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INEQUALITIES INVOLVING THE KHATRI–RAO PRODUCT OF MATRICES

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Dedicated to Professor Josip Pečarić on the occasion of his 60th birthday

Abstract. We shall show several complementary inequalities to Jensen's type inequality involving the Khatri-Rao product of maps on positive definite matrices. They are applied to extend some known inequalities involving powers of the Khatri-Rao product. Finally, we have generalized some known results for the Hadamard product of operators.

1. Introduction

Let $\mathbb{M}_n$ be the space of $n \times n$ matrices with complex entries. Let $\mathbb{H}_n$ be the real space of $n \times n$ Hermitian matrices and $\mathbb{H}_n^+$ be its open subset of positive definite matrices.

We denote by $\bigcirc_{i=1}^{k} A_i$, $\bigotimes_{i=1}^{k} A_i$, $\ast_{i=1}^{k} A_i$ and $\bigodot_{i=1}^{k} A_i$ the Hadamard, Kronecker, Khatri-Rao and Tracy-Singh product, respectively, of matrices $A_i \in \mathbb{M}_{n(i)}$, $i = 1,\dots,k$. The Hadamard product is an operation on matrices of the same size, the Tracy-Singh product is an operation on partitioned matrices and the Khatri-Rao product is an operation on compatibly partitioned matrices [6]. The Khatri-Rao product can be viewed as a generalization of the Hadamard product and the Tracy-Singh product as a generalization of the Kronecker product, since $A * B = A \otimes B$ and $A \odot B = A \circ B$ hold for nonpartitioned matrices A and B.

Let $A_i \in \mathbb{H}_{n(i)}$, $i = 1,\dots,k$, be partitioned as $A_i = \left(A_{jl}^{(i)}\right)_{jl}$ where $A_{jj}^{(i)} \in \mathbb{H}_{n(i)_j}$ for $i = 1,\dots,k$, $j = 1,\dots,t$ and $\sum_{j=1}^{t} n(i)_j = n(i)$ for $i = 1,\dots,k$. Then we say that the k-tuple $(A_1,\dots,A_k)$ of matrices $A_i \in \mathbb{H}_{n(i)}$ is *compatibly partitioned.*

The following relationship between the Khatri-Rao and the Tracy-Singh product holds (for special cases involving the Hadamard and Kronecker products) [2, Corollary 2.2], [12, Lemma 2.2]:

$$\mathop{\ast}_{i=1}^{k} A_i = Z^T \left(\bigodot_{i=1}^{k} A_i \right) Z, \tag{1}$$

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where an k-tuple $(A_1,\dots,A_k)$ is compatibly partitioned and Z is a matrix of zeros and ones such that $Z^T Z = I$. Also, the following relationship between the Tracy-Singh and the Kronecker product holds [5]:

$$\bigodot_{i=1}^{k} A_i = P^T \left(\bigotimes_{i=1}^{k} A_i \right) P, \tag{2}$$

where P is a permutation square matrix.

For a matrix $A \in \mathbb{H}_n^+$, we denote by $\lambda_1(A)$ and $\lambda_n(A)$ the largest and smallest eigenvalue of A, respectively. If $(A_1,\dots,A_k)$ is an k-tuple of compatibly partitioned matrices $A_i \in \mathbb{H}_{n(i)}^+$, then [2, Lemma 2.5.]:

$$\bigodot_{i=1}^{k} A_i \in \mathbb{H}_n^+, \quad \lambda_1\left(\bigodot_{i=1}^{k} A_i\right) = \prod_{i=1}^{k} \lambda_1(A_i), \quad \lambda_n\left(\bigodot_{i=1}^{k} A_i\right) = \prod_{i=1}^{k} \lambda_n(A_i).$$

S. Liu [7] showed Jensen's type inequality on the Khatri-Rao product and its converses for functions $f(t) = t^{-1}$ and $f(t) = t^2$. X. Cao, Z.-P. Zhang and C.-G. Yang in [2] generalized these results which are given in the next theorem:

THEOREM A. [2, Theorem 3.1] *Let $(A_1,\dots,A_k)$ be an k-tuple of compatibly partitioned matrices $A_i \in \mathbb{H}_{n(i)}^+$. Then*

i)
$$\left(\mathop{\ast}_{i=1}^{k} A_i^s \right)^{1/s} \geqslant \left(\mathop{\ast}_{i=1}^{k} A_i^r \right)^{1/r},$$

where r and s are two real numbers such that $s > r$, and either $s \notin (-1,1)$ and $r \notin (-1,1)$ or $1/2 \leqslant r \leqslant 1 \leqslant s$ or $r \leqslant -1 \leqslant s \leqslant -1/2$;

ii)
$$\left(\mathop{\ast}_{i=1}^{k} A_i^s \right)^{1/s} \leqslant \bar{\Delta}(r,s) \left(\mathop{\ast}_{i=1}^{k} A_i^r \right)^{1/r},$$

where r and s are two real numbers such that $s > r$, and either $s \notin (-1,1)$ or $r \notin (-1,1)$, $\bar{\Delta}(r,s) = \left\{ \frac{r(\delta^s - \delta^r)}{(s-r)(\delta^r - 1)} \right\}^{1/s} \left\{ \frac{s(\delta^r - \delta^s)}{(r-s)(\delta^s - 1)} \right\}^{-\frac{1}{r}}$, $\delta = \frac{W}{w}$, $w = \prod_{i=1}^{k} \lambda_1(A_i)$, $W = \prod_{i=1}^{k} \lambda_{m(i)}(A_i)$;

iii)
$$\left(\mathop{\ast}_{i=1}^{k} A_i^s \right)^{1/s} - \left(\mathop{\ast}_{i=1}^{k} A_i^r \right)^{1/r} \leqslant \Delta(r,s) I,$$

where $\Delta(r,s) = \max_{\theta \in [0,1]} \left\{ [\theta W^s + (1-\theta) w^s]^{1/s} - [\theta W^r + (1-\theta) w^r]^{1/r} \right\}$, and r, s, w, W and δ are as in (ii).

