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# ACCURATE APPROXIMATIONS FOR THE RIEMANN–STIELTJES INTEGRAL VIA THEORY OF INEQUALITIES

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*Dedicated to Professor Josip Pečarić*
*on the occasion of his 60th birthday*

*Abstract.* Accurate approximations for the Riemann-Stieltjes integral by the use of various recent inequalities for the generalised Čebyšev functional introduced in 1998 by Dragomir & Fedotov are surveyed. Applications in deriving sharp inequalities of Grüss' type are also given.

## 1. Introduction

In 1998, Dragomir and Fedotov [21], in order to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ with the simpler expression

$$\frac{1}{b-a}[u(b)-u(a)]\int_a^b f(t)\,dt,$$

introduced the following error functional

$$D(f,u;a,b) := \int_a^b f(t)\,du(t) - \frac{1}{b-a}[u(b)-u(a)]\int_a^b f(t)\,dt \tag{1.1}$$

provided that both the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist.

If $u(t) = \int_a^t g(s)\,ds,\ t \in [a,b]$, with $g$ continuous on $[a,b]$, then

$$\begin{aligned} D(f,u;a,b) &= \int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \int_a^b g(t)\,dt \\ &= (b-a)\,T(f,g;a,b), \end{aligned} \tag{1.2}$$

where $T(\cdot,\cdot;a,b)$ is the well-known *Čebyšev functional*. Therefore $D(f,u;a,b)$ can be see as a generalised *Čebyšev type functional*.

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The natural connection provided by the equality (1.2) also motivates the study of the functional $D(\cdot,\cdot;a,b)$ since there are numerous results in the literature concerning bounds for the Čebyšev functional for which we only mention the following ones:

$$|T(f,g;a,b)| \leqslant \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma) \qquad \text{(Grüss 1935, [23])} \tag{1.3}$$

provided $\varphi \leqslant f(x) \leqslant \Phi,\ \gamma \leqslant g(x) \leqslant \Gamma$ for each $x \in [a,b]$;

$$|T(f,g;a,b)| \leqslant \frac{1}{12}\cdot(b-a)^2 \left\|f'\right\|_\infty \left\|g'\right\|_\infty \qquad \text{(Čebyšev 1882, [7])} \tag{1.4}$$

if $f$, $g$ are absolutely continuous on $[a,b]$ and $f',\ g' \in L_\infty[a,b]$;

$$|T(f,g;a,b)| \leqslant \frac{1}{8}(b-a)(\Phi-\phi)\left\|g'\right\|_\infty \qquad \text{(Ostrowski 1970, [26])} \tag{1.5}$$

provided $\varphi \leqslant f(x) \leqslant \Phi$ for any $x \in [a,b]$ and $g' \in L_\infty[a,b]$, and

$$|T(f,g;a,b)| \leqslant \frac{1}{\pi^2}(b-a)\left\|f'\right\|_2 \left\|g'\right\|_2 \qquad \text{(Lupaş 1973, [25])} \tag{1.6}$$

provided $f',\ g' \in L_2[a,b]$. The multiplicative constants $\frac{1}{4}, \frac{1}{12}, \frac{1}{8}$ and $\frac{1}{\pi^2}$ are the best possible in the sense that they cannot be replaced by smaller quantities.

Recently, Cerone and Dragomir [3], proved the following result:

$$|T(f,g;a,b)| \leqslant \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \tag{1.7}$$

provided $f \in L[a,b]$ and $g \in L_\infty[a,b]$.

As particular cases of (1.7), we can state the results:

$$|C(f,g;a,b)| \leqslant \|g\|_\infty \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \tag{1.8}$$

if $g \in L_\infty[a,b]$ and $f \in L[a,b]$, and

$$|C(f,g;a,b)| \leqslant \frac{1}{2}(M-m)\frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt, \tag{1.9}$$

where $m \leqslant g(x) \leqslant M$ for $x \in [a,b]$. The constants $1$ in (1.8) and $\frac{1}{2}$ in (1.9) are the best possible. The inequality (1.9) has been obtained before in a different way by Cheng & Sun in [8]. However, they did not consider the problem of sharpness.

For generalizations of (1.9) in abstract Lebesgue spaces, best constants and discrete versions, see [4] in both preprint and final form.

## 2. Error Bounds for $D(f,u;a,b)$

### 2.1. Bounds for Lipschitzian Integrators

In this section we assume that in the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$, the integrator $u$ is $L$-Lipschitzian, i.e.,

$$|u(t)-u(s)| \leqslant L|t-s| \qquad \text{for each } t,s \in [a,b]. \tag{2.1}$$

It is well known that, in this case, the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists provided the integrand $f:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$.

THEOREM 1. (Dragomir-Fedotov 1998, [21]) *If $u$ is $L$-Lipschitzian on $[a,b]$ and $f$ is Riemann integrable on $[a,b]$, then*

$$|D(f,u;a,b)| \leqslant L\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt. \tag{2.2}$$

*The inequality (2.2) is the best possible.*

*Moreover, if there exist the constants $m,M\in\mathbb{R}$ such that*

$$m \leqslant f(t) \leqslant M \qquad \text{for any } t\in[a,b], \tag{2.3}$$

*then*

$$|D(f,u;a,b)| \leqslant \frac{1}{2}L(M-m)(b-a). \tag{2.4}$$

*The constant $\frac{1}{2}$ is the best possible in (2.4).*

A function $w$ is said to be of *bounded variation* if for any *division* $I_n$ of $[a,b]$, $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, the variation of $w$ on $I_n$ is finite, which means that

$$\sum_{i=0}^{n-1} |w(x_{i+1}) - w(x_i)| < \infty. \tag{2.5}$$

The *total variation* of $w$ on $[a,b]$ is denoted by $\bigvee_a^b(w)$, where

$$\bigvee_a^b(w) := \sup\left\{ \sum_{i=0}^{n-1} |w(x_{i+1}) - w(x_i)|, \ I_n \text{ is a division of } [a,b] \right\}. \tag{2.6}$$

