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APPLICATIONS OF CERTAIN DIFFERENTIAL INEQUALITIES TO THE UNIVALENCE OF AN INTEGRAL OPERATOR

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Dedicated to Professor Josip Pečarić
on the occasion of his 60th birthday

Abstract. In [1] we have introduced the integral operator denoted by $I(f_1, f_2, \ldots, f_m)$ given in Definition 2. Also, certain sufficient conditions of univalence were given for this operator. In this paper we take a different approach for proving the univalence of this operator.

1. Introduction and preliminaries

Let U denote the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\}$$

and

$$\overline{U} = \{z \in \mathbb{C} : \ |z| \leqslant 1\}.$$

Let $\mathscr{H}(U)$ denote the space of holomorphic functions in U and let

$$A_n = \{f \in \mathscr{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + \ldots, \ z \in U\}$$

with $A_1 = A$.

Let

$$S = \{f \in A : \ f \text{ is univalent in } U\}.$$

DEFINITION 1. (St. Ruscheweyh [3]). For $f \in A$, $n \in \mathbb{N} \cup \{0\}$, let R^n be the operator defined by $R^n : A \to A$

$$R^0 f(z) = f(z)$$
$$(n+1)R^{n+1} f(z) = z[R^n f(z)]' + nR^n f(z), \ z \in U.$$

REMARK 1. If $f \in A$

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in U$$

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then

$$R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^n a_j z^j, \quad z \in U. \tag{1}$$

In [1] we have defined the following integral operator:

DEFINITION 2. [1] Let $n,m \in \mathbb{N} \cup \{0\}$, $i \in \{1,2,3,\dots,m\}$, $\alpha_i \in \mathbb{C}$, $\alpha \in \mathbb{C}$, $\operatorname{Re} \alpha > 0$, $A^m = \underbrace{A \times A \times \dots \times A}_{m\ times}$. We let $I : A^m \to A$ be the integral operator given by

$$I(f_1, f_2, \dots, f_m)(z) = F(z) \tag{2}$$

$$= \left[\alpha \int_0^z t^{\alpha-1} \left(\frac{R^n f_1(t)}{t}\right)^{\alpha_1} \dots \left(\frac{R^n f_m(t)}{t}\right)^{\alpha_m} dt\right]^{\frac{1}{\alpha}},$$

where $f_i \in A$, $i \in \{1,2,3,\dots,n\}$ and R^n is the Ruscheweyh differential operator (Definition 1).

In order to prove our main results, we shall use the following lemma:

LEMMA A. [2, Theorem 5] *Let $n,m \in \mathbb{N} \cup \{0\}$, α be a complex number with $\operatorname{Re} \alpha > 0$ and c a complex number with $|c| \leqslant 1$, $c \neq -1$. If $f_k \in A$, $\alpha_k \in \mathbb{C}$, $k \in \{1,2,\dots,m\}$ and if*

$$\left| c|z|^{2\alpha} + (1-|z|^{2\alpha}) \frac{z}{\alpha} \cdot \frac{f''(z)}{f'(z)} \right| \leqslant 1,$$

holds for all $z \in U$, where f is given by

$$f(z) = \int_0^z \left(\frac{R^n f_1(t)}{t}\right)^{\alpha_1} \dots \left(\frac{R^n f_m(t)}{t}\right)^{\alpha_m} dt, \quad z \in U,$$

with R^n given by Definition 1, then the function F given by (2), belongs to the class S.

REMARK 2. In [1], the author has shown that this operator is a generalization of other operators.

REMARK 3. In [1], the author has stated and proven the following theorems:

THEOREM 1. *Let $n,m \in \mathbb{N} \cup \{0\}, \alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0$, $f_i \in A$, $\alpha_i \in \mathbb{C}$, $i \in \{1,2,\dots,m\}$ with $|\alpha_1| + |\alpha_2| + \dots + |\alpha_m| \leqslant 1$.*

If

$$\left| \frac{z(R^n f_i(z))'}{R^n f_i(z)} - 1 \right| \leqslant 1, \quad z \in U, \ i \in \{1,2,\dots,m\}$$

then $F(z)$ given by (2) belongs to the class S.