The aim of this paper is to obtain Jensen's type inequality on the Khatri-Rao product of maps and its converses. Next, we can obtain bounds of a power mean version on the Khatri-Rao product for any two real numbers $r,s \neq 0$, which extend the results given in Theorem A. Finally, we shall generalize results given in [1, 3, 10, 4] for the Hadamard product of operators.

2. Jensen's type inequality and its converses

We recall that a real function f is *supermultiplicative* (resp. *submultiplicative*) *on an interval* I if $f(xy) \geqslant f(x)f(y)$ (resp. $f(xy) \leqslant f(x)f(y)$) for every $x,y \in I$.

Let $(A_1,\dots,A_k)$ be an k-tuple of compatibly partitioned matrices $A_i \in \mathbb{H}_{n(i)}$ and $(\Phi_1,\dots,\Phi_k)$ be an k-tuple of normalized positive linear maps $\Phi_i : \mathbb{H}_{n(i)} \to \mathbb{H}_{\tilde{n}(i)}$. If $(\Phi_1(A_1),\dots,\Phi_k(A_k))$ is an k-tuple of compatibly partitioned matrices, then we say that $(\Phi_1,\dots,\Phi_k)$ is an k-tuple of *compatible maps*.

In the next two theorems we give Jensen's type inequality on the Khatri-Rao product and its converses.

THEOREM 2.1. *Let* $(A_1,\dots,A_k)$ *be an* k*-tuple of compatibly partitioned matrices* $A_i \in \mathbb{H}^+_{n(i)}$ *and* $(\Phi_1,\dots,\Phi_k)$ *be an* k*-tuple of compatible maps* $\Phi_i : \mathbb{H}^+_{n(i)} \to \mathbb{H}^+_{\tilde{n}(i)}$*. If* f *is a submultiplicative matrix convex function on* $(0,\infty)$*, then*

$$f\left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i)\right) \leqslant \mathop{\ast}_{i=1}^{k} \Phi_i(f(A_i)). \tag{3}$$

In the dual case (when f *is a supermultiplicative matrix concave function on* $(0,\infty)$*) the opposite inequality holds in* (3)*.*

Proof. We show the submultiplicative matrix convex case only. Using [4, Lemma 6.2] we obtain

$$f\left(\bigotimes_{i=1}^{k} A_i\right) \leqslant \bigotimes_{i=1}^{k} f(A_i). \tag{4}$$

Using Jensen's inequality for a matrix map [4, Theorem 1.20] we have $f(\Phi_i(A_i)) \leqslant \Phi_i(f(A_i))$. Next, using (4) and monotonity of Kronecker product we obtain

$$f\left(\bigotimes_{i=1}^{k} \Phi_i(A_i)\right) \leqslant \bigotimes_{i=1}^{k} f(\Phi_i(A_i)) \leqslant \bigotimes_{i=1}^{k} \Phi_i(f(A_i)).$$

It follows from (2) that

$$\begin{aligned} f\left(\bigodot_{i=1}^{k} \Phi_i(A_i)\right) &\leqslant f\left(P^T\left(\bigotimes_{i=1}^{k} \Phi_i(A_i)\right)P\right) \leqslant P^T f\left(\bigotimes_{i=1}^{k} \Phi_i(A_i)\right)P \\ &\leqslant P^T\left(\bigotimes_{i=1}^{k} \Phi_i(f(A_i))\right)P = \bigodot_{i=1}^{k} \Phi_i(f(A_i)). \end{aligned}$$

Finally, using (1) we obtain the desired inequality (3). □

We introduce some notations. For an k-tuple $(A_1,\dots,A_k)$ positive definite matrices $A_i \in \mathbb{H}^+_{n(i)}$ with $\mathsf{Sp}(A_i) \subseteq [w_i,W_i]$, $0 < w_i \leqslant W_i$, we denote:

$$w = \prod_{i=1}^{k} w_i, \qquad W = \prod_{i=1}^{k} W_i, \qquad X_\omega = \bigcup_{i=1}^{k} [w_i,W_i] \cup [w,W].$$

For a real valued function f we define:

$$\mu_f = \frac{f(W)-f(w)}{W-w} \qquad \text{and} \qquad \nu_f = \frac{Wf(w)-wf(W)}{W-w}. \tag{5}$$