THEOREM 2. (Cerone-Dragomir 2006, [2]) *Let $u:[a,b]\to\mathbb{R}$ be $L$-Lipschitzian on $[a,b]$.*

*(i) If $f$ is of bounded variation on $[a,b]$, then*

$$|D(f,u;a,b)| \leqslant \frac{3}{4}L(b-a)\bigvee_a^b(f); \tag{2.7}$$

*(ii) If* $f:[a,b]\to\mathbb{R}$ *is of* $r-H-$*Hölder type, i.e.,*

$$|f(t)-f(s)|\leqslant H|t-s|^{r} \tag{2.8}$$

*for each* $t,s\in[a,b]$*, where* $H>0$ *and* $r\in(0,1]$ *are given, then*

$$|D(f,u;a,b)|\leqslant\frac{2HL(b-a)^{r+1}}{(r+1)(r+2)}; \tag{2.9}$$

*(iii) If* $f:[a,b]\to\mathbb{R}$ *is absolutely continuous, then*

$$|D(f,u;a,b)|\leqslant\begin{cases}\frac{1}{3}L(b-a)^{2}\|f'\|_{\infty}, \text{ if } f'\in L_{\infty}[a,b];\\[2mm] \frac{2^{\frac{1}{q}}L(b-a)^{\frac{1}{q}+1}\|f'\|_{p}}{(q+1)^{\frac{1}{q}}(q+2)^{\frac{1}{q}}}, \text{ if } f'\in L_{p}[a,b], \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2mm] \frac{3}{4}L(b-a)\|f'\|_{1}.\end{cases} \tag{2.10}$$

REMARK 1. It is an open question whether or not the multiplicative constants $\frac{3}{4},2,\frac{1}{3},\frac{2^{1/q}}{(q+1)^{1/q}(q+2)^{1/q}}$ and $\frac{3}{4}$ in the inequalities (2.7) – (2.10) are the best possible.

## 2.2. Bounds for $(l,L)$-Lipschitzian Integrators

The following lemma may be stated:

LEMMA 1. *Let* $u:[a,b]\to\mathbb{R}$ *and* $l,L\in\mathbb{R}$ *with* $L>l$*. The following statements are equivalent:*

*(i) The function* $u-\frac{l+L}{2}\cdot e$*, where* $e(t)=t,\ t\in[a,b]$*, is* $\frac{1}{2}(L-l)-$*Lipschitzian;*

*(ii) We have the inequalities*

$$l\leqslant\frac{u(t)-u(s)}{t-s}\leqslant L \quad \text{for each } t,s\in[a,b], \quad \text{with } t\neq s; \tag{2.11}$$

*(iii) We have the inequalities*

$$l(t-s)\leqslant u(t)-u(s)\leqslant L(t-s) \quad \text{for each } t,s\in[a,b], \quad \text{with } t>s. \tag{2.12}$$

Following [24], we can introduce the definition of $(l,L)$-Lipschitzian functions:

DEFINITION 1. The function $u:[a,b]\to\mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) from Lemma 1 is said to be $(l,L)$-Lipschitzian on $[a,b]$.

If $L>0$ and $l=-L$, then $(-L,L)$-Lipschitzian means $L$-Lipschitzian in the classical sense.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides examples of $(l,L)$-Lipschitzian functions.

PROPOSITION 1. *Let* $u:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* $(a,b)$. *If* $-\infty<l=\inf_{t\in(a,b)}u'(t)$ *and* $\sup_{t\in(a,b)}u'(t)=L<\infty$, *then* $u$ *is* $(l,L)$-*Lipschitzian on* $[a,b]$.

THEOREM 3. (Liu 2004, [24]) *If* $u$ *is* $(l,L)$-*Lipschitzian on* $[a,b]$ *and* $f$ *is Riemann integrable on* $[a,b]$ *then*

$$|D(f,u;a,b)|\leqslant\frac{1}{2}(L-l)\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt. \tag{2.13}$$

*The constant* $\frac{1}{2}$ *is the best possible in* (2.13).

*Moreover, if there exist constants* $m,M\in\mathbb{R}$ *such that*

$$m\leqslant f(t)\leqslant M \qquad \text{for any } t\in[a,b], \tag{2.14}$$

*then*

$$|D(f,u;a,b)|\leqslant\frac{1}{4}(L-l)(M-m)(b-a). \tag{2.15}$$

*The constant* $\frac{1}{4}$ *is the best possible in* (2.15).

REMARK 2. It is clear that Liu's results above provide a refinement for the inequality (2.2) when the function $u$ is $(l,L)$-Lipschitzian.

The following different results for $(l,L)$-Lipschitzian integrators can be stated as well:

THEOREM 4. (Dragomir 2007, [18]) *Let* $f,u:[a,b]\to\mathbb{R}$ *be such that* $u$ *is* $(l,L)$-*Lipschitzian on* $[a,b]$.

*(i) If* $f$ *is of bounded variation, then*

$$|D(f,u;a,b)|\leqslant\frac{1}{4}(L-l)(b-a)\bigvee_a^b(f). \tag{2.16}$$

*The constant* $\frac{1}{4}$ *is the best possible in* (2.16).

