Journal of
Mathematical
Inequalities
Volume 3, Number 4 (2009), 559–566

ESTIMATIONS OF THE ERROR FOR GENERAL SIMPSON TYPE FORMULAE VIA PRE–GRÜSS INEQUALITY

A. VUKELIĆ

Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. Generalizations of estimations of general Simpson type formulae are given, by using the pre-Grüss inequality.

1. Introduction

In the recent paper [8] N. Ujevic used the generalization of pre-Grüss inequality to derive some better estimations of the error for Simpson's quadrature rule. In fact, he proved the next three theorems:

THEOREM 1. *Let $I \subset \mathbf{R}$ be a closed interval and $a, b \in \operatorname{Int} I$, $a < b$. If $f : I \to \mathbf{R}$ is an absolutely continuous function with $f' \in L_2(a,b)$ then we have*

$$\left|\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\int_a^b f(t)dt\right| \leqslant \frac{(b-a)^{3/2}}{6}K_1, \tag{1.1}$$

where

$$K_1^2 = \|f'\|_2^2 - \frac{1}{b-a}\left(\int_a^b f'(t)dt\right)^2 - \left(\int_a^b f'(t)\Psi_0(t)dt\right)^2 \tag{1.2}$$

and $\Psi(t) = t - \frac{a+b}{2}, \Psi_0(t) = \Psi(t)/\|\Psi\|_2$.

THEOREM 2. *Let $I \subset \mathbf{R}$ be a closed interval and $a, b \in \operatorname{Int} I$, $a < b$. If $f : I \to \mathbf{R}$ is such that f' is an absolutely continuous function with $f'' \in L_2(a,b)$ then we have*

$$\left|\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\int_a^b f(t)dt\right| \leqslant \frac{(b-a)^{5/2}}{12\sqrt{30}}K_2, \tag{1.3}$$

where

$$K_2^2 = \|f''\|_2^2 - \frac{1}{b-a}\left(\int_a^b f''(t)dt\right)^2 - \left(\int_a^b f''(t)\Psi_0(t)dt\right)^2, \tag{1.4}$$

Mathematics subject classification (2000): 26D10, 41A55.
Keywords and phrases: Pre-Grüss inequality, generalization, general Simpson type formulae.

Paper JMI-03-54

$$\Psi(t) = \begin{cases} 1, & t \in \left[a, \frac{a+b}{2}\right] \\ -1, & t \in \left(\frac{a+b}{2}, b\right] \end{cases} \tag{1.5}$$

and $\Psi_0(t) = \Psi(t)/\|\Psi\|_2$.

THEOREM 3. *Let* $I \subset \mathbf{R}$ *be a closed interval and* $a,b \in \operatorname{Int} I$, $a < b$. *If* $f : I \to \mathbf{R}$ *is such that* f'' *is an absolutely continuous function with* $f''' \in L_2(a,b)$ *then we have*

$$\left| \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \int_a^b f(t)dt \right| \leqslant \frac{(b-a)^{7/2}}{48\sqrt{105}} K_3, \tag{1.6}$$

where

$$K_3^2 = \|f'''\|_2^2 - \frac{1}{b-a}\left(\int_a^b f'''(t)dt\right)^2 - \left(\int_a^b f'''(t)\Psi_0(t)dt\right)^2, \tag{1.7}$$

$$\Psi(t) = \begin{cases} t - \frac{7a+3b}{10}, & t \in \left[a, \frac{a+b}{2}\right] \\ t - \frac{3a+7b}{10}, & t \in \left(\frac{a+b}{2}, b\right] \end{cases} \tag{1.8}$$

and $\Psi_0(t) = \Psi(t)/\|\Psi\|_2$.

In this paper we will unify and generalize these results so that we will give the results for general Euler-Simpson formula and for functions whose derivative of order n, $n \geqslant 1$, is from $L_2(0,1)$ space. We will also give related results for the general dual Euler-Simpson formula. We will use interval $[0,1]$ because of simplicity and since it involves no loss in generality.