THEOREM 2.2. *Let $(A_1,\ldots,A_k)$ be an k-tuple of compatibly partitioned matrices $A_i \in \mathbb{H}^+_{n(i)}$ and $\mathsf{Sp}(A_i) \subseteq [w_i, W_i]$ for some scalars $0 < w_i \leqslant W_i$. Let $(\Phi_1,\ldots,\Phi_k)$ be an k-tuple of compatible maps $\Phi_i : \mathbb{H}^+_{n(i)} \to \mathbb{H}^+_{\tilde{n}(i)}$. Let $f \in \mathscr{C}(X_\omega)$, $g \in \mathscr{C}([w,W])$ and $F(u,v)$ be a real valued continuous function defined on $U \times V$, matrix monotone in u, where $U \supset \{\prod_{i=1}^k f(t_i) : t_i \in [w_i,W_i]\}$, $V \supset \{g(s) : s \in [w,W]\}$. If f is a supermultiplicative convex function on X_ω, then*

$$F\left[\mathop{\ast}_{i=1}^{k} \Phi_i(f(A_i)), g\left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i)\right)\right] \leqslant \max_{t\in[w,W]} F[\mu_f t + \nu_f, g(t)]I. \tag{6}$$

In the dual case (when f is submultiplicative concave) the opposite inequality holds in (6) with min *instead of* max.

Proof. We obtain this theorem by using Mond-Pečarić method and two connections (1) and (2). We give a proof for the sake of completeness. We consider the case when f is a supermultiplicative convex function. Using $f(A)\otimes f(B) \leqslant f(A\otimes B)$ (see [4, Lemma 6.2]), we obtain:

$$\bigotimes_{i=1}^{k} \Phi_i(f(A_i)) \leqslant f\left(\bigotimes_{i=1}^{k} \Phi_i(A_i)\right).$$

Since f is convex, then $f(t) \leqslant \mu_f t + \nu_f$ holds for any $t \in [w,W]$. It follows that

$$f\left(\bigotimes_{i=1}^{k} \Phi_i(A_i)\right) \leqslant \mu_f \left(\bigotimes_{i=1}^{k} \Phi_i(A_i)\right) + \nu_f I,$$

because $w_i I \leqslant \Phi_i(A_i) \leqslant W_i I$ imply $wI \leqslant \bigotimes_{i=1}^{k} \Phi_i(A_i) \leqslant WI$. It follows from (2) that

$$\begin{aligned}\mathop{\odot}_{i=1}^{k} \Phi_i(f(A_i)) &= P^T \left(\bigotimes_{i=1}^{k} \Phi_i(f(A_i))\right) P \leqslant P^T f\left(\bigotimes_{i=1}^{k} \Phi_i(A_i)\right) P \\ &\leqslant P^T \left(\mu_f \left(\bigotimes_{i=1}^{k} \Phi_i(A_i)\right) + \nu_f I\right) P = \mu_f \left(\mathop{\odot}_{i=1}^{k} \Phi_i(A_i)\right) + \nu_f I.\end{aligned}$$

Next, using (1), we obtain

$$\mathop{\ast}_{i=1}^{k} \Phi_i(f(A_i)) \leqslant \mu_f \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i)\right) + \nu_f I.$$

Finally, using the monotonicity of $F(\cdot,v)$ we obtain the desired inequality (6). $\square$

Applying Theorem 2.2 for the function $F(u,v) = \alpha u - v$, we obtain the following corollary.

COROLLARY 2.3. *Let* $(A_1,\ldots,A_k)$ *and* $(\Phi_1,\ldots,\Phi_k)$ *be as in Theorem 2.2. Let* $f \in \mathscr{C}(X_\omega)$ *be a supermultiplicative convex function and* $\alpha \neq 0$ *be a real number. If* $\alpha g : [m,M] \to \mathbb{R}$ *is a strictly convex differentiable function, then the inequality*

$$\mathop{\ast}_{i=1}^{k} \Phi_i(f(A_i)) \leqslant \alpha g\left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i)\right) + \beta I \tag{7}$$

holds for $\beta = \mu_f t_0 + \nu_f - \alpha g(t_0)$, *where*

$$t_0 = \begin{cases} \text{the unique solution of } g'(t) = \frac{\mu_f}{\alpha} & \text{if} \quad \alpha g'(w) \leqslant \mu_f \leqslant \alpha g'(W), \\ W & \text{if} \quad \mu_f > \alpha g'(W), \\ w & \text{if} \quad \alpha g'(w) > \mu_f. \end{cases}$$

If αg *is a concave function, then* (7) *holds for*

$$t_0 = \begin{cases} W & \text{if} \quad \mu_f \geqslant \alpha \mu_g, \\ w & \text{if} \quad \mu_f < \alpha \mu_g. \end{cases}$$

In the case when f *is a submultiplicative concave function the opposite inequality holds in* (7) *with the same* β *but the opposite condition while determining* t_0.

3. Inequalities involving powers of the Khatri-Rao product

In this section, we shall give a power mean version on the Khatri-Rao product and we find the best bounds among those we obtain by using the Mond-Pečarić method.

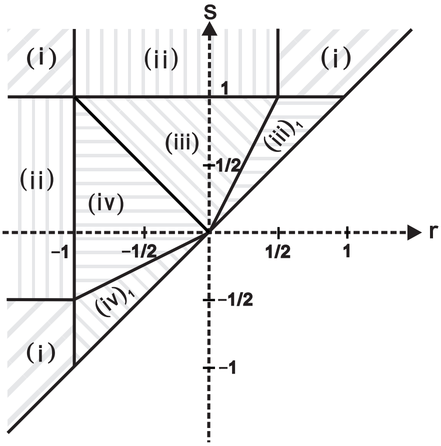
The constant μ_p and ν_p are the constants μ_f and μ_f associated with the function $f(t) = t^p$ in (5).