THEOREM 2. *Let $n,m \in \mathbb{N} \cup \{0\}, \alpha, \beta \in \mathbb{C}$, with $\operatorname{Re} \beta \geqslant \operatorname{Re} \alpha > 0$, let $f_i \in A$ and let $\alpha_i \in \mathbb{C}$, $i \in \{1,2,\dots,m\}$, with $|\alpha_1| + |\alpha_2| + \dots + |\alpha_m| \leqslant 1$.*

If

$$\left| \frac{z(R^n f_i(z))'}{R^n f_i(z)} - 1 \right| \leqslant 1, \quad z \in U, \ i \in \{1,2,\dots,m\}$$

where R^n *is the Ruschweyh differential operator, then the function given by*

$$(2') \qquad F_\beta(z) = \left[\beta \int_0^z t^{\beta-1}\left(\frac{R^n f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{R^n f_m(t)}{t}\right) dt\right]^{\frac{1}{\beta}}$$

belongs to the class S.

THEOREM 3. *Let* $n,m \in \mathbb{N}\cup\{0\}, \mu > 0, \alpha \in \mathbb{C}$ *with* $\operatorname{Re}\alpha > 0$, $f_i \in A$, $\alpha_i \in \mathbb{C}$, $i \in \{1,2,\ldots,m\}$ *with* $|\alpha_1| + |\alpha_2| + \ldots + |\alpha_m| \leqslant \dfrac{1}{2\mu+1}$.

If

(i) $|R^n f_i(z)| \leqslant \mu$,

(ii) $\left|\dfrac{z^2(R^n f_i(z))'}{[R^n f_i(z)]^2} - 1\right| \leqslant 1,\ z \in U,\ i \in \{1,2,\ldots,m\}$

where R^n *is the Ruschweyh differential operator, then the function* $F(z)$ *given by (2) belongs to the class* S.

THEOREM 4. *Let* $n,m \in \mathbb{N}\cup\{0\}, \mu > 0, \alpha,\beta \in \mathbb{C}$, *with* $\operatorname{Re}\beta \geqslant \operatorname{Re}\alpha > 0$, *let* $f_i \in A$ *and let* $\alpha_i \in \mathbb{C}$, $i \in \{1,2,\ldots,m\}$, *with* $|\alpha_1| + |\alpha_2| + \ldots + |\alpha_m| \leqslant \dfrac{1}{2\mu+1}$.

If

(i) $|R^n f_i(z)| \leqslant \mu$,

(ii) $\left|\dfrac{z^2(R^n f_i(z))'}{[R^n f_i(z)]^2} - 1\right| \leqslant 1,\ z \in U,\ i \in \{1,2,\ldots,m\}$

where R^n *is the Ruschweyh differential operator, then the function* $F(z)$ *given by (2') belongs to the class* S.

2. Main results

THEOREM 5. *Let* $n,m \in \mathbb{N}\cup\{0\}, \mu \geqslant 0, \alpha$ *and* c *be complex numbers with* $\operatorname{Re}\alpha > 0$, $|c| \leqslant 1$, $c \neq -1$ *and let* $f_k \in A$, $\alpha_k \in \mathbb{C}$, $k \in \{1,2,\ldots,m\}$.

If

$$\begin{aligned} &i)\ |\alpha_1| + |\alpha_2| + \ldots + |\alpha_m| \leqslant (1-|c|)\cdot\frac{|\alpha|}{1+\mu(1+|\alpha|)} \\ &ii)\ |R^n f_k(z)| \leqslant \mu \\ &iii)\ \left|\frac{1-|z|^{2\alpha}}{\alpha}\left[\frac{z^2(R^n f_k(z))'}{(R^n f_k(z))^2} - 1\right]\right| \leqslant 1,\ z \in U,\ k \in \{1,2,\ldots,m\}, \end{aligned} \tag{3}$$

where R^n *is the Ruscheweyh differential operator, then the function* F *given by (2) belongs to the class* S.

Proof. Let

$$f(z) = \int_0^z \left[\frac{R^n f_1(t)}{t}\right]^{\alpha_1} \cdots \left[\frac{R^n f_m(t)}{t}\right]^{\alpha_m} dt, \quad z \in U. \tag{4}$$

By differentiating (4), we obtain

$$f'(z)=\left[\frac{R^nf_1(z)}{z}\right]^{\alpha_1}\dots\left[\frac{R^nf_m(z)}{z}\right]^{\alpha_m}=1+p_2z+p_3z^2+\dots, \tag{5}$$

$z\in U$.