2. Estimations of the error for general Euler-Simpson formula

In the recent paper [6] the following identity, named the general Euler-Simpson formula, has been proved. For $n \geqslant 1$ and every $t \in [0,1]$ we have

$$\int_0^1 f(t)\mathrm{d}t = D(u,v) - T_n(u,v) + S_n(f) \tag{2.1}$$

where

$$D(u,v) = \frac{1}{2u+v}\left[uf(0) + vf\left(\frac{1}{2}\right) + uf(1)\right],$$

$T_0(u,v) = 0$ and

$$T_m(u,v) = \frac{1}{2u+v}\sum_{k=1}^{m} \frac{\tilde{B}_k}{k!}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right], \tag{2.2}$$

for $1 \leqslant m \leqslant n$, while

$$\tilde{B}_k = uB_k(0) + vB_k\left(\frac{1}{2}\right) + uB_k(1),\ k \geqslant 1,$$

$$S_n(x) = \frac{1}{(2u+v)(n!)} \int_0^1 G_n(t) f^{(n)}(t) dt$$

and

$$G_n(t) = 2uB_n^*(1-t) + vB_n^*\left(\frac{1}{2}-t\right),\ t \in \mathbf{R}.$$

The identity holds for every function $f : [0,1] \to \mathbf{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$. $u,v \in \mathbf{Z}^+$ and the greatest common divisor of u and v is 1. The functions $B_k(t)$ are the Bernoulli polynomials, $B_k = B_k(0)$ are the Bernoulli numbers, and $B_k^*(t)$, $k \geqslant 0$, are periodic functions of period 1, related to the Bernoulli polynomials as

$$B_k^*(t) = B_k(t),\ 0 \leqslant t < 1 \quad \text{and} \quad B_k^*(t+1) = B_k^*(t),\ t \in \mathbf{R}.$$

The Bernoulli polynomials $B_k(t)$, $k \geqslant 0$ are uniquely determined by the following identities

$$B_k'(t) = kB_{k-1}(t),\ k \geqslant 1;\ B_0(t) = 1,\ B_k(t+1) - B_k(t) = kt^{k-1},\ k \geqslant 0.$$

For some further details on the Bernoulli polynomials and the Bernoulli numbers see for example [1] or [2]. We have $B_0^*(t) = 1$ and $B_1^*(t)$ is a discontinuous function with a jump of -1 at each integer. It follows that $B_k(1) = B_k(0) = B_k$ for $k \geqslant 2$, so that $B_k^*(t)$ are continuous functions for $k \geqslant 2$. We get

$$B_k^{*\prime}(t) = kB_{k-1}^*(t),\ k \geqslant 1 \tag{2.3}$$

for every $t \in \mathbf{R}$ when $k \geqslant 3$, and for every $t \in \mathbf{R} \setminus \mathbf{Z}$ when $k = 1,2$.

In the proof of our main result we shell use the following result of N. Ujević ([8]):

LEMMA 1. *If* $g,h,\Psi \in L_2(0,1)$ *and* $\int_0^1 \Psi(t)dt = 0$ *then we have*

$$|S_\Psi(g,h)| \leqslant S_\Psi(g,g)^{1/2} S_\Psi(h,h)^{1/2}, \tag{2.4}$$

where

$$S_\Psi(g,h) = \int_0^1 g(t)h(t)dt - \int_0^1 g(t)dt \int_0^1 h(t)dt - \int_0^1 g(t)\Psi_0(t)dt \int_0^1 h(t)\Psi_0(t)dt$$

and $\Psi_0(t) = \Psi(t)/\|\Psi\|_2$.

THEOREM 4. *If* $f : [0,1] \to \mathbf{R}$ *is such that* $f^{(n-1)}$ *is absolutely continuous function with* $f^{(n)} \in L_2(0,1)$ *then we have*

$$\left| \int_0^1 f(t)\mathrm{d}t - D(u,v) + T_n(u,v) \right| \tag{2.5}$$

$$\leqslant \frac{1}{2u+v} \left[\frac{(-1)^{n-1}}{(2n)!} \left[4u^2 + v^2 - 4uv(1-2^{1-2n})\right] B_{2n} \right]^{1/2} K,$$

where

$$K^2=\|f^{(n)}\|_2^2-\left(\int_0^1 f^{(n)}(t)dt\right)^2-\left(\int_0^1 f^{(n)}(t)\Psi_0(t)dt\right)^2. \tag{2.6}$$