For the sake of convenience, we denote areas from (i) to (iv) as in Figure 1.

3.1. Difference type inequalities

THEOREM 3.1. *Let* $(A_1,\ldots,A_k)$ *be an* k*-tuple of compatibly partitioned matrices* $A_i \in \mathbb{H}^+_{n(i)}$ *and* $\mathsf{Sp}(A_i) \subseteq [w_i,W_i]$ *for some scalars* $0 < w_i \leqslant W_i$. *Let* $(\Phi_1,\ldots,\Phi_k)$ *be an* k*-tuple of compatible maps* $\Phi_i : \mathbb{H}^+_{n(i)} \to \mathbb{H}^+_{\tilde{n}(i)}$. *Let* $r,s \in \mathbb{R}$, $r \leqslant s$ *and* $rs \neq 0$.
If (i), *then*

$$0 \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s} - \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)\right)^{1/r} \leqslant \tilde{\Delta}(w,W,0)I,$$



(i)	$r \leq s$, $s \notin (-1,1)$, $r \notin (-1,1)$ **or** $1/2 \leq r \leq 1 \leq s$ **or** $r \leq -1 \leq s \leq -1/2$,
(ii)	$s \geq 1$, $-1 < r < 1/2$, $r \neq 0$ **or** $r \leq -1$, $-1/2 < s < 1$, $s \neq 0$,
(iii)	$-s \leq r \leq s/2$, $r \neq 0$, $0 < s \leq 1$,
$(\text{iii})_1$	$r \leq s \leq 2r$, $0 < s \leq 1$,
(iv)	$r/2 \leq s \leq -r$, $s \neq 0$, $-1 \leq r < 0$,
$(\text{iv})_1$	$2s \leq r \leq s$, $-1 \leq r < 0$.

Figure 1. Areas

if (ii) *and*

$$\frac{W^s - w^s}{W^r - w^r} \cdot \min\{w^r, W^r\} + \left(1 - \frac{s}{r}\right)\left(\frac{r}{s}\frac{W^s - w^s}{W^r - w^r}\right)^{\frac{s}{s-r}} > 0,$$

$$\frac{W^r - w^r}{W^s - w^s} \cdot \min\{w^s, W^s\} + \left(1 - \frac{r}{s}\right)\left(\frac{s}{r}\frac{W^r - w^r}{W^s - w^s}\right)^{\frac{r}{r-s}} > 0,$$

then

$$\tilde{\Delta}(w,W,d)I \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s} - \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)\right)^{1/r} \leqslant \tilde{\Delta}(w,W,0)I,$$

if (iii), *then*

$$-C(w^r, W^r, 1/r)I \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s} - \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)\right)^{1/r} \leqslant \left(\tilde{\Delta}(w,W,0) + C(w^s, W^s, 1/s)\right)I,$$

if (iv) *or* $(\text{iii})_1$ *or* $(\text{iv})_1$, *then*

$$-C(w^s, W^s, 1/s)I \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s} - \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)\right)^{1/r} \leqslant \left(\tilde{\Delta}(w,W,0) + C(w^s, W^s, 1/s)\right)I,$$

where

$$\tilde{\Delta}(m,M,0) = \max_{\theta \in [0,1]} \left\{[\theta M^s + (1-\theta)m^s]^{1/s} - [\theta M^r + (1-\theta)m^r]^{1/r}\right\},$$

$$\tilde{\Delta}(m,M,d) = \min_{\theta \in [0,1] \cup [d/(M^r - m^r), 1 + d/(M^r - m^r)]} \left\{[\theta M^s + (1-\theta)m^s]^{1/s} - [\theta M^r + (1-\theta)m^r - d]^{1/r}\right\},$$

$$d \equiv d(m,M,r,s) = \frac{M^s m^r - M^r m^s}{M^s - m^s} - \left(1 - \frac{r}{s}\right)\left(\frac{s}{r}\frac{M^r - m^r}{M^s - m^s}\right)^{r/(r-s)}$$

and a constant $C(w,W,p)$ [4, §2.7, Lemma 2.59] *is defined as*

$$(*)\qquad C(w,W,p)=(p-1)\left(\frac{1}{p}\frac{W^p-w^p}{W-w}\right)^{p/(p-1)}+\frac{Ww^p-wW^p}{W-w}\qquad \textit{for all } p\in\mathbb{R}.$$

In order to prove Theorem 3.1, we need some preliminary results.

