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A SURVEY OF SOME BOUNDS FOR GAUSS' HYPERGEOMETRIC FUNCTION AND RELATED BIVARIATE MEANS

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Abstract. We give an expository summary of a collection of inequalities involving Gauss' hypergeometric function ${}_2F_1$ and the closely-related power mean (and certain other bivariate means). Two conjectures involving simultaneous sharp bounds for the hypergeometric function are included. Sharpness for the corresponding zero-balanced case is observed.

1. Introduction

This investigation begins with the fundamental concept of a homogeneous bivariate *mean* which is defined here as a continuous function $\mathscr{M} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ satisfying $\min(x,y) \leqslant \mathscr{M}(x,y) \leqslant \max(x,y)$ and $\mathscr{M}(\lambda x, \lambda y) = \lambda \mathscr{M}(x,y)$ for all $x, y, \lambda > 0$. The arithmetic mean $\mathscr{A}(x,y) \equiv \frac{x+y}{2}$ and the geometric mean $\mathscr{G}(x,y) \equiv \sqrt{xy}$ are special cases of the family of power means (or Hölder means) given by

$$\mathscr{A}_\lambda(x,y) \equiv \left(\frac{x^\lambda + y^\lambda}{2}\right)^{1/\lambda} \qquad (\lambda \neq 0),$$

with $\mathscr{A}_0(x,y) \equiv \sqrt{xy}$. A standard argument can be used to show that the function $\lambda \mapsto \mathscr{A}_\lambda$ is increasing. From this follows one of many proofs (see [9]) of the well-known arithmetic mean - geometric mean inequality: $\mathscr{G} = \mathscr{A}_0 \leqslant \mathscr{A}_1 = \mathscr{A}$ where each is evaluated at (x,y). Other interesting (but perhaps less familiar) means include the *logarithmic mean* $\mathscr{L}(x,y) \equiv (x-y)/(\ln x - \ln y)$ and the *identric mean* $\mathscr{I}(x,y) \equiv \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{1/(x-y)}$ for $x \neq y$ (with $\mathscr{L}(x,x) \equiv x \equiv \mathscr{I}(x,x)$ preserving continuity). The previous inequality has been refined (e.g., see [9]) by these two means to yield

$$\mathscr{G} \leqslant \mathscr{L} \leqslant \mathscr{I} \leqslant \mathscr{A}. \tag{1}$$

(It is worth noting here that (1) has an elegant generalization given by Páles' Comparison Theorem [14] below.) The elliptical arc length function

$$E(x,y) \equiv \int_0^{\pi/2} \sqrt{x^2\cos^2 t + y^2 \sin^2 t}\, dt$$

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gives rise to the mean $\hat{E}(x,y) \equiv \frac{2}{\pi}E(x,y)$. The important *complete elliptic integral* of the second kind $\mathscr{E}(r) \equiv E(1,\sqrt{1-r^2})$ is a much-studied function arising in mathematical physics (e.g., see [16]) and can be expressed in terms of the Gaussian hypergeometric function ${}_2F_1$ defined by

$$ {}_2F_1(\alpha,\beta;\gamma;z) \equiv \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n, \quad |z|<1, $$

$(\alpha)_n \equiv \Gamma(\alpha+n)/\Gamma(\alpha) = \alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n \in \mathbb{N}$, $(\alpha)_0 \equiv 1$. The representation of $\mathscr{E}$ in terms of ${}_2F_1$ is

$$ \mathscr{E}(r) \equiv \int_0^{\pi/2} (1-r^2\sin^2\theta)^{1/2} d\theta = \frac{\pi}{2} {}_2F_1(-1/2,1/2;1;r^2) $$

and can be established by way of Euler's integral representation for ${}_2F_1$:

$$ {}_2F_1(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)\Gamma(\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt $$

for $\gamma > \beta > 0$ and $z \in \mathbb{C}\backslash[1,\infty)$ (see [16]). By the earlier discussion involving $\hat{E}$, it follows that ${}_2F_1(-1/2,1/2;1;1-r^2) = \hat{E}(1,r)$. Given this representation, it is natural to seek relationships between ${}_2F_1$ and other means.