For n even

$$\Psi(t)=\begin{cases}1, & t\in\left[0,\frac{1}{2}\right],\\ -1, & t\in\left(\frac{1}{2},1\right],\end{cases}$$

while for n odd we have

$$\Psi(t)=\begin{cases}t+\frac{2^{1-n}u-2u+v}{2^{1-n}v-2^{2-n}u+8u-4v}, & t\in\left[0,\frac{1}{2}\right],\\ t+\frac{2^{1-n}(u-v)+3v-6u}{2^{1-n}v-2^{2-n}u+8u-4v}, & t\in\left(\frac{1}{2},1\right].\end{cases}$$

Proof. It is not difficult to verify that

$$\int_0^1 G_n(t)dt=0, \tag{2.7}$$

$$\int_0^1 \Psi(t)dt=0, \tag{2.8}$$

$$\int_0^1 G_n(t)\Psi(t)dt=0. \tag{2.9}$$

From (2.1), (2.7) and (2.9) it follows that

$$\begin{aligned}&\int_0^1 f(t)\mathrm{d}t-D(u,v)+T_n(u,v)\\ &=\frac{1}{(2u+v)(n!)}\int_0^1 G_n(t)f^{(n)}(t)dt-\frac{1}{(2u+v)(n!)}\int_0^1 G_n(t)dt\int_0^1 f^{(n)}(t)dt\\ &\quad-\frac{1}{(2u+v)(n!)}\int_0^1 G_n(t)\Psi_0(t)dt\int_0^1 f^{(n)}(t)\Psi_0(t)dt\\ &=\frac{1}{(2u+v)(n!)}S_\Psi(G_n,f^{(n)}).\end{aligned} \tag{2.10}$$

Using (2.10) and (2.4) we get

$$\left|\int_0^1 f(t)\mathrm{d}t-D(u,v)+T_n(u,v)\right|\leqslant\frac{1}{(2u+v)(n!)}S_\Psi(G_n,G_n)^{1/2}S_\Psi(f^{(n)},f^{(n)})^{1/2}. \tag{2.11}$$

We also have (see [6])

$$\begin{aligned}S_\Psi(G_n,G_n)&=\|G_n\|_2^2-\left(\int_0^1 G_n(t)dt\right)^2-\left(\int_0^1 G_n(t)\Psi_0(t)dt\right)^2\\ &=(-1)^{n-1}\frac{(n!)^2}{(2n)!}\left[4u^2+v^2-4uv(1-2^{1-2n})\right]B_{2n}\end{aligned} \tag{2.12}$$

and

$$S_\Psi(f^{(n)},f^{(n)}) = \|f^{(n)}\|_2^2 - \left(\int_0^1 f^{(n)}(t)dt\right)^2 - \left(\int_0^1 f^{(n)}(t)\Psi_0(t)dt\right)^2 = K^2. \quad (2.13)$$

From (2.11)–(2.13) we easily get (2.5). □

REMARK 1. Function $\Psi(t)$ can be any function which satisfies conditions $\int_0^1 \Psi(t)dt = 0$ and $\int_0^1 G_n(t)\Psi(t)dt = 0$. Since $G_n(1-t) = (-1)^n G_n(t)$ (see [6]), for n we can even take function $\Psi(t)$ such that $\Psi(1-t) = -\Psi(t)$. For n odd we have to calculate $\Psi(t)$ and without lost of generality in our theorem we take the form $\Psi(t) = \begin{cases} t+a,\ t\in\left[0,\frac{1}{2}\right], \\ t+b,\ t\in\left(\frac{1}{2},1\right]. \end{cases}$

REMARK 2. The inequality (2.5) achieves minimum of $\left[\frac{(-1)^{n-1}}{(2n)!}2^{-2n}B_{2n}\right]^{1/2}$ for $u=1$ and $v=2$ which is bitrapezoid formula (see [4]). For $n=1$ it is $1/4\sqrt{3}$.