LEMMA 3.2. *Let* $(A_1,\dots,A_k)$ *and* $(\Phi_1,\dots,\Phi_k)$ *be as in Theorem* 3.1. *If* $0<p\leqslant 1$, *then*

$$\mu_p \mathop{\ast}_{i=1}^{k} \Phi_i(A_i)+\nu_p I \leqslant \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^p) \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i)\right)^p, \tag{8}$$

if $-1\leqslant p<0$ *or* $1\leqslant p\leqslant 2$, *then*

$$\left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i)\right)^p \leqslant \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^p) \leqslant \mu_p \mathop{\ast}_{i=1}^{k} \Phi_i(A_i)+\nu_p I, \tag{9}$$

while if $p<-1$ *or* $p>2$, *then*

$$\mu_p \mathop{\ast}_{i=1}^{k} \Phi_i(A_i)+(1-p)\,(\mu_p/p)^{p/(p-1)} I \leqslant \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^p) \leqslant \mu_p \mathop{\ast}_{i=1}^{k} \Phi_i(A_i)+\nu_p I. \tag{10}$$

Proof. The the right hand inequality in (8) and the left hand inequality in (9) follows from Theorem 2.1. The left hand inequality in (8) and the right hand inequality in (9) and (10) follows from Corollary 2.3. Finally, the left hand inequality in (10) follows by using inequality $A_i^p-\mu_p A_i\geqslant \beta I$ if $p<-1$ or $p>2$, $(i=1,\dots,k)$, where $\beta=\max\limits_{g_s\leqslant f}\min\limits_{w\leqslant t\leqslant W}\left\{g_s(t)-\mu_p t\right\}$, $g_s(t)=f(s)+f'(s)(t-s)$. □

LEMMA 3.3. *Let* $(A_1,\dots,A_k)$ *and* $(\Phi_1,\dots,\Phi_k)$ *be as in Theorem* 3.1.
(a) *If* $r\leqslant s\leqslant -1$ *or* $1\leqslant s\leqslant -r$ *or* $0<r\leqslant s\leqslant 2r$, $s\geqslant 1$, *then*

$$\left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)\right)^{1/r} \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s} \leqslant \left(\tilde{\mu} \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)+\tilde{\nu} I\right)^{1/s}.$$

(b) *If* $0<-r\leqslant s$, $s\geqslant 1$ *or* $0<2r\leqslant s$, $s\geqslant 1$, *and*

$$\frac{W^s-w^s}{W^r-w^r}\cdot\min\{w^r,W^r\}+\left(1-\frac{s}{r}\right)\left(\frac{r}{s}\frac{W^s-w^s}{W^r-w^r}\right)^{\frac{s}{s-r}}>0,$$

then

$$\left(\tilde{\mu} \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)+\tilde{\nu}^* I\right)^{1/s} \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s} \leqslant \left(\tilde{\mu} \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)+\tilde{\nu} I\right)^{1/s}.$$

(c) *If* $r \leqslant s,\ -1 \leqslant s < 0$ *or* $s \leqslant -r,\ 0 < s \leqslant 1$ *or* $0 < r \leqslant s \leqslant 2r,\ s \leqslant 1$, *then*

$$\left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)\right)^{1/r} - C(w^s, W^s, 1/s)I \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s}$$

$$\leqslant \left(\tilde{\mu} \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r) + \tilde{\nu} I\right)^{1/s} + C(w^s, W^s, 1/s)I.$$

(d) *If* $0 < -r \leqslant s \leqslant 1$ *or* $0 < 2r \leqslant s \leqslant 1$, *and*

$$\frac{W^s - w^s}{W^r - w^r} \cdot \min\{w^r, W^r\} + \left(1 - \frac{s}{r}\right) \left(\frac{r}{s} \frac{W^s - w^s}{W^r - w^r}\right)^{\frac{s}{s-r}} > 0,$$

then

$$\left(\tilde{\mu} \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r) + \tilde{\nu}^* I\right)^{1/s} - C(w^s, W^s, 1/s)I \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s}$$

$$\leqslant \left(\tilde{\mu} \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r) + \tilde{\nu} I\right)^{1/s} + C(w^s, W^s, 1/s)I,$$

where we denote

$$\tilde{\mu} = \frac{W^s - w^s}{W^r - w^r}, \quad \tilde{\nu} = \frac{W^r w^s - W^s w^r}{W^r - w^r} \quad \text{and} \quad \tilde{\nu}^* = \left(1 - \frac{s}{r}\right) \left(\frac{r}{s} \tilde{\mu}\right)^{s/(s-r)}.$$

Proof. This lemma follows from Lemma 3.2 and the following inequalities which preserve or reverse the operator order [9, 11]:

– If $A \geqslant B > 0$ and the spectrum $\mathsf{Sp}(B) \subseteq [w, W]$ for some scalars $0 < w < W$, then

$$A^p + C(w, W, p)\,\mathbf{1} \geqslant B^p \qquad \text{for all} \qquad p \geqslant 1, \tag{11}$$

where the constant $C(w, W, p)$ defined by $(*)$.

– If $A \geqslant B > 0$ and the spectrum $\mathsf{Sp}(A) \subseteq [w, W]$, $0 < w < W$, then

$$B^p + C(w, W, p)\,\mathbf{1} \geqslant A^p \qquad \text{for all} \qquad p \leqslant -1. \tag{12}$$

– The function $f(t) = t^p$ is operator monotone for $p \in [0, 1]$ (the Löwner-Heinz theorem).

We use the same technique as in the proof of [8, Theorem 3.1] and we give the short proof for the sake of completeness.