By differentiating (5), after a short calculation, we have

$$\frac{zf''(z)}{f'(z)}=\alpha_1\left[\frac{z(R^nf_1(z))'}{R^nf_1(z)}-1\right]+\dots+\alpha_m\left[\frac{z(R^nf_m(z))'}{R^nf_m(z)}-1\right],\quad z\in U. \tag{6}$$

In order to prove the theorem, using (6), we evaluate

$$\begin{aligned}\left|\frac{1-|z|^{2\alpha}}{\alpha}\cdot\frac{zf''(z)}{f'(z)}\right| &= \left|\frac{1-|z|^{2\alpha}}{\alpha}\alpha_1\left[\frac{z(R^nf_1(z))'}{R^nf_1(z)}-1\right]+\dots\right. \\ &\quad\left.+\frac{1-|z|^{2\alpha}}{\alpha}\alpha_m\left[\frac{z(R^nf_m(z))'}{R^nf_m(z)}-1\right]\right| \\ &\leqslant |\alpha_1|\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|\left|\frac{z(R^nf_1(z))'}{R^nf_1(z)}-1\right|+\dots \\ &\quad+|\alpha_m|\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|\left|\frac{z(R^nf_m(z))'}{R^nf_m(z)}-1\right| \\ &\leqslant |\alpha_1|\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|\left[\left|\frac{z(R^nf_1(z))'}{R^nf_1(z)}\right|+1\right]+\dots \\ &\quad+|\alpha_m|\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|\left[\left|\frac{z(R^nf_m(z))'}{R^nf_m(z)}\right|+1\right] \\ &= |\alpha_1|\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|\left[\left|\frac{z^2(R^nf_1(z))'}{(R^nf_1(z))^2}\right|\cdot\left|\frac{R^nf_1(z)}{z}\right|+1\right]+\dots \\ &\quad+|\alpha_m|\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|\left[\left|\frac{z^2(R^nf_m(z))'}{(R^nf_m(z))^2}\right|\cdot\left|\frac{R^nf_m(z)}{z}\right|+1\right] \\ &\leqslant |\alpha_1|\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|\left[\left|\frac{z^2(R^nf_1(z))'}{(R^nf_1(z))^2}\right|\cdot\mu+1\right]+\dots \\ &\quad+|\alpha_m|\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|\left[\left|\frac{z^2(R^nf_m(z))'}{(R^nf_m(z))^2}\right|\cdot\mu+1\right] \\ &= (|\alpha_1|+\dots+|\alpha_m|)\left|\frac{1-|z|^{2\alpha}}{\alpha}\right| \\ &\quad+|\alpha_1|\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|\left|\frac{z^2(R^nf_1(z))'}{(R^nf_1(z))^2}\right|\cdot\mu+\dots \\ &\quad+|\alpha_m|\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|\left|\frac{z^2(R^nf_m(z))'}{(R^nf_m(z))^2}\right|\cdot\mu \\ &= (|\alpha_1|+|\alpha_2|+\dots+|\alpha_m|)\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|\end{aligned} \tag{7}$$

$$
\begin{aligned}
&+|\alpha_1|\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|\left[\left|\frac{z^2(R^nf_1(z))'}{(R^nf_1(z))^2}\right|-1+1\right]\cdot\mu+\ldots\\
&+|\alpha_m|\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|\left[\left|\frac{z^2(R^nf_m(z))'}{(R^nf_m(z))^2}\right|-1+1\right]\cdot\mu\\
\leqslant\ &(|\alpha_1|+|\alpha_2|+\ldots+|\alpha_m|)\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|\\
&+\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|\left[|\alpha_1|\left|\frac{z^2(R^nf_1(z))'}{(R^nf_1(z))^2}-1\right|\cdot\mu+\ldots\right.\\
&\left.+|\alpha_m|\left|\frac{z^2(R^nf_m(z))'}{(R^nf_m(z))^2}-1\right|\right]\cdot\mu\\
&+\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|\cdot\mu[|\alpha_1|+|\alpha_2|+\ldots+|\alpha_m|]\\
=\ &(|\alpha_1|+|\alpha_2|+\ldots+|\alpha_m|)\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|(\mu+1)+\mu(|\alpha_1|+\ldots+|\alpha_m|)\\
\leqslant\ &(|\alpha_1|+|\alpha_2|+\ldots+|\alpha_m|)\frac{1}{|\alpha|}(\mu+1)+\mu(|\alpha_1|+\ldots+|\alpha_m|)\\
=\ &(|\alpha_1|+\ldots+|\alpha_m|)\left[\frac{\mu(1+|\alpha|)+1}{|\alpha|}\right]\leqslant 1-|c|.
\end{aligned}
$$