*(ii) If* $f$ *is* $K$-*Lipschitzian on* $[a,b]$, *then*

$$|D(f,u;a,b)|\leqslant\frac{1}{6}K(L-l)(b-a)^2. \tag{2.17}$$

*(iii) If* $f$ *is nondecreasing, then*

$$|D(f,u;a,b)|\leqslant 2\cdot\frac{L-l}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt \tag{2.18}$$

$$\leqslant\begin{cases}\frac{1}{2}(L-l)\max\{|f(a)|,|f(b)|\}(b-a);\\[2mm] \frac{1}{(q+1)^{\frac{1}{q}}}(L-l)\|f\|_p(b-a)^{\frac{1}{q}} \quad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2mm] (L-l)\|f\|_1.\end{cases} \tag{2.19}$$

*The constants* $2$ *and* $\frac{1}{2}$ *are the best possible in (*2.18*).*

REMARK 3. It is an open question whether or not the multiplicative constant $\frac{1}{6}$ is the best possible in (2.17).

### 2.3. Bounds for Integrators of Bounded Variation

THEOREM 5. (Dragomir-Fedotov 2001, [22]) *If* $u$ *is of bounded variation on* $[a,b]$ *and* $f$ *is continuous on* $[a,b]$*, then*

$$|D(f,u;a,b)| \leqslant \bigvee_a^b (u) \max_{t\in[a,b]} \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|. \tag{2.20}$$

*The inequality (*2.20*) is sharp.*

*Moreover, if* $f$ *is* $K$*-Lipschitzian, then*

$$|D(f,u;a,b)| \leqslant \frac{1}{2}K(b-a)\bigvee_a^b (u). \tag{2.21}$$

*The constant* $\frac{1}{2}$ *is the best possible in (*2.21*).*

If other information is available about the integrand $f$, then other bounds can be obtained as well.

THEOREM 6. (Cerone-Dragomir 2006, [2]) *Let* $u : [a,b] \to \mathbb{R}$ *be a function of bounded variation on* $[a,b]$.

*(i) If* $f$ *is continuous and of bounded variation on* $[a,b]$*, then*

$$|D(f,u;a,b)| \leqslant \bigvee_a^b (f)\bigvee_a^b (u); \tag{2.22}$$

*(ii) If* $f$ *is of* $r-H-$*Hölder type (with* $r \in (0,1]$ *and* $H > 0$*), then*

$$|D(f,u;a,b)| \leqslant \frac{H}{r+1}(b-a)^r\bigvee_a^b (u); \tag{2.23}$$

*(iii) If* $f : [a,b] \to \mathbb{R}$ *is absolutely continuous, then*

$$|D(f,u;a,b)| \leqslant \begin{cases} \frac{1}{2}(b-a)\|f'\|_\infty \bigvee_a^b (u), & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{(q+1)^{\frac{1}{q}}}(b-a)^{\frac{1}{q}}\|f'\|_p \bigvee_a^b (u), & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ & \text{if } f' \in L_p[a,b]; \\ \|f'\|_1 \bigvee_a^b (u). \end{cases} \tag{2.24}$$

REMARK 4. It is an open problem whether or not the multiplicative constants $1, \frac{1}{r+1}, \frac{1}{2}, \frac{1}{(q+1)^{1/q}}$ and $1$ in (2.22) – (2.24) are the best possible.

### 2.4. Bounds for Monotonic Integrators

The following result holds.

THEOREM 7. (Dragomir 2004, [15]) *If* $f:[a,b]\to\mathbb{R}$ *is* $L$*-Lipschitzian on* $[a,b]$ *and* $u$ *is nondecreasing on* $[a,b]$, *then*

$$\begin{aligned} |D(f,u;a,b)| &\leqslant \frac{1}{2}L(b-a)\left[u(b)-u(a)-K(u;a,b)\right] \\ &\leqslant \frac{1}{2}L(b-a)\left[u(b)-u(a)\right], \end{aligned} \tag{2.25}$$

*where*

$$K(u;a,b):=\frac{4}{(b-a)^2}\int_a^b u(t)\left(t-\frac{a+b}{2}\right)dt\geqslant 0. \tag{2.26}$$

*The constant* $\frac{1}{2}$ *is the best possible in both inequalities.*

Another result may be stated as:

THEOREM 8. (Dragomir 2004, [15]) *Let* $f:[a,b]\to\mathbb{R}$ *be a function of bounded variation on* $[a,b]$, $u:[a,b]\to\mathbb{R}$ *a nondecreasing function on* $[a,b]$ *such that the Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$ *exists. Then*

$$\begin{aligned} |D(f,u;a,b)| &\leqslant \left[u(b)-u(a)-Q(u;a,b)\right]\bigvee_a^b(f) \\ &\leqslant \left[u(b)-u(a)\right]\bigvee_a^b(f), \end{aligned} \tag{2.27}$$

*where*

$$Q(u;a,b):=\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)u(t)\,dt\geqslant 0.$$

*The first inequality in (2.27) is sharp.*

### 2.5. Bounds for Convex Integrators

We recall that the function $u:[a,b]\to\mathbb{R}$ is convex on $[a,b]$ if $u(\lambda t+(1-\lambda)s)\leqslant \lambda u(t)+(1-\lambda)u(s)$ for each $t,s\in[a,b]$ and $\lambda\in[0,1]$.

THEOREM 9. (Dragomir 2007, [17]) *Let* $u:[a,b]\to\mathbb{R}$ *be a continuous convex function on* $[a,b]$.