2. Bounds for the Bivariate Hypergeometric Mean

A unifying cornerstone for much of the work in this pursuit is due to B.C. Carlson. In a series of seminal papers in the 1960's (see [10] and the references therein), Carlson et al. investigated the general *hypergeometric mean* value which is defined as an extension of Euler's integral representation. A specific case of the hypergeometric mean of interest here is given by

$$ \mathscr{H}_a(x,y) \equiv x\cdot[{}_2F_1(-a,1/2;1;1-y/x)]^{\frac{1}{a}}. $$

Thus, $\mathscr{H}_a(x^p,y^p)^{1/p} = x\cdot[{}_2F_1(-a,1/2;1;1-y^p/x^p)]^{\frac{1}{ap}}$ with $a=1/2$ and $p=2$ again reveals the mean $\mathscr{H}_{1/2}(1,r^2)^{1/2} = {}_2F_1(-1/2,1/2;1;1-r^2)$. Carlson's work in this area is widely cited and his insights in relating ${}_2F_1$ to the power mean motivated several conjectures and subsequent discoveries, some of which are highlighted below.

M. Vuorinen [19] conjectured that $\mathscr{H}_{1/2}(1,r) = {}_2F_1(-1/2,1/2;1;1-r)^2$ is sharply bounded below by the power mean of order 3/4. This was proven by the authors in [5]. Later, H. Alzer et al. [3] found a sharp upper bound. Together, these results become

THEOREM A. [3, 5] *For all* $r \in (0,1)$,

$$ \mathscr{A}_\lambda(1,r) \leqslant {}_2F_1(-1/2,1/2;1;1-r)^2 \leqslant \mathscr{A}_\mu(1,r), $$

if $\lambda \leqslant 3/4$ *(sharp*[1]*) and* $\mu \geqslant \ln(\sqrt{2})/\ln(\pi/2)$ *(sharp).*

(Remark: Precursors of the inequalities in Theorem A date back to astronomers like Kepler who sought estimates of elliptical arc length. Ramanujan's interest in $\mathscr{E}$

[1](i.e., best possible)

led him to construct his own estimates (see [1]). Theorem A provides the best lower and upper power mean approximations to $\mathscr{E}$. Other computable bounds for the complete elliptic integrals have been discovered by H. Kazi and E. Neuman in [12].) The *arithmetic-geometric mean* due to Gauss (see [6, 16]) is given by

$$\mathscr{AG}(1,r) = \frac{1}{_2F_1(1/2,1/2;1;1-r^2)}.$$

It is known that

$$\mathscr{A}_0(x,y) \leqslant \mathscr{AG}(x,y) \leqslant \mathscr{A}_{1/2}(x,y) \quad \text{for all } x,y>0. \tag{2}$$

Vamanamurthy and Vuorinen [18] showed that the order $1/2$ is sharp in (2). It should also be noted that the complete elliptical integral of the first kind corresponds to this case of the hypergeometric function:

$$\mathscr{K}(r) \equiv \int_0^{\pi/2} (1-r^2\sin^2\theta)^{-1/2} d\theta = \frac{\pi}{2} {}_2F_1(1/2,1/2;1;r^2),$$

Sharp bounds for $\mathscr{K}$ follow directly from (2) which can be restated as

$$\mathscr{A}_\lambda(1,r) \leqslant \frac{1}{_2F_1(1/2,1/2;1;1-r^2)} \leqslant \mathscr{A}_\mu(1,r), \tag{3}$$

where $\lambda \leqslant 0$ (sharp), $\mu \geqslant 1/2$ (sharp) and $r \in (0,1)$. Related known inequalities involving the logarithmic mean include

$$\mathscr{L}(1,r) \leqslant \frac{1}{_2F_1(1/2,1/2;1;1-r^2)} \leqslant \mathscr{L}_{3/2}(1,r), \tag{4}$$

where $\mathscr{L}_p(x,y) \equiv \mathscr{L}(x^p,y^p)^{1/p}$. The first inequality in (4) is due to Carlson and Vuorinen [11] and the second is due to Borwein and Borwein [7]. Kazi and Neuman (see Theorem 4.1 in [12]) provided the following improvement of (4) with

$$\mathscr{L}(\mathscr{A}(1,r),\mathscr{G}(1,r)) \leqslant \frac{1}{_2F_1(1/2,1/2;1;1-r^2)} \leqslant \mathscr{L}_{3/2}(\mathscr{A}(1,r),\mathscr{G}(1,r)).$$