Putting $p = s/r$ in (8)–(10) and replacing A_i by A_i^r $(i = 1, \ldots, n)$, we obtain the following statements:

(I) If $r \leqslant s < 0$, then

$$\tilde{\mu} \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r) + \tilde{\nu}^* I \leqslant \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s) \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r) \right)^{s/r}.$$

(II) If $0 < s \leqslant -r$ or $0 < r \leqslant s \leqslant 2r$, then

$$\left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r) \right)^{s/r} \leqslant \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s) \leqslant \tilde{\mu} \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r) + \tilde{\nu}^* I.$$

(III) If $0 < -r \leqslant s$ or $0 < 2r \leqslant s$, then

$$\tilde{\mu} \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r) + \tilde{\nu}^* I \leqslant \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s) \leqslant \tilde{\mu} \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r) + \tilde{\nu} I.$$

Now, applying the Löwner-Heinz theorem for the function $f(t) = t^{1/s}$, $s \geqslant 1$ and $s \leqslant -1$, we obtain (a) and (b), respectively. Using (11) for $p = 1/s > 1$ we obtain (c), since

$$w^s I \leqslant \tilde{\mu} \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r) \leqslant W^s I \quad \text{and} \quad w^s I \leqslant \tilde{\mu} \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r) + \tilde{\nu} I \leqslant W^s I.$$

Similarly, using (12) for $p = 1/s < -1$ we obtain (d). □

REMARK 3.4. Putting $p = r/s$ in (8)–(10) and replacing A_i by A_i^s $(i = 1, \ldots, n)$, we can obtain similar inequalities as in Lemma 3.3. For example,

(a_1) If $1 \leqslant r \leqslant s$ or $-s \leqslant r \leqslant -1$ or $2s \leqslant r \leqslant s < 0$, $r \leqslant -1$, then

$$\left(\bar{\mu} \mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s) + \bar{\nu} I \right)^{1/r} \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r) \right)^{1/r} \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s) \right)^{1/s},$$

where

$$\bar{\mu} = \frac{W^r - w^r}{W^s - w^s}, \quad \bar{\nu} = \frac{W^s w^r - W^r w^s}{W^s - w^s}$$

etc.

Proof of Theorem 3.1. Using Lemma 3.3 in the case (a) we obtain the following statement:

If $r \leqslant s \leqslant -1$ or $1 \leqslant s \leqslant -r$ or $0 < r \leqslant s \leqslant 2r$, $s \geqslant 1$, then

$$0 \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s) \right)^{1/s} - \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r) \right)^{1/r} \leqslant \max_{t \in \bar{T}_1} \left\{ (\tilde{\mu}\, t + \tilde{\nu})^{1/s} - t^{1/r} \right\} I$$

holds, where $\bar{T}_1$ denotes the closed interval joining w^r to W^r. Setting $t = \theta W^r + (1-\theta)w^r$ for some $\theta \in [0,1]$, we obtain $\max_{t\in \bar{T}_1} \left\{ (\tilde{\mu}\, t + \tilde{\nu})^{1/s} - t^{1/r} \right\} = \tilde{\Delta}(w,W,0)$.

Similarly, we can obtain inequalities in areas (b), (c), (d) of Lemma 3.3 and in all areas in Remark 3.4.

Now, comparing all obtained inequalities and using the fact that a function (Figure 2)

$$C(r) \equiv C(m^r, M^r, 1/r) := \frac{1-r}{r}\left(r\frac{M-m}{M^r - m^r}\right)^{1/(1-r)} + \frac{M^r m - m^r M}{M^r - m^r}$$

is strictly decreasing for all $r \in \mathbb{R}$ and $M > m > 0$ [8, Lemma 3.2],

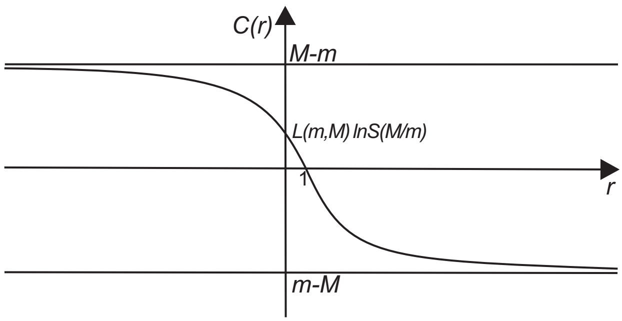


Figure 2. Function $C(r) \equiv C(m^r, M^r, 1/r)$

we obtain desired inequalities. □

REMARK 3.5. (Added in proofs.) We can obtain bounds in Theorem 3.1 in the case (ii) without the conditions $\frac{W^s - w^s}{W^r - w^r} \cdot \min\{w^r, W^r\} + \left(1 - \frac{s}{r}\right)\left(\frac{r}{s}\frac{W^s - w^s}{W^r - w^r}\right)^{\frac{s}{s-r}} > 0$ and $\frac{W^r - w^r}{W^s - w^s} \cdot \min\{w^s, W^s\} + \left(1 - \frac{r}{s}\right)\left(\frac{s}{r}\frac{W^r - w^r}{W^s - w^s}\right)^{\frac{r}{r-s}} > 0$.

