_t A_2^p)^{\frac{1}{p}} \|$ is monotone increasing for $p>0$. For $n \geqslant 3$, we have the following result:

LEMMA 3.2. *Let $A_1,A_2,\cdots,A_n$ be positive invertible operators such that $m \leqslant A_i \leqslant M$ for $i=1,2,\cdots,n$ and some scalars $0<m\leqslant M$. Put $h=\frac{M}{m}$. Then for each $0<q<p$*

$$\begin{aligned}&S(h^p)^{-\frac{2}{p}} \ \| G[n,t](A_1^p,\cdots,A_n^p)^{\frac{1}{p}} \| \\ &\qquad \leqslant \| G[n,t](A_1^q,\cdots,A_n^q)^{\frac{1}{q}} \| \leqslant S(h^p)^{\frac{2}{p}} \ \| G[n,t](A_1^p,\cdots,A_n^p)^{\frac{1}{p}} \|,\end{aligned}$$

where $S(h)$ is defined as (2.2).

Proof. By the arithmetic-geometric mean inequality, it follows that for each $0<q<p$

$$\begin{aligned}G[n,t](A_1^{\frac{q}{p}},\cdots,A_n^{\frac{q}{p}}) &\leqslant A[n,t](A_1^{\frac{q}{p}},\cdots,A_n^{\frac{q}{p}}) \\ &\leqslant A[n,t](A_1,\cdots,A_n)^{\frac{q}{p}} \quad \text{by the concavity of } t^{\frac{q}{p}} \text{ and } 0<\tfrac{q}{p}<1 \\ &\leqslant S(h)^{\frac{2q}{p}} G[n,t](A_1,\cdots,A_n)^{\frac{q}{p}} \quad \text{by (2.3) and Löwner-Heinz Theorem.}\end{aligned}$$

Replacing A_i by A_i^p, we have

$$G[n,t](A_1^q,\cdots,A_n^q) \leqslant S(h^p)^{\frac{2q}{p}} G[n,t](A_1^p,\cdots,A_n^p)^{\frac{q}{p}}.$$

Also,

$$G[n,t](A_1^{-q},\cdots,A_n^{-q}) \leqslant S(h^{-p})^{\frac{2q}{p}} G[n,t](A_1^{-p},\cdots,A_n^{-p})^{\frac{q}{p}}$$

and hence

$$G[n,t](A_1^q,\cdots,A_n^q) \geqslant S(h^p)^{-\frac{2q}{p}} G[n,t](A_1^p,\cdots,A_n^p)^{\frac{q}{p}}.$$

Therefore we have for all $q>0$

$$\begin{aligned}&S(h^p)^{-\frac{2}{p}} \ \| G[n,t](A_1^p,\cdots,A_n^p)^{\frac{1}{p}} \| \\ &\qquad \leqslant \| G[n,t](A_1^q,\cdots,A_n^q)^{\frac{1}{q}} \| \leqslant S(h^p)^{\frac{2}{p}} \ \| G[n,t](A_1^p,\cdots,A_n^p)^{\frac{1}{p}} \|.\end{aligned}$$ □

By Lemma 3.2, we show n-variable versions of a complement of the Golden-Thompson-Segal type inequality due to Ando-Hiai:

THEOREM 3.3. *Let $H_1, H_2, \cdots, H_n$ be selfadjoint operators such that $m \leqslant H_i \leqslant M$ for $i = 1,2,\cdots,n$ and some scalars $m \leqslant M$. Then*

$$S(e^{p(M-m)})^{-\frac{2}{p}} \ \| G[n,t](e^{pH_1},\cdots,e^{pH_n})^{\frac{1}{p}} \| \leqslant \| e^{A[n,t](H_1,\cdots,H_n)} \| \leqslant S(e^{p(M-m)})^{\frac{2}{p}} \ \| G[n,t](e^{pH_1},\cdots,e^{pH_n})^{\frac{1}{p}} \| \quad (3.1)$$

for all $p > 0$ and the both-hand sides of (3.1) *converge to the middle-hand side as $p \downarrow 0$, where the Specht ratio $S(h)$ is defined as* (2.2).

Proof. If we replace A_i by e^{H_i} in Lemma 3.2, then it follows that

$$S(e^{p(M-m)})^{-\frac{2}{p}} \ \| G[n,t](e^{pH_1},\cdots,e^{pH_n})^{\frac{1}{p}} \| \leqslant \| G[n,t](e^{qH_1},\cdots,e^{qH_n})^{\frac{1}{q}} \| \leqslant S(e^{p(M-m)})^{\frac{2}{p}} \ \| G[n,t](e^{pH_1},\cdots,e^{pH_n})^{\frac{1}{p}} \|$$

for all $0 < q < p$. Hence we have (3.1) as $q \downarrow 0$ by Lemma 3.1.

The latter part of Theorem follows from $S(e^{p(M-m)})^{\frac{2}{p}} \to 1$ as $p \downarrow 0$ by Lemma 2.1. □

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