Results of the type in (3) and in Theorem A motivate a search for generalizations applicable when $(\pm a,b;c;\cdot)$ replaces $(\pm 1/2,1/2;1;\cdot)$. Guidance in this direction is provided by numerical evidence and by Carlson's [10] work on the hypergeometric mean and weighted power means given by

$$\mathscr{A}_\lambda(\omega;x,y) \equiv \left(\omega x^\lambda + (1-\omega)y^\lambda\right)^{1/\lambda} \quad (\lambda \neq 0)$$

$\mathscr{A}_0(\omega;x,y) \equiv x^\omega y^{1-\omega}$, with weights $\omega, 1-\omega > 0$. (Note: Throughout the remaining discussion, the equally-weighted mean is implied if the weights are omitted: $\mathscr{A}_\lambda(x,y) = \mathscr{A}_\lambda(1/2;x,y)$.) Carlson [10] verified results that imply

$$\mathscr{A}_a(1-b/c;1,1-r) \leqslant {}_2F_1(-a,b;c;r)^{1/a}, \quad \forall r \in (0,1) \tag{5}$$

if $1 \geqslant a$ and $c > b > 0$. Carlson [10] also showed that the inequality in (5) reverses if $a > 1$. Notice that, when $a = 1/2 = b = c/2$, the sharp power mean order is $3/4$ rather than a. This naturally inspires a quest to replace the order of the power mean a in (5) by the best possible value. Efforts to find *sharp power mean orders* and an intrinsic generalization of Theorem A resulted in the following:

THEOREM B. [15] *Suppose* $1 \geqslant a$, $b > 0$ *and* $c > \max(-a,b)$. *If* $c \geqslant \max(1-2a, 2b)$, *then*

$$\mathscr{A}_\lambda(1-b/c;1,1-r) \leqslant {}_2F_1(-a,b;c;r)^{1/a}, \quad \forall r \in (0,1)$$

if and only if $\lambda \leqslant \frac{a+c}{1+c}$. *If* $c \leqslant \min(1-2a, 2b)$, *then*

$$\mathscr{A}_\mu(1-b/c;1,1-r) \geqslant {}_2F_1(-a,b;c;r)^{1/a}, \quad \forall r \in (0,1)$$

if and only if $\mu \geqslant \frac{a+c}{1+c}$.

Borwein et al. [8] studied hypergeometric analogues of $\mathscr{AG}$ that take the form $1/{}_2F_1(1/2-s, 1/2+s; 1; 1-r^p)^q$. Simultaneous sharp bounds extending (3) for these analogues of $\mathscr{AG}$ are given by (see [4, 10])

$$\mathscr{A}_\lambda(\alpha;1,r) \leqslant \frac{1}{{}_2F_1(\alpha,1-\alpha;1;1-r^p)^{\frac{1}{\alpha p}}} \leqslant \mathscr{A}_\mu(\alpha;1,r), \ \forall r \in (0,1)$$

if $\lambda \leqslant 0$ (sharp) and $\mu \geqslant p(1-\alpha)/2$ (sharp) where $0 < \alpha \leqslant 1/2$, $p > 0$.

The next corollary highlights the corresponding simultaneous sharp bounds for the *zero-balanced* hypergeometric function of the form ${}_2F_1(a,b;a+b;\cdot)$ (also studied in [2, 4, 7, 8]). The first inequality in (6) follows from Theorem B. The second inequality in (6) follows directly from the work of Carlson on the R-hypergeometric functions (see (2.15) and (3.4) in [10]). Here, we verify the sharpness of the second inequality and note that the upper bound can also be obtained using elementary series techniques (we prove the stronger result that $r \mapsto \mathscr{A}_0\left(\frac{b}{a+b};1,\frac{1}{(1-r)^a}\right) - {}_2F_1(a,b;a+b;r)$ is absolutely monotonic under the stated conditions).