*(i) If* $f:[a,b]\to\mathbb{R}$ *is a function of bounded variation on* $[a,b]$, *then*

$$|D(f,u;a,b)|\leqslant \frac{1}{4}\left[u'_-(b)-u'_+(a)\right](b-a)\bigvee_a^b(f). \tag{2.28}$$

*The constant* $\frac{1}{4}$ *is the best possible in* (2.28)*.*

*(ii) If* $f:[a,b]\to\mathbb{R}$ *is a nondecreasing function on* $[a,b]$*, then*

$$\begin{aligned} 0 &\leqslant D(f,u;a,b) \\ &\leqslant 2\cdot\frac{u'_-(b)-u'_+(a)}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt \\ &\leqslant \left[u'_-(b)-u'_+(a)\right]\times\begin{cases}\frac{1}{2}\max\{|f(a)|,|f(b)|\}(b-a); \\ \frac{1}{(q+1)^{\frac{1}{q}}}\|f\|_p(b-a)^{\frac{1}{q}} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \|f\|_1.\end{cases}\end{aligned} \tag{2.29}$$

*The constants* 2 *and* $\frac{1}{2}$ *are the best possible.*

*(iii) If* $f$ *is an* $L$*-Lipschitzian function on* $[a,b]$*, then:*

$$|D(f,u;a,b)|\leqslant\frac{1}{6}L\left[u'_-(b)-u'_+(a)\right](b-a)^2. \tag{2.30}$$

REMARK 5. It is an open question whether or not $\frac{1}{6}$ is the best constant in (2.30).

## 3. Integral Representation and Other Error Bounds

For the integrator $u:[a,b]\to\mathbb{R}$ consider the following auxiliary mappings $\Phi_u,\Gamma_u$ and $\Delta_u$ that have been introduced in [15] (see also [16] and [17]):

$$\Phi_u(t):=\frac{(t-a)u(b)+(b-t)u(a)}{b-a}-u(t),\qquad t\in[a,b]; \tag{3.1}$$

$$\Gamma_u(t):=(t-a)[u(b)-u(t)]-(b-t)[u(t)-u(a)],\qquad t\in[a,b] \tag{3.2}$$

and

$$\Delta_u(t):=\frac{u(b)-u(t)}{b-t}-\frac{u(t)-u(a)}{t-a},\qquad t\in(a,b). \tag{3.3}$$

### 3.1. Integral Representation and Other Bounds

The following representation result was essentially established in [15], (see also [16]).

THEOREM 10. (Dragomir 2004, [15]) *Let* $f,u:[a,b]\to\mathbb{R}$ *be such that the Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$ *and the Riemann integral* $\int_a^b f(t)\,dt$ *exist. Then*

$$\begin{aligned} D(f,u;a,b) &= \int_a^b\Phi_u(t)\,df(t)=\frac{1}{b-a}\int_a^b\Gamma_u(t)\,df(t) \\ &= \frac{1}{b-a}\int_a^b(t-a)(b-t)\Delta_u(t)\,df(t). \end{aligned} \tag{3.4}$$

The following bounds for the functional $D(f,u;a,b)$ can then be stated:

THEOREM 11. (Dragomir 2004, [15]) *Assume that* $f,u:[a,b]\to\mathbb{R}$.

*(i) If* $f$ *is of bounded variation and* $u$ *is continuous on* $[a,b]$, *then*

$$|D(f,u;a,b)| \leqslant \begin{cases} \sup\limits_{t\in[a,b)} |\Phi_u(t)| \bigvee_a^b(f), \\ \frac{1}{b-a} \sup\limits_{t\in[a,b]} |\Gamma_u(t)| \bigvee_a^b(f), \\ \frac{1}{b-a} \sup\limits_{t\in(a,b)} [(t-a)(b-t)|\Delta_u(t)|] \bigvee_a^b(f). \end{cases} \tag{3.5}$$

*(ii) If* $f$ *is* $L$*-Lipschitzian and* $u$ *is Riemann integrable on* $[a,b]$, *then*

$$|D(f,u;a,b)| \leqslant \begin{cases} L\int_a^b |\Phi_u(t)|\,dt, \\ \frac{L}{b-a}\int_a^b |\Gamma_u(t)|\,dt, \\ \frac{L}{b-a}\int_a^b (t-a)(b-t)|\Delta_u(t)|\,dt. \end{cases} \tag{3.6}$$

*(iii) If* $f$ *is nondecreasing on* $[a,b]$ *and* $u$ *is continuous on* $[a,b]$, *then*

$$|D(f,u;a,b)| \leqslant \begin{cases} \int_a^b |\Phi_u(t)|\,df(t), \\ \frac{1}{b-a}\int_a^b |\Gamma_u(t)|\,df(t), \\ \frac{1}{b-a}\int_a^b (t-a)(b-t)|\Delta_u(t)|\,df(t). \end{cases} \tag{3.7}$$

COROLLARY 1. (Dragomir 2004, [15]) *Let* $f,u:[a,b]\to\mathbb{R}$.