By using inequality $f(t) \geqslant f(y) + l(y)(t-y)$ for every $t, y \in [w,W]$ for convex function, we give the following generalization of (10):

$$py^{p-1} \mathop{\circledast}_{i=1}^{k} \Phi_i(A_i) + (1-p)(1-p)y^p I \leqslant \mathop{\circledast}_{i=1}^{k} \Phi_i(A_i^p) \leqslant \mu_p \mathop{\circledast}_{i=1}^{k} \Phi_i(A_i) + \nu_p I, \qquad (13)$$

$$p < -1 \text{ or } p > 2.$$

Putting $y = (\mu_p/p)^{1/(p-1)} \in [w,W]$ we obtain (10), but putting $y = w$ or $y = W$ we obtain that the operator in LHS of (13) is positive. By using the same technique as in proofs of Lemma 3.3 and Theorem 3.1, we obtain the following inequalities:

If $s \geqslant 1$, $-1 < r < 1/2$, $r \neq 0$, then

$$C_1 I \leqslant \left(\mathop{\circledast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s} - \left(\mathop{\circledast}_{i=1}^{k} \Phi_i(A_i^r)\right)^{1/r} \leqslant \tilde{\Delta}(w,W,0)I,$$

where $C_1 = m\left(\frac{s}{r}\frac{M^r}{m^r}+1-\frac{s}{r}\right)^{1/s} - M$, and if $r \leqslant -1$, $-1/2 < s < 1$, $s \neq 0$, then

$$C_2 I \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s} - \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)\right)^{1/r} \leqslant \tilde{\Delta}(w,W,0)I,$$

where $C_2 := m - M\left(\frac{r}{s}\frac{m^s}{M^s}+1-\frac{r}{s}\right)^{1/r}$.

So if (r,s) belongs to (ii), then

$$\min\{C_1,C_2\}I \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s} - \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)\right)^{1/r} \leqslant \tilde{\Delta}(w,W,0)I.$$

Under the assumptions of Theorem 3.1 it can be proven that the bounds C_1 and C_2 are worse than the lower bound $\tilde{\Delta}(w,W,d)$ in this theorem (see [8, Corollary 3.4]).

3.2. Ratio type inequalities

THEOREM 3.6. *Let $(A_1,\ldots,A_k)$ be an k-tuple of compatibly partitioned matrices $A_i \in \mathbb{H}_{n(i)}^+$ and $\mathsf{Sp}(A_i) \subseteq [w_i,W_i]$ for some scalars $0 < w_i \leqslant W_i$. Let $(\Phi_1,\ldots,\Phi_k)$ be an k-tuple of compatible maps $\Phi_i : \mathbb{H}_{n(i)}^+ \to \mathbb{H}_{\tilde{n}(i)}^+$. Let $r,s \in \mathbb{R}$, $r \leqslant s$ and $rs \neq 0$ and areas* (i) – (iv) *be as in Figure 1.*
If (i)*, then*

$$\Delta(h,r,s)^{-1}\left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s} \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)\right)^{1/r} \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s},$$

if (ii)*, then*

$$\Delta(h,r,s)^{-1}\left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s} \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)\right)^{1/r} \leqslant \Delta(h,r,s)\left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s},$$

if (iii)*, then*

$$\Delta(h,s,1)^{-1}\Delta(h,r,s)^{-1}\left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s} \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)\right)^{1/r} \leqslant \Delta(h,r,1)\left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s},$$

if (iv) *or* $(\mathrm{iii})_1$ *or* $(\mathrm{iv})_1$*, then*

$$\Delta(h,s,1)^{-1}\Delta(h,r,s)^{-1}\left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s} \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)\right)^{1/r} \leqslant \Delta(h,s,1)\left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s},$$

where a generalized Specht ratio $\Delta(h,r,s)$ [4, § 2.7] *is defined as*

$$\Delta(h,r,s) = \left\{\frac{r(h^s-h^r)}{(s-r)(h^r-1)}\right\}^{1/s}\left\{\frac{s(h^r-h^s)}{(r-s)(h^s-1)}\right\}^{-1/r}, \qquad h = \frac{W}{w}.$$

In order to prove Theorem 3.6, we need the following lemma.

LEMMA 3.7. *Let* $(A_1,\dots,A_k)$ *and* $(\Phi_1,\dots,\Phi_k)$ *be as in Theorem* 3.6.
(a) *If* $0<p\leqslant 1$, *then*

$$K(w,W,p)\left(\mathop{\ast}_{i=1}^{k}\Phi_i(A_i)\right)^p \leqslant \mathop{\ast}_{i=1}^{k}\Phi_i(A_i^p)\leqslant\left(\sum_{i=1}^{n}\mathop{\ast}_{i=1}^{k}\right)^p.$$

(b) *If* $-1\leqslant p<0$ *or* $1\leqslant p\leqslant 2$, *then*

$$\left(\mathop{\ast}_{i=1}^{k}\Phi_i(A_i)\right)^p \leqslant \mathop{\ast}_{i=1}^{k}\Phi_i(A_i^p)\leqslant K(w,W,p)\left(\mathop{\ast}_{i=1}^{k}\Phi_i(A_i)\right)^p.$$

(c) *If* $p<-1$ *or* $p>2$, *then*

$$K(w,W,p)^{-1}\left(\mathop{\ast}_{i=1}^{k}\Phi_i(A_i)\right)^p \leqslant \mathop{\ast}_{i=1}^{k}\Phi_i(A_i^p)\leqslant K(w,W,p)\left(\mathop{\ast}_{i=1}^{k}\Phi_i(A_i)\right)^p,$$

where a generalized Kantorovich constant $K(w,W,p)$ [4, §2.7] *is defined as*

$$K(w,W,p):=\frac{wW^p-Ww^p}{(p-1)(W-w)}\left(\frac{p-1}{p}\frac{W^p-w^p}{wW^p-Ww^p}\right)^p \qquad \text{for all } p\in\mathbb{R}. \qquad (**)$$