COROLLARY 1. (see also [4, 10]) *Suppose* $1+a \geqslant b \geqslant a > 0$. *Then for all* $r \in (0,1)$

$$\mathscr{A}_\rho\left(\frac{a}{a+b};1,\frac{1}{(1-r)^a}\right) \leqslant {}_2F_1(a,b;a+b;r) \leqslant \mathscr{A}_\sigma\left(\frac{a}{a+b};1,\frac{1}{(1-r)^a}\right) \tag{6}$$

if $\rho \leqslant \frac{-b}{a(1+a+b)}$ *(sharp) and* $\sigma \geqslant 0$ *(sharp).*

Proof. The first inequality in (6) follows directly from Theorem B by replacing a by $-a$ and c by $a+b$. Hence

$$\mathscr{A}_\mu(\alpha;1,1-r)^{-a} \leqslant {}_2F_1(a,b;a+b;r), \quad \text{for all } r \in (0,1)$$

if and only if $\mu \geqslant b/(a+b+1)$. Since $\mathscr{A}_\mu(\omega;1,1-r)^{-a} = \mathscr{A}_{-\mu/a}(\omega;1,(1-r)^{-a})$, we obtain the first inequality in (6) for $\rho = -\mu/a$. An elementary proof of the second inequality in (6) is as follows. By the monotonicity of $\sigma \mapsto \mathscr{A}_\sigma$, it suffices to prove it for the simple case that $\sigma = 0$. It follows by induction that $\left(\frac{ab}{a+b}\right)_n > \frac{(a)_n(b)_n}{(a+b)_n}$ for all $n-1 \in \mathbb{N}$. Thus $\mathscr{A}_0\left(\frac{b}{a+b};1,\frac{1}{(1-r)^a}\right) = (1-r)^{-ab/(a+b)} = \sum_{n=0}^{\infty}\left(\frac{ab}{a+b}\right)_n \frac{r^n}{n!} > \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(a+b)_n}r^n = {}_2F_1(a,b;a+b;r)$, for all $r \in (0,1)$. Sharpness follows from the observation that for $\hat{\sigma} < 0$, $\mathscr{A}_{\hat{\sigma}}(\omega;1,(1-r)^{-a})$ has a positive finite limit as $r \to 1^-$ while ${}_2F_1(a,b;a+b;r) \to \infty$ as $r \to 1^-$ (see [16], p. 111). Thus, for $\hat{\sigma} < 0$ and r sufficiently close to and less than 1, it follows that

$$ {}_2F_1(a,b;a+b;r) > \mathscr{A}_{\hat{\sigma}}\left(\frac{b}{a+b};1,\frac{1}{(1-r)^a}\right). $$

That is, $\sigma \geqslant 0$ is sharp for (6). □

3. Conjectures and Related Results

Note that either lower or upper bounds are guaranteed by Theorem B under essentially complementary conditions on the parameters. It is also desirable to find simultaneous upper and lower bounds of the form

$$ \mathscr{A}_\lambda(1-b/c;1,1-r) \leqslant {}_2F_1(-a,b;c;r)^{1/a} \leqslant \mathscr{A}_\mu(1-b/c;1,1-r), \qquad (7) $$

Using Alzer's [3] approach to the upper bound in Theorem A and numerical evidence, we conjecture the following companion to Theorem B:

CONJECTURE I. *Let* $1 \geqslant a$, $c > b > 0$ *and* $c > b-a$.
Suppose $c \geqslant \max(1-2a,2b)$. *Then*

$$ {}_2F_1(-a,b;c;r)^{1/a} \leqslant \mathscr{A}_\mu(1-b/c;1,1-r), \text{ for all } r \in (0,1) \qquad (8) $$

if $\mu \geqslant \frac{a\ln(1-b/c)}{\ln\left(\frac{\Gamma(c+a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(c+a)}\right)}$ *(sharp). Suppose* $c \leqslant \min(1-2a,2b)$. *Then the inequality in* (8) *reverses if* $\mu \leqslant \frac{a\ln(1-b/c)}{\ln\left(\frac{\Gamma(c+a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(c+a)}\right)}$ *(sharp).*