*(i) If* $f$ *is of bounded variation and* $u$ *is continuous, then*

$$|D(f,u;a,b)| \leqslant \frac{1}{4}(b-a)\|\Delta_u\|_\infty \bigvee_a^b(f); \tag{3.8}$$

*(ii) If* $f$ *is* $L$*-Lipschitzian and* $u$ *is Riemann integrable on* $[a,b]$, *then*

$$|D(f,u;a,b)| \leqslant \begin{cases} \frac{1}{6}L(b-a)^2\|\Delta_u\|_\infty, \\ L(b-a)^{1+\frac{1}{q}}[B(q+1,q+1)]^{\frac{1}{q}}\|\Delta_u\|_p, \text{ if } \Delta_u\in L_p[a,b] \\ \qquad\qquad \text{and } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{4}L(b-a)\|\Delta_u\|_1, \end{cases} \tag{3.9}$$

*where* $B(\cdot,\cdot)$ *is Euler's Beta function;*

*(iii) If $f$ is nondecreasing on $[a,b]$ and $u$ is continuous, then*

$$|D(f,u;a,b)| \leqslant \begin{cases} \frac{1}{4}(b-a)\int_a^b |\Delta_u(t)|\,df(t), \\ \frac{1}{b-a}\left(\int_a^b [(b-t)(t-a)]^q\,df(t)\right)^{\frac{1}{q}}\left(\int_a^b |\Delta_u(t)|^p\,df(t)\right)^{\frac{1}{p}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{b-a}\|\Delta_u\|_\infty \int_a^b (t-a)(b-t)\,df(t). \end{cases} \tag{3.10}$$

REMARK 6. It is an open problem whether or not the multiplicative constants in (3.8) – (3.10) are the best possible.

Utilising the first representation in (3.4), the following sharp estimate of the error $D(f,u;a,b)$ can be stated.

THEOREM 12. (Dragomir 2005. [16]) *Let $f,u:[a,b]\to\mathbb{R}$ be of bounded variation on $[a,b]$ and $u:[a,b]\to\mathbb{R}$ such that there exist constants $n,N\in\mathbb{R}$ such that*

$$n \leqslant u(t) \leqslant N \quad \text{for any } t\in[a,b] \tag{3.11}$$

*and the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists. Then*

$$|D(f,u;a,b)| \leqslant (N-n)\bigvee_a^b (f). \tag{3.12}$$

*The multiplicative constant* 1 *on the right hand side of (*3.12*) is the best possible.*

COROLLARY 2. (Dragomir 2005. [16]) *Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation and $u:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. Then*

$$|D(f,u;a,b)| \leqslant \left[\max_{t\in[a,b]} u(t) - \min_{t\in[a,b]} u(t)\right]\bigvee_a^b (f). \tag{3.13}$$

*The inequality (*3.13*) is sharp.*

### 3.2. Double Integral Representations and More Bounds

For a function $g:[a,b]\to\mathbb{R}$, consider the *generalised trapezoid error transform* $\Phi_g:[a,b]\to\mathbb{R}$ given by (3.1), and if $g$ is Lebesgue integrable, the *Ostrowski transform*, which is the error of approximating the function by its integral mean, defined by:

$$\Theta_g(t) := g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds, \quad t\in[a,b]. \tag{3.14}$$

We also define the kernel $Q:[a,b]^2 \to \mathbb{R}$,

$$Q(t,s) := \begin{cases} t-b \text{ if } a \leqslant s \leqslant t \leqslant b, \\ \\ t-a \text{ if } a \leqslant t < s \leqslant b. \end{cases} \tag{3.15}$$

The following representation result in terms of $\Theta_g$ and $Q$ may be stated:

LEMMA 2. (Dragomir 2007, [19]) *If $f,u:[a,b] \to \mathbb{R}$ are bounded functions and such that the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist, then we have the representation:*

$$D(f,u;a,b) = \int_a^b \Theta_f(s)\,du(s) = \frac{1}{b-a}\int_a^b \left(\int_a^b Q(t,s)\,df(t)\right) du(s). \tag{3.16}$$

Another representation of $D(f,u;a,b)$ is incorporated in:

LEMMA 3. (Dragomir 2007, [19]) *With the assumptions in Lemma 2, we have*

$$D(f,u;a,b) = \int_a^b \Phi_u(t)\,df(t) = \frac{1}{b-a}\int_a^b \left(\int_a^b Q(t,s)\,du(s)\right) df(t), \tag{3.17}$$

*where $Q$ is defined by (3.15).*

The following lemma is of interest in itself [19].

LEMMA 4. *If $f:[a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and $v:[a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$, then*

$$\left|\int_a^b f(t)\,dv(t)\right| \leqslant \int_a^b |f(t)|\,d\left(\bigvee_a^t(v)\right) \tag{3.18}$$

$$\leqslant \left[\bigvee_a^b(v)\right]^{\frac{1}{q}} \left\{\int_a^b |f(t)|^p\,d\left[\bigvee_a^t(v)\right]\right\}^{\frac{1}{p}} \leqslant \max_{t\in[a,b]}|f(t)|\bigvee_a^b(v),$$

*where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.*

The first inequality in the above Lemma 4 can be utilized to provide other bounds for the error functional $D(f,u;a,b)$ as follows:

THEOREM 13. (Dragomir 2007, [19]) *If $u:[a,b] \to \mathbb{R}$ is of bounded variation and $f:[a,b] \to \mathbb{R}$ is $L$-Lipschitzian, then*

$$|D(f,u;a,b)| \leqslant L\left[\frac{1}{2}(b-a)\bigvee_a^b(u) - \frac{2}{b-a}\int_a^b \left(\bigvee_a^s(u)\right)\left(s-\frac{a+b}{2}\right)ds\right] \tag{3.19}$$

$$\leqslant \frac{1}{2}L(b-a)\bigvee_a^b(u).$$

*The constant $\frac{1}{2}$ is the best possible in both inequalities.*

REMARK 7. The inequality between the first and last term in (3.19) was firstly discovered by Dragomir and Fedotov in [22] where they also showed the sharpness of the constant $\frac{1}{2}$.

