

BOUNDEDNESS FOR MULTILINEAR OPERATORS OF PSEUDO-DIFFERENTIAL OPERATORS FOR THE EXTREME CASES

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Abstract. We prove the boundedness of the multilinear operators associated to the pseudo-differential operator for the extreme cases.

1. Introduction and Preliminaries

It is well known that the singular integral operators and their commutators and multilinear operators have been well studied (see [3–5], [7], [12–13]). In [10], authors obtain the boundedness properties of the commutators for the extreme values of p . The purpose of this paper is to introduce some multilinear operators associated to pseudo-differential operators and prove the boundedness properties of the multilinear operators for the extreme cases.

First, let us introduce some preliminaries. Throughout this paper, $Q = Q(x, d)$ will denote a cube of R^n with sides parallel to the axes, whose center is x and side length is d . For a locally integrable function b , the sharp function of b is defined by

$$b^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy,$$

where, and in what follows, $b_Q = |Q|^{-1} \int_Q b(x) dx$. It is well-known that (see [9], [12])

$$b^\#(x) \approx \sup_{Q \ni x} \inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q |b(y) - c| dy$$

and

$$\|b - b_{2^k Q}\|_{BMO} \leq Ck \|b\|_{BMO} \text{ for } k \geq 1.$$

We say that b belongs to $BMO(R^n)$ if $b^\#$ belongs to $L^\infty(R^n)$ and $\|b\|_{BMO} = \|b^\#\|_{L^\infty}$. We also define the central BMO space by $CMO(R^n)$, which is the space of those functions $f \in L_{loc}(R^n)$ such that

$$\|f\|_{CMO} = \sup_{r>1} |Q(0, r)|^{-1} \int_Q |f(x) - f_Q| dx < \infty.$$

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$$\|f\|_{CMO} \approx \sup_{r>1} \inf_{c \in \mathbb{C}} |Q(0, r)|^{-1} \int_Q |f(x) - c| dx.$$

DEFINITION. Let $1 < p < \infty$. We shall call $B_p(R^n)$ the space of those functions f on R^n such that

$$\|f\|_{B_p} = \sup_{r>1} |Q(0, r)|^{-1/p} \|f \chi_{Q(0,r)}\|_{L^p} < \infty.$$

2. Theorems

In this paper, we will study the following multilinear pseudo-differential operators. We say a symbol $\sigma(x, \xi)$ is in the class $S_{\rho, \delta}^m$ and write $\sigma \in S_{\rho, \delta}^m$, if for $x, \xi \in R^n$,

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial \xi^\beta} \sigma(x, \xi) \right| \leq C_{\alpha, \beta} (1 + |\xi|)^{m - \rho|\beta| + \delta|\alpha|}.$$

A pseudo-differential operator with symbol $\sigma(x, \xi) \in S_{\rho, \delta}^m$ is defined by

$$T(f)(x) = \int_{R^n} e^{2\pi i x \cdot \xi} \sigma(x, \xi) \hat{f}(\xi) d\xi,$$

where f is a Schwartz function and \hat{f} denotes the Fourier transform of f . We know there exists a kernel $K(x, y)$ such that

$$T(f)(x) = \int_{R^n} K(x, x - y) f(y) dy,$$

where, formally,

$$K(x, y) = \int_{R^n} e^{2\pi i(x-y) \cdot \xi} \sigma(x, \xi) d\xi.$$

In [8], the boundedness of the pseudo-differential operators with symbol $\sigma \in S_{1-\theta, \delta}^{-\beta}$ ($\beta < n\theta/2, 0 \leq \delta < 1 - \theta$) are obtained. In [11], the boundedness of the pseudo-differential operators with symbol of order 0 and $-\infty$ are obtained. In [1], the sharp function estimate of the pseudo-differential operators with symbol $\sigma \in S_{1-\theta, \delta}^{-n\theta/2}$ ($0 < \theta < 1, 0 \leq \delta < 1 - \theta$) are obtained. In [14], the boundedness of the pseudo-differential operators and their commutators with symbol $\sigma \in S_{1-\theta, \delta}^{-n\theta/2}$ ($0 < \theta < 1, 0 \leq \delta < 1 - \theta$) are obtained. Our result is motivated by these papers.

Suppose T is a pseudo-differential operator with symbol $\sigma(x, \xi) \in S_{\rho, \delta}^m$. Let m_j be the positive integers ($j = 1, \dots, l$), $m_1 + \dots + m_l = m$ and b_j be the functions on R^n ($j = 1, \dots, l$). Set, for $1 \leq j \leq m$,

$$R_{m_j+1}(b_j; x, y) = b_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^\alpha b_j(y) (x - y)^\alpha.$$

The multilinear operator associated to T is defined by

$$T^b(f)(x) = \int_{\mathbb{R}^n} \frac{\prod_{j=1}^l R_{m_j+1}(b_j; x, y)}{|x-y|^m} K(x, x-y) f(y) dy.$$

Note that when $m = 0$, T^b is just the higher order commutators of T and b (see [13]). It is well-known that multilinear operator, as a non-trivial extension of commutator, is of great interest in harmonic analysis and has been widely studied by many authors (see [3–5]). In [2], the weak (H^1, L^1) -boundedness of the multilinear operator related to some singular integral operator are obtained. In this paper, we will study the boundedness properties of the multilinear operators T^b for the extreme cases.

We shall prove the following theorems in Section 3.

THEOREM 1. *Let $D^\alpha b_j \in BMO(\mathbb{R}^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \dots, l$. Suppose that T is the pseudo-differential operator with symbol $\sigma \in S_{1-\theta, \delta