REMARK 3. For $u=1$ and $v=4$ in Theorem 4 we get Euler-Simpson formula (see [3]) and then we have

$$\left|\int_0^1 f(t)\mathrm{d}t - D(1,4) + T_n(1,4)\right| \leqslant \frac{1}{3}\left[\frac{(-1)^{n-1}}{(2n)!}\left(1+2^{3-2n}\right)B_{2n}\right]^{1/2} K, \quad (2.14)$$

where

$$D(1,4) = \frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right],$$

and

$$T_n(1,4) = \sum_{k=2}^{\lfloor n/2\rfloor} \frac{1}{3(2k)!}(1-2^{2-2k})B_{2k}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right].$$

For n even

$$\Psi(t) = \begin{cases} 1,\ t\in\left[0,\frac{1}{2}\right], \\ -1,\ t\in\left(\frac{1}{2},1\right], \end{cases}$$

while for n odd we have

$$\Psi(t) = \begin{cases} t+\frac{2^{-n}+1}{4(2^{-1-n}-1)},\ t\in\left[0,\frac{1}{2}\right], \\ t+\frac{3(1-2^{-n})}{4(2^{-1-n}-1)},\ t\in\left(\frac{1}{2},1\right]. \end{cases}$$

For $n=1$, 2 and 3 in the inequality (2.14) we get inequalities (1.1), (1.3) and (1.6) respectively.

3. Estimations of the error for general dual Euler-Simpson formula

In the recent paper [7] the following identity, named the general dual Euler-Simpson formula, has been proved. For $n \geqslant 1$ and every $t \in [0,1]$ we have

$$\int_0^1 f(t)\mathrm{d}t = F(u,v) - T_n^D(u,v) + R_n(f) \tag{3.1}$$

where

$$F(u,v) = \frac{1}{2u-v}\left[uf\left(\frac{1}{4}\right) - vf\left(\frac{1}{2}\right) + uf\left(\frac{3}{4}\right)\right],$$

$T_0^D(u,v) = 0$ and

$$T_m^D(u,v) = \frac{1}{2u-v}\sum_{k=1}^{m}\frac{\tilde{B}_k^D}{k!}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right], \tag{3.2}$$

for $1 \leqslant m \leqslant n$, while

$$\tilde{B}_k^D = uB_k\left(\frac{1}{4}\right) - vB_k\left(\frac{1}{2}\right) + uB_k\left(\frac{3}{4}\right),\ k \geqslant 1,$$

$$R_n(x) = \frac{1}{(2u-v)(n!)}\int_0^1 G_n^D(t)\, f^{(n)}(t)dt$$

and

$$G_n^D(t) = uB_n^*\left(\frac{1}{4}-t\right) - vB_n^*\left(\frac{1}{2}-t\right) + uB_n^*\left(\frac{3}{4}-t\right),\ t \in \mathbf{R}.$$

The identity holds for every function $f:[0,1]\to\mathbf{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$. $u,v \in \mathbf{Z}^+$, $v < 2u$ and the greatest common divisor of u and v is 1.

THEOREM 5. *If $f:[0,1]\to\mathbf{R}$ is such that $f^{(n-1)}$ is absolutely continuous function with $f^{(n)} \in L_2(0,1)$ then we have*

$$\left|\int_0^1 f(t)\mathrm{d}t - F(u,v) + T_n^D(u,v)\right| \tag{3.3}$$

$$\leqslant \frac{1}{2u-v}\left[\frac{(-1)^{n-1}}{(2n)!}\left[2u^2+v^2-(2u^2-uv\cdot 2^{2-2n})(1-2^{1-2n})\right]B_{2n}\right]^{1/2} K,$$

where

$$K^2 = \|f^{(n)}\|_2^2 - \left(\int_0^1 f^{(n)}(t)dt\right)^2 - \left(\int_0^1 f^{(n)}(t)\Psi_0(t)dt\right)^2. \tag{3.4}$$

For n even

$$\Psi(t) = \begin{cases} 1, & t \in \left[0,\frac{1}{2}\right], \\ -1, & t \in \left(\frac{1}{2},1\right], \end{cases}$$

while for n *odd we have*

$$\Psi(t)=\begin{cases} t+\frac{2^{-n}u(1-2^{-n})+v}{4v(2^{-n-1}-1)}, & t\in\left[0,\frac{1}{2}\right], \\ t+\frac{v(3-2^{-n+1})-2^{-n}u(1-2^{-n})}{4v(2^{-n-1}-1)}, & t\in\left(\frac{1}{2},1\right]. \end{cases}$$