By the work of Alzer [3] in verifying the second inequality in Theorem A, the conjecture holds when $a = b = c/2 = 1/2$. It is also interesting to note that, with a replaced by $-a$, the sharp value of $\mu_c \equiv \frac{-a\ln(1-b/c)}{\ln\left(\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(c-a)}\right)}$ in Conjecture I has the property that $\mu_c \to 0$ as c approaches $a+b$ from above. Thus Corollary 1 provides an example of (7) and a verification of Conjecture I in the zero-balanced case.

CONJECTURE II. *If* $a > 1$ *and* $b > 0$, *then*

$$ \mathscr{A}_\lambda(1/2;1,1-r) \leqslant {}_2F_1(-a,b;2b;r)^{1/a}, \text{ for all } r \in (0,1) $$

if $\lambda \leqslant \frac{a\ln(2)}{\ln\left(\frac{\Gamma(b)\Gamma(2b+a)}{\Gamma(2b)\Gamma(b+a)}\right)}$ *(sharp).*

As evidence for these conjectures leading to simultaneous bounds of the type in (7), we have the following two propositions (discussed in [17] and presented here for the reader's convenience) which address the above conjectures in the case that $b = 1$.

PROPOSITION 1. *Suppose $a \in (-1,1)$. If $a > -1/2$, then for all $r \in (0,1)$*

$$\mathscr{A}_\lambda(1/2;1,1-r) \leqslant {}_2F_1(-a,1;2;r)^{1/a} \leqslant \mathscr{A}_\mu(1/2;1,1-r) \tag{9}$$

if $\lambda \leqslant \frac{a+2}{3}$ (sharp) and $\mu \geqslant \frac{a\ln(2)}{\ln(1+a)}$ (sharp). If $a < -1/2$, then (9) holds for all $r \in (0,1)$ if $\lambda \leqslant \frac{a\ln(2)}{\ln(1+a)}$ (sharp) and $\mu \geqslant \frac{a+2}{3}$ (sharp).

PROPOSITION 2. *If $a > 1$, then*

$$\mathscr{A}_\lambda(1/2;1,1-r) \leqslant {}_2F_1(-a,1;2;r)^{1/a} \leqslant \mathscr{A}_\mu(1/2;1,1-r) \quad \forall r \in (0,1)$$

if $\lambda \leqslant \frac{a\ln(2)}{\ln(1+a)}$ (sharp) and $\mu \geqslant \frac{a+2}{3}$ (sharp).

Proofs for both of these propositions can be obtained by applying Páles' Comparison Theorem [14] discussed below. This theorem involves the Stolarsky mean which is defined for distinct $x,y > 0$ and $ab(a-b) \neq 0$ as

$$\mathscr{D}_{a,b}(x,y) \equiv \left(\frac{b(x^a-y^a)}{a(x^b-y^b)}\right)^{1/(a-b)}$$

and continuously extended when $ab(a-b) = 0$ or $x = y$. (Interestingly, the Stolarsky mean has sharp bounds in terms of the identric mean as shown in [13].)

PÁLES' COMPARISON THEOREM. [14] *The inequality $\mathscr{D}_{(a,b)}(x,y) \leqslant \mathscr{D}_{(c,d)}(x,y)$ holds true for all $x,y > 0$ if and only if $a+b \leqslant c+d$ and*

$$\begin{array}{ll} \mathscr{L}(a,b) \leqslant \mathscr{L}(c,d) & \text{if } 0 \leqslant \min(a,\ b,\ c,\ d) \\ \dfrac{|a|-|b|}{a-b} \leqslant \dfrac{|c|-|d|}{c-d} & \text{if } \min(a,\ b,\ c,\ d) < 0 < \max(a,\ b,\ c,\ d) \\ -\mathscr{L}(-a,-b) \leqslant -\mathscr{L}(-c,-d) & \text{if } \max(a,\ b,\ c,\ d) \leqslant 0. \end{array}$$