Using (7), we evaluate

$$
\left|c|z|^{2\alpha}+\frac{1-|z|^{2\alpha}}{\alpha}\cdot\frac{zf''(z)}{f'(z)}\right|\leqslant|c|\cdot|z|^{2\alpha}+\left|\frac{1-|z|^{2\alpha}}{\alpha}\cdot\frac{zf''(z)}{f'(z)}\right|
$$
$$
\leqslant|c|+1-|c|=1.
$$

By applying Lemma A, we obtain that function F given by (2) belongs to the class S. □

EXAMPLE 1. Let $n\in\mathbb{N}\cup\{0\}$, $m=2$, $\mu=2$, $\alpha=3-4i$, $|\alpha|=5$, $c=\frac{1}{3}+i\frac{\sqrt{3}}{3}$, $|c|=\frac{2}{3}$, $\alpha_1=\frac{1}{4}+i\frac{\sqrt{3}}{4}$, $\alpha_2=\frac{\sqrt{3}}{3}-i\frac{1}{3}$, $f_1(z)=z+az^2$, $R^nf_1(z)=z+(n+1)az^2$, $f_2(z)=z+bz^2$, $R^nf_2(z)=z+(n+1)bz^2$, where $|a|<\frac{1}{2(n+1)}$, $|b|<\frac{1}{2(n+1)}$, $z\in U$.

The conditions given by the hypothesis of Theorem 5 become

$$
\begin{aligned}
&i)\ |\alpha_1|+|\alpha_2|\leqslant\frac{7}{6}<(1-|c|)\frac{|\alpha|}{1+\mu(|\alpha|+1)}=\frac{5}{39}\\
&ii)\ \left|z+(n+1)az^2\right|\leqslant|z|\,[1+(n+1)\,|a|\,|z|]\leqslant 1+(n+1)\,|a|\,|z|\leqslant 2,\\
&\quad\ \left|z+(n+1)bz^2\right|\leqslant|z|\,[1+(n+1)\,|b|\,|z|]\leqslant 1+(n+1)\,|b|\,|z|\leqslant 2,\\
&iii)\ \left|\frac{1-|z|^{2\alpha}}{\alpha}\left[\frac{z^2(1+2(n+1)az^2)}{z^2(1+(n+1)az)}-1\right]\right|=\left|\frac{1-|z|^{2\alpha}}{\alpha}\right|\cdot\left|\frac{(n+1)az}{1+(n+1)az}\right|
\end{aligned}
$$

$$\leqslant \frac{1-|z|^{2\alpha}}{|\alpha|} \cdot \frac{(n+1)\,|a|\,|z|}{1-(n+1)\,|a|\,|z|} \leqslant \frac{1-|z|^{2\alpha}}{|\alpha|} \leqslant \frac{1}{|\alpha|} < 1.$$

$$\left|\frac{1-|z|^{2\alpha}}{\alpha}\left[\frac{z^2(1+2(n+1)az^2)}{z^2(1+(n+1)az)}-1\right]\right| \leqslant \frac{1-|z|^{2\alpha}}{|\alpha|} \cdot \left|\frac{(n+1)bz}{1+(n+1)bz}\right| \leqslant$$
$$\leqslant \frac{1-|z|^{2\alpha}}{|\alpha|} \cdot \frac{(n+1)|b||z|}{1-(n+1)|b||z|} \leqslant \frac{1-|z|^{2\alpha}}{|\alpha|} \leqslant 1.$$

From Theorem 5, we have

$$F(z)=\{(3-4i)\int_0^z t^{2-4i}[1+(n+1)at]^{\frac{1}{4}+i\frac{\sqrt{3}}{4}}[1+(n+1)bt]^{\frac{\sqrt{3}}{3}-i\frac{1}{3}}dt\}^{\frac{1}{3-4i}} \in S,$$

for all $z \in U$.