Proof. The the right hand inequality in (a) and the left hand inequality in (b) follows from Theorem 2.1. The left hand inequality in (a) and the right hand inequality in (b) and (c) follows from Corollary 2.3. Finally, the left hand inequality in (c) follows by using inequality $A_i^p\geqslant s^pI+ps^{p-1}(A_i-sI)$ if $p<-1$ or $p>2$, $(i=1,\dots,k)$. □

Proof of Theorem 3.6. This theorem follows from Lemma 3.7, the Löwner-Heinz theorem and the following inequalities which preserve or reverse the operator order [9, 11]: If $A,B\in\mathscr{B}_+(H)$, $A\geqslant B>0$ such that $\mathsf{Sp}(A)\subseteq[n,N]$ and $\mathsf{Sp}(B)\subseteq[m,M]$ for some scalars $0<n<N$ and $0<m<M$, then

$$K(n,N,p)\,A^p\geqslant B^p>0 \qquad \text{for all} \quad p>1, \qquad (14)$$
$$K(m,M,p)A^p\geqslant B^p>0 \qquad \text{for all} \quad p>1, \qquad (15)$$
$$K(n,N,p)\,B^p\geqslant A^p>0 \qquad \text{for all} \quad p<-1, \qquad (16)$$
$$K(m,M,p)B^p\geqslant A^p>0 \qquad \text{for all} \quad p<-1. \qquad (17)$$

Finally, we choose better bounds by using the fact that a function (Figure 3)

$$\Delta(r)\equiv\Delta(h,r):=\frac{r(h-h^r)}{(1-r)(h^r-1)}\left(\frac{h^r-h}{(r-1)(h-1)}\right)^{-1/r}, \qquad h=\frac{M}{m}, \qquad (18)$$

is strictly decreasing for all $r\in\mathbb{R}$ and $M>m>0$ [8, Lemma 3.2].

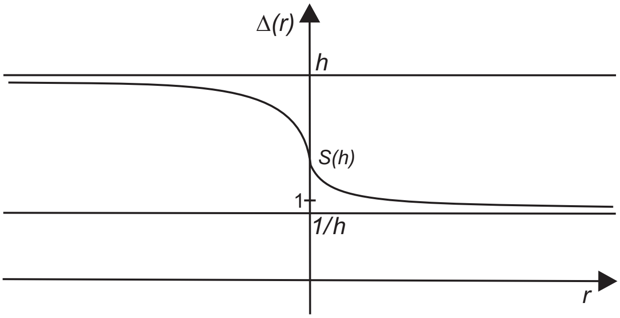


Figure 3. Function $\Delta(r) \equiv \Delta(h,r,1)$

We use the same technique as in the proof of [8, Theorem 3.2] and we give the short proof for the sake of completeness.

Applying Lemma 3.7 for $p = s/r$ and using (14)–(17) and $K(m^r, M^r, s/r)^{1/s} = K(M^r, m^r, s/r)^{1/s} = \Delta(h,r,s)$, we obtain the following statement:

If $r \leqslant s \leqslant -1$ or $1 \leqslant s \leqslant -r$ or $0 < r \leqslant s \leqslant 2r,\ s \geqslant 1$, then

$$\left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)\right)^{1/r} \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s} \leqslant \Delta(h,r,s) \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)\right)^{1/r}.$$

But, applying Lemma 3.7 for $p = r/s$ and using (14)–(17) and $K(m^s, M^s, r/s)^{1/r} = (M^s, m^s, r/s)^{1/r} = \Delta(h,r,s)^{-1}$, we obtain the following:

If $1 \leqslant r \leqslant s$ or $-s \leqslant r \leqslant -1$ or $2s \leqslant r \leqslant s < 0,\ r \leqslant -1$, then

$$\Delta(h,r,s)^{-1} \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s} \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^r)\right)^{1/r} \leqslant \left(\mathop{\ast}_{i=1}^{k} \Phi_i(A_i^s)\right)^{1/s}.$$

Similarly, we can obtain inequalities in other areas.

Finally, we choose better bounds by using monotonicity of the function (18). □

4. Remarks for the Hadamard product of operators

We recall that all results given for the Khatri-Rao product of matrices lead to results involving the Hadamard product of matrices, as a special case, for nonpartitioned matrices.

We assume that H and K are Hilbert spaces and $\mathscr{B}(H)$ and $\mathscr{B}(K)$ are C*-algebras of all bounded linear operators on the appropriate Hilbert space.

All results given in Theorem 2.1, Theorem 2.2, Corollary 2.3, Theorem 3.1 and Theorem 3.6 hold for the Hadamard product of operators. Instead of the Khatri-Rao product, k-tuple $(A_1,\ldots,A_k)$ of compatibly partitioned matrices and k-tuple $(\Phi_1,\ldots,\Phi_k)$ of compatible maps, we place the Hadamard product, k-tuple $(A_1,\ldots,A_k)$ of positive operators $A_i \in \mathscr{B}(H)$ and k-tuple $(\Phi_1,\ldots,\Phi_k)$ of normalized positive linear maps $\Phi_i : \mathscr{B}(H) \to \mathscr{B}(K)$, respectively. The proofs are omitted. We use the same technique as in the sections above.

These results are a generalization of results [1, 3, 10] and [4, Theorem 6.28].

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