When certain conditions around the end points are imposed, then the following results may be stated as well:

THEOREM 14. (Dragomir 2007, [19]) *Assume that* $f:[a,b]\to\mathbb{R}$ *is of bounded variation on* $[a,b]$. *If* $u:[a,b]\to\mathbb{R}$ *is continuous and such that there exist constants* $L_a, L_b > 0$ *and* $\alpha,\beta>0$ *with the properties that:*

$$|u(t)-u(a)| \leqslant L_a (t-a)^{\alpha}, \qquad |u(t)-u(b)| \leqslant L_b (b-t)^{\beta} \tag{3.20}$$

*for any* $t\in[a,b]$, *then*

$$\begin{aligned}
&|D(f,u;a,b)| \\
&\leqslant \frac{1}{b-a}L_a\left[\int_a^b\left(\bigvee_a^t(f)\right)(t-a)^{\alpha}\,dt-\alpha\int_a^b\left(\bigvee_a^t(f)\right)(b-t)(t-a)^{\alpha-1}\,dt\right] \\
&\quad+\frac{1}{b-a}L_b\left[\beta\int_a^b\left(\bigvee_a^t(f)\right)(t-a)(b-t)^{\beta-1}\,dt-\int_a^b\left(\bigvee_a^t(f)\right)(b-t)^{\beta}\,dt\right].
\end{aligned} \tag{3.21}$$

The following particular result may be useful for applications.

COROLLARY 3. (Dragomir 2007, [19]) *If* $f:[a,b]\to\mathbb{R}$ *is of bounded variation and* $u:[a,b]\to\mathbb{R}$ *is* $K$*-Lipschitzian, then*

$$|D(f,u;a,b)| \leqslant \frac{4}{b-a}\cdot K\int_a^b\left(t-\frac{a+b}{2}\right)\cdot\bigvee_a^t(f)\,dt \tag{3.22}$$

$$\leqslant\begin{cases} K(b-a)\bigvee_a^b(f); \\ \frac{2(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}K\left(\int_a^b\left[\bigvee_a^t(f)\right]^p dt\right)^{\frac{1}{p}}, \ \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ 2K\int_a^b\left(\bigvee_a^t(f)\right)dt. \end{cases}$$

*The multiplication constant* $4$ *is the best possible.*

Finally for the section we have the following result as well:

THEOREM 15. (Dragomir 2007, [19]) *Let* $f:[a,b]\to\mathbb{R}$ *be a function of bounded variation and* $u:[a,b]\to\mathbb{R}$ *an* $(l,L)$*-Lipschitzian function. Then*

$$|D(f,u;a,b)| \leqslant \frac{2}{b-a}(L-l)\int_a^b \left(t-\frac{a+b}{2}\right)\cdot\bigvee_a^t(f)\,dt \tag{3.23}$$

$$\leqslant \begin{cases} \frac{1}{2}(L-l)(b-a)\bigvee_a^b(f); \\ \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}(L-l)\left(\int_a^b\left[\bigvee_a^t(f)\right]^p dt\right)^{\frac{1}{p}}, \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ (L-l)\int_a^b\left(\bigvee_a^t(f)\right)dt. \end{cases}$$

*The constant* 2 *in the first inequality is the best possible.*

## 3.3. Bounds in the Case when $u'$ is of Bounded Variation

In [15], by considering the *kernel* $\Phi_u:[a,b]\to\mathbb{R}$ given by (3.1), the author has obtained the following *integral representation*:

$$D(f,u;a,b)=\int_a^b \Phi_u(t)\,df(t), \tag{3.24}$$

where $u,f:[a,b]\to\mathbb{R}$ are bounded functions such that the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist.

We have the following integral representation of $\Phi_u$.

LEMMA 5. (Dragomir 2007, [20]) *Assume that* $u:[a,b]\to\mathbb{R}$ *is absolutely continuous on* $[a,b]$ *and such that the derivative* $u'$ *exists on* $[a,b]$ *(eventually except in a finite number of points). If* $u'$ *is Riemann integrable on* $[a,b]$, *then*

$$\Phi_u(t):=\frac{1}{b-a}\int_a^b K(t,s)\,du'(s), \quad t\in[a,b], \tag{3.25}$$

*where the kernel* $K:[a,b]^2\to\mathbb{R}$ *is given by*

$$K(t,s):=\begin{cases}(b-t)(s-a) \text{ if } s\in[a,t], \\ (t-a)(b-s) \text{ if } s\in(t,b].\end{cases} \tag{3.26}$$

Utilising the above representation for the kernel $\Phi_u$ and the identity (3.24), we can start with the following results:

THEOREM 16. (Dragomir 2007, [20]) *Assume that* $u:[a,b]\to\mathbb{R}$ *is as in Lemma* 5.
*(i) If* $u'$ *and* $f$ *are of bounded variation on* $[a,b]$, *then*

$$|D(f,u;a,b)|\leqslant\frac{1}{4}(b-a)\bigvee_a^b(u')\cdot\bigvee_a^b(f), \tag{3.27}$$

*and the constant* $\frac{1}{4}$ *is the best possible in* (3.27)*.*

*(ii) If the derivative* $u'$ *is of bounded variation on* $[a,b]$ *while* $f$ *is* $L$*-Lipschitzian on* $[a,b]$*, then*

$$|D(f,u;a,b)| \leqslant \frac{1}{6}L(b-a)^2 \bigvee_a^b (u'). \tag{3.28}$$

*(iii) If the derivative* $u'$ *is of bounded variation on* $[a,b]$ *and* $f$ *is nondecreasing on* $[a,b]$*, then*

$$|D(f,u;a,b)| \leqslant 2 \cdot \frac{\bigvee_a^b (u')}{b-a} \cdot \left( \int_a^b t - \frac{a+b}{2} \right) f(t)\,dt \tag{3.29}$$

$$\leqslant \begin{cases} \frac{1}{2}\bigvee_a^b (u') \max\{|f(a)|,|f(b)|\}(b-a); \\ \frac{1}{(q+1)^{1/q}} \bigvee_a^b (u') \|f\|_p (b-a)^{1/q} \quad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \bigvee_a^b (u') \|f\|_1 . \end{cases}$$

*The constants* 2 *and* $\frac{1}{2}$ *are the best possible in* (3.29)*.*

REMARK 8. It is an open question whether or not $\frac{1}{6}$ is the best constant in (3.28).