THEOREM 6. *Let $n,m \in \mathbb{N}\cup\{0\}, \alpha$ be a complex number, with* Re $\alpha > 0$, $|\alpha| \geqslant \frac{1}{1-|c|}$ *and c a complex number with $|c| \leqslant 1$, $c \neq -1$, $\alpha_k \in \mathbb{C}$, $f_k \in A$, $k \in \{1,2,\ldots,m\}$. If*

$$\left|(1-|z|^{2\alpha})\frac{zf''(z)}{f'(z)}\right| \leqslant 1 \tag{8}$$

for all $z \in U$, where $\frac{zf''(z)}{f'(z)}$ is given by (6), then the function F given by (2) belongs to the class S.

Proof. If $z \in U$, $|z| \leqslant 1$, we obtain

$$|z|^{2\alpha} \leqslant 1.$$

In order to prove the theorem, using (6) and (8), we evaluate

$$\left|c|z|^{2\alpha}+(1-|z|^{2\alpha})\frac{zf''(z)}{\alpha f'(z)}\right| \leqslant \left|cz^{2\alpha}\right| + \left|(1-|z|^{2\alpha})\frac{zf''(z)}{\alpha f'(z)}\right|$$
$$= |c||z|^{2\alpha} + \left|(1-|z|^{2\alpha})\frac{zf''(z)}{f'(z)}\right|\frac{1}{|\alpha|} \leqslant |c| + \frac{1}{\frac{1}{1-|c|}} \leqslant 1.$$

By applying Lemma A, we obtain that function F given by (2) belongs to the class S. □

EXAMPLE 2. Let $n \in \mathbb{N}\cup\{0\}$, $m=2$, $\alpha = 5-2i$, $|\alpha| = \sqrt{29}$, $c = \frac{1}{5} - i\frac{\sqrt{3}}{5}$, $|c| = \frac{2}{5}$, $\alpha_1 = 1-i$, $\alpha_2 = 1+i$, $f_1(z) = z+az^2$, $R^n f_1(z) = z+(n+1)az^2$, $f_2(z) = z+bz^2$, $R^n f_2(z) = z+(n+1)bz^2$, where

$$|a| < \frac{1}{(n+1)(2\sqrt{2}+1)}, \quad |b| < \frac{1}{(n+1)(2\sqrt{2}+1)}$$

and

$$\frac{zf''(z)}{f'(z)} = (1-i)\left[\frac{1+2(n+1)az^2}{1+(n+1)az} - 1\right] + (1+i)\left[\frac{1+2(n+1)bz}{1+(n+1)bz)} - 1\right]$$
$$= (1-i)\frac{(n+1)az}{1+(n+1)az} + (1+i)\frac{(n+1)bz}{1+(n+1)bz}.$$

We evaluate

$$\left|(1-|z|^{2\alpha})\cdot\left[(1-i)\frac{(n+1)az}{1+(n+1)az} + (1+i)\frac{(n+1)bz}{1+(n+1)bz}\right]\right|$$
$$\leqslant \left|1-|z|^{2\alpha}\right|\cdot\left|(1-i)\frac{(n+1)az}{1+(n+1)az} + (1+i)\frac{(n+1)bz}{1+(n+1)bz}\right|$$
$$\leqslant |1-i|\frac{(n+1)\,|a|\,|z|}{1-(n+1)\,|a|\,|z|} + |1+i|\frac{(n+1)\,|b|\,|z|}{1-(n+1)\,|b|\,|z|}$$
$$= \sqrt{2}\cdot\left[\frac{(n+1)\,|a|\,|z|}{1-(n+1)\,|a|\,|z|} + \frac{(n+1)\,|b|\,|z|}{1-(n+1)\,|b|\,|z|}\right] \leqslant \frac{\sqrt{2}}{2\sqrt{2}} + \frac{\sqrt{2}}{2\sqrt{2}} = 1.$$

From Theorem 6 we deduce

$$F(z) = \{(5-2i)\int_0^z t^{4-2i}\,[1+(n+1)at]^{1-i}\,[1+(n+1)bt]^{1+i}dt\}^{\frac{1}{5-2i}} \in S,$$

for all $z \in U$.

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