The connection between the work by Páles and the hypergeometric function becomes clear by noting that

$$\begin{aligned} {}_2F_1(-a,1;2;r)^{1/a} &= \left(\sum_{n=0}^{\infty} \frac{(1-\sigma)_n(1)_n}{(2)_n n!} r^n\right)^{1/(\sigma-1)} \quad (\text{substituting } \sigma - 1 \text{ for } a) \\ &= \left(\frac{1-(1-r)^\sigma}{\sigma r}\right)^{1/(\sigma-1)} = \mathscr{D}_{1+a,1}(1,1-r) \end{aligned}$$

We will also use the following basic lemma:

LEMMA. *Suppose $f(\sigma) \equiv \frac{\sigma+1}{3} - \frac{\ln(2)(\sigma-1)}{\ln(\sigma)}$ when $\sigma \neq 1$ and $f(1) \equiv \frac{2}{3} - \ln(2)$. Then $f(\sigma) > 0$ $\forall \sigma \in (0,1/2) \cup (2,\infty)$ and $f(\sigma) < 0$ $\forall \sigma \in (1/2,2)$.*

Proof. Let $f(\sigma) = \frac{\sigma-1}{3\ln(\sigma)} g(\sigma)$ where $g(\sigma) = \left[\left(\frac{\sigma+1}{\sigma-1}\right)\ln(\sigma) - 3\ln 2\right]$. It can be shown that $g'(\sigma) = \frac{1}{\sigma(\sigma-1)^2} h(\sigma)$ where $h(\sigma) = \sigma^2 - 1 - 2\sigma\ln(\sigma)$, which is increasing on $(0,\infty)$ with $h(1)=0$. Thus, $g'(\sigma)<0$ on $(0,1)$ and $g'(\sigma)>0$ on $(1,\infty)$. Thus, g is decreasing on $(0,1)$ and increasing on $(1,\infty)$ with $g(1/2)=0=g(2)$ and $g(1)<0$. Therefore, $f(\sigma)=\frac{\sigma+1}{3}-\frac{\ln(2)(\sigma-1)}{\ln(\sigma)}>0$ on $(0,1/2)\cup(2,\infty)$ and $f(\sigma)<0$ on $(1/2,2)$, which proves the lemma. □

Proof of Proposition 1. First suppose $1>a>-1/2$. The left-hand side of (9) is a special case of Theorem B. Páles' Comparison Theorem implies

$$_2F_1(-a,1;2;r)^{1/a}=\mathscr{D}_{a+1,1}(1,1-r)\leqslant\mathscr{D}_{2\mu,\mu}(1,1-r)=\mathscr{A}_\mu(1,1-r)$$

if and only if (i) $a+2\leqslant 3\mu$ and (ii) $\mathscr{L}(a+1,1)\leqslant\mathscr{L}(2\mu,\mu)$. Let $\sigma=a+1$. Simplifying we find that (i) and (ii) hold if and only if $\mu\geqslant\max\left(\frac{\sigma+1}{3},\frac{(\sigma-1)\ln(2)}{\ln(\sigma)}\right)$. Since $1>a>-1/2$, we have $2>\sigma>1/2$ and hence $\frac{(\sigma-1)\ln(2)}{\ln(\sigma)}>\frac{\sigma+1}{3}$ by the above lemma. Therefore, the right-hand side inequality in (9) holds when $\mu\geqslant\frac{a\ln(2)}{\ln(a+1)}$.

Now suppose $-1<a<-1/2$. Then $0<\sigma<1/2$ and hence the left-hand side in (9) holds if and only if $\lambda\leqslant\frac{a\ln(2)}{\ln(a+1)}=\min\left(\frac{\sigma+1}{3},\frac{(\sigma-1)\ln(2)}{\ln(\sigma)}\right)$, by the preceding lemma. The right-hand side of (9) in this case is again a special case of Theorem B. □

The proof of Proposition 2 follows in a similar manner.

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