### 3.4. Bounds in the Case when $u'$ is Lipschitzian

The following result can be stated as well:

THEOREM 17. (Dragomir 2007, [20]) *Let* $u:[a,b]\to\mathbb{R}$ *be absolutely continuous on* $[a,b]$ *with the property that* $u'$ *is* $S$*-Lipschitzian on* $(a,b)$*.*

*(i) If* $f$ *is of bounded variation, then*

$$|D(f,u;a,b)| \leqslant \frac{1}{8}(b-a)^2 S \bigvee_a^b (f). \tag{3.30}$$

*The constant* $\frac{1}{8}$ *is the best possible in* (3.30)*.*

*(ii) If* $f$ *is* $L$*-Lipschitzian on* $[a,b]$*, then*

$$|D(f,u;a,b)| \leqslant \frac{1}{12}(b-a)^3 SL. \tag{3.31}$$

*The constant* $\frac{1}{12}$ *is the best possible in* (3.31)*.*

*(iii) If* $f$ *is nondecreasing, then*

$$|D(f,u;a,b)| \leqslant S \int_a^b \left( t - \frac{a+b}{2} \right) f(t)\,dt \tag{3.32}$$

$$\leqslant \begin{cases} \frac{1}{4} S \max\{|f(a)|,|f(b)|\}(b-a)^2; \\ \frac{1}{2(q+1)^{1/q}} S \|f\|_p (b-a)^{1+1/q} \quad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}(b-a) S \|f\|_1 . \end{cases}$$

*The first inequality is sharp. The constant* $\frac{1}{4}$ *is the best possible in* (3.32)*.*

## 4. Grüss' Type Inequalities

Now, assume that $g:[a,b]\to\mathbb{R}$ is Lebesgue integrable on $[a,b]$ and $-\infty<m\leqslant g(t)\leqslant M<\infty$ for a.e. $t\in[a,b]$. Then the function $u(t):=\int_a^t g(s)\,ds$ is $(m,M)$-Lipschitzian on $[a,b]$ and, by (3.1),

$$\Phi_u(t)=\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds,\quad t\in[a,b].$$

On utilising Theorem 4, the following result for the Čebyšev functional can be stated:

PROPOSITION 2. (Dragomir 2007, [18]) *If* $f:[a,b]\to\mathbb{R}$ *is of bounded variation on* $[a,b]$ *and* $g:[a,b]\to\mathbb{R}$ *is Lebesgue integrable and satisfies the bounds*

$$-\infty<m\leqslant g\leqslant M<\infty\quad a.e.\ on\ [a,b], \tag{4.1}$$

*then*

$$|C(f,g;a,b)|\leqslant\frac{1}{4}(M-m)\bigvee_a^b(f). \tag{4.2}$$

*The constant* $\frac{1}{4}$ *is the best possible.*

*Moreover, if* $f:[a,b]\to\mathbb{R}$ *is nondecreasing on* $[a,b]$, *then*

$$|C(f,g;a,b)|\leqslant 2\cdot\frac{(M-m)}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt \tag{4.3}$$
$$\leqslant\begin{cases}\frac{1}{2}(M-m)\max\{|f(a)|,|f(b)|\};\\[2ex] \frac{1}{(q+1)^{\frac{1}{q}}}(M-m)\|f\|_p(b-a)^{-\frac{1}{p}} & if\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ (M-m)\frac{1}{b-a}\|f\|_1.\end{cases}$$

*The constants* 2 *and* $\frac{1}{2}$ *are the best possible.*

On utilising Theorem 9, the following result may be stated as well:

PROPOSITION 3. (Dragomir 2007, [17]) *If* $f,g$ *are nondecreasing functions, then*

$$0\leqslant C(f,g;a,b) \tag{4.4}$$
$$\leqslant 2\cdot\frac{g(b)-g(a)}{b-a}\cdot\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt$$
$$\leqslant\begin{cases}\frac{1}{2}[g(b)-g(a)]\max\{|f(a)|,|f(b)|\};\\[2ex] \frac{1}{(q+1)^{\frac{1}{q}}}[g(b)-g(a)]\|f\|_p(b-a)^{-\frac{1}{p}}\ if\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{g(b)-g(a)}{b-a}\|f\|_1.\end{cases}$$

*The constants* $2$ *and* $\frac{1}{2}$ *are the best possible.*

*If* $g$ *is nondecreasing on* $[a,b]$ *and* $f$ *is of bounded variation on* $[a,b]$, *then*

$$|C(f,g;a,b)| \leqslant \frac{1}{4}[g(b)-g(a)]\bigvee_a^b(f). \tag{4.5}$$

*The constant* $\frac{1}{4}$ *is the best possible in (*4.5*).*

Notice that these two inequalities can be obtained from Proposition 2 as well.

PROPOSITION 4. (Dragomir 2007, [16]) *If we assume that for the Lebesgue integrable function* $g$, $t \mapsto \int_a^t g(s)\,ds$ *satisfies the condition*

$$\gamma \leqslant \int_a^t g(s)\,ds \leqslant \Gamma \quad \textit{for any} \quad t \in [a,b],$$

*then*

$$|C(f,g;a,b)| \leqslant (\Gamma-\gamma)\bigvee_a^b(f), \tag{4.6}$$

*where* $f:[a,b]\to\mathbb{R}$ *is of bounded variation on* $[a,b]$. *The inequality is sharp.*

The proof of (4.6) follows from Theorem 12.

PROPOSITION 5. (Dragomir 2007, [19]) *If* $-\infty < \phi \leqslant g(t) \leqslant \Phi$ *for a.e.* $t\in[a,b]$, *and* $f:[a,b]\to\mathbb{R}$ *is of bounded variation on* $[a,b]$, *then we have the inequalities*

$$|C(f;g)| \leqslant \frac{2}{(b-a)^2}(\Phi-\phi)\int_a^b\left(t-\frac{a+b}{2}\right)\bigvee_a^t(f)\,dt \tag{4.7}$$
$$\leqslant \begin{cases} \frac{1}{2}(\Phi-\phi)\bigvee_a^b(f); \\ \frac{(b-a)^{\frac{1}{q}-1}}{(q+1)^{\frac{1}{q}}}(\Phi-\phi)\left(\int_a^b\left[\bigvee_a^t(f)\right]^p dt\right)^{\frac{1}{p}}, \ \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\Phi-\phi}{b-a}\int_a^b\left(\bigvee_a^t(f)\right)dt. \end{cases}$$

Finally, we mention the following results from [20]:

PROPOSITION 6. (Dragomir 2007, [20]) *Assume that* $g$ *is bounded variation on* $[a,b]$.

*(i) If* $f$ *is of bounded variation on* $[a,b]$, *then*

$$|C(f,g;a,b)| \leqslant \frac{1}{4}\bigvee_a^b(g)\cdot\bigvee_a^b(f). \tag{4.8}$$

*The constant* $\frac{1}{4}$ *is the best possible in (*4.8*).*

*(ii) If $f$ is nondecreasing, then*

$$|C(f,g;a,b)| \leqslant 2\bigvee_a^b (g) \cdot \frac{1}{(b-a)^2} \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt \tag{4.9}$$
$$\leqslant \begin{cases} \frac{1}{2} \cdot \bigvee_a^b (g) \max\{|f(a)|, |f(b)|\}; \\ \frac{1}{(q+1)^{1/q}} \bigvee_a^b (g) \|f\|_p (b-a)^{-1/p} \quad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{b-a} \bigvee_a^b (g) \|f\|_1 . \end{cases}$$

*The multiplicative constants* 2 *and* $\frac{1}{2}$ *are the best possible in (*4.9*).*

PROPOSITION 7. (Dragomir 2007, [20]) *Assume that $g$ is $K$-Lipschitzian on $[a,b]$.*

*(i) If $f$ is of bounded variation, then*

$$|C(f,g;a,b)| \leqslant \frac{1}{8} \cdot (b-a) K \bigvee_a^b (f). \tag{4.10}$$

*The constant $\frac{1}{8}$ is the best possible.*

*(ii) If $f$ is $L$-Lipschitzian, then*

$$|C(f,g;a,b)| \leqslant \frac{1}{12} (b-a)^2 KL. \tag{4.11}$$

*The constant $\frac{1}{12}$ is the best possible in (*4.11*).*

*(iii) If $f$ is nondecreasing, then*

$$|C(f,g;a,b)| \leqslant K \cdot \frac{1}{b-a} \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt \tag{4.12}$$
$$\leqslant \begin{cases} \frac{1}{4} K (b-a) \max\{|f(a)|, |f(b)|\}; \\ \frac{1}{2(q+1)^{1/q}} K (b-a)^{1/q} \|f\|_p \quad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2} K \|f\|_1 . \end{cases}$$

*The first inequality is sharp. The constant $\frac{1}{4}$ is the best possible.*

REMARK 9. Inequalities (4.8) and (4.10) were obtained by Cerone and Dragomir in [5, Corollary 3.5] in a different context. However, the sharpness of the constants $\frac{1}{4}$ and $\frac{1}{8}$ was not discussed there.

## 5. Other Error Functionals

In 2003 in order to approximate the Riemann-Stieltjes integral of a product the author introduced in [10] the following *generalised Čebyšev functional* for Riemann-Stieltjes integrals:

$$T(f,g;u) := \frac{1}{u(b)-u(a)}\int_a^b f(t)g(t)\,du(t) - \frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t)\cdot\frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t), \quad (5.1)$$

provided the involved integrals exist and $u(b) \neq u(a)$. Since then, many sharp error bounds for this functional have been obtained, see [11], [12] and [6].

From a different view point, in 2000, see [13] the author introduced the following *generalised Ostrowski functional* for the Riemann-Stieltjes integral

$$O(f;u) := \int_a^b f(t)\,du(t) - [u(b)-u(a)]f(x), \quad x \in [a,b] \quad (5.2)$$

and pointed out various bounds which provided sharp inequalities of Ostrowski type. Since then, many other sharp error bounds have been obtained for different classes of integrands and integrators, see [14], [1] and [9].

For other error functionals and sharp bounds, see the *Research Report Collection of RGMIA* at

http://rgmia.vu.edu.au/reports.html.

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