Proof. Similar as in Theorem 4. □

REMARK 4. For $u=2$ and $v=1$ in Theorem 5 we get the dual Euler-Simpson formula (see [5]) and then we have

$$\left|\int_0^1 f(t)\mathrm{d}t-F(2,1)+T_n^D(2,1)\right|\leqslant\frac{1}{3}\left[\frac{(-1)^{n-1}}{(2n)!}\left[9-(8-2^{3-2n})(1-2^{1-2n})\right]B_{2n}\right]^{1/2}K, \tag{3.5}$$

where

$$F(2,1)=\frac{1}{3}\left[2f\left(\frac{1}{4}\right)-f\left(\frac{1}{2}\right)+2f\left(\frac{3}{4}\right)\right],$$

and

$$T_n^D(2,1)=\sum_{k=2}^{\lfloor n/2\rfloor}\frac{1}{3(2k)!}\left(8\cdot 2^{-4k}-6\cdot 2^{-2k}+1\right)B_{2k}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right].$$

For n even

$$\Psi(t)=\begin{cases} 1, & t\in\left[0,\frac{1}{2}\right], \\ -1, & t\in\left(\frac{1}{2},1\right], \end{cases}$$

while for n odd we have

$$\Psi(t)=\begin{cases} t+\frac{2^{1-n}(1-2^{-n})+1}{4(2^{-1-n}-1)}, & t\in\left[0,\frac{1}{2}\right], \\ t+\frac{3-2^{2-n}+2^{1-2n}}{4(2^{-1-n}-1)}, & t\in\left(\frac{1}{2},1\right]. \end{cases}$$

For $n=1,\ 2$ and 3 we get inequalities

$$\left|\frac{1}{3}\left[2f\left(\frac{1}{4}\right)-f\left(\frac{1}{2}\right)+2f\left(\frac{3}{4}\right)\right]-\int_0^1 f(t)dt\right|\leqslant\frac{1}{3\sqrt{2}}K_1,$$

$$\left|\frac{1}{3}\left[2f\left(\frac{1}{4}\right)-f\left(\frac{1}{2}\right)+2f\left(\frac{3}{4}\right)\right]-\int_0^1 f(t)dt\right|\leqslant\frac{\sqrt{13}}{48\sqrt{15}}K_2$$

and

$$\left|\frac{1}{3}\left[2f\left(\frac{1}{4}\right)-f\left(\frac{1}{2}\right)+2f\left(\frac{3}{4}\right)\right]-\int_0^1 f(t)dt\right|\leqslant\frac{\sqrt{13}}{192\sqrt{70}}K_3$$

respectively.

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (Eds.), *Handbook of mathematical functions with formulae,graphs and mathematical tables*, National Bureau of Standards, Applied Math.Series 55, 4th printing, Washington 1965.

[2] I.S. BEREZIN AND N.P. ZHIDKOV, *Computing methods*, Vol. I, Pergamon Press, Oxford-London-Edinburgh-New York-Paris-Frankfurt, 1965.

[3] LJ. DEDIĆ, M. MATIĆ, J. PEČARIĆ, *On Euler-Simpson formulae*, PanAmer. Math. J., **11**, 2 (2001), 47–64.

[4] LJ. DEDIĆ, M. MATIĆ, J. PEČARIĆ AND A. VUKELIĆ, *Hadamard-type inequalities via some Euler-type identities- Euler bitrapezoid formulae*, Nonlinear Studies, **8**, 3 (2001), 343–372.

[5] LJ. DEDIĆ, M. MATIĆ, J. PEČARIĆ, *On dual Euler-Simpson formulae*, Bull. Belg. Math. Soc., **8** (2001), 479–504.

[6] I. FRANJIĆ, J. PEČARIĆ, A. VUKELIĆ, *General Euler-Simpson formulae*, Indian Journal of Mathematics, **47**, 2–3 (2005), 139–157.

[7] J. PEČARIĆ, A. VUKELIĆ, *General dual Euler-Simpson formula*, Journal of Mathematical Inequalities **2**, 4 (2008), 511–526.

[8] N. UJEVIĆ, *New bounds for Simpson's inequality*, Tamkang J. of Math., **33**, 2 (2002), 129–138.

(Received October 27, 2008)

A. Vukelić
Faculty of Food Technology and Biotechnology
Mathematics department
University of Zagreb
Pierottijeva 6
10000 Zagreb
Croatia
e-mail: avukelic@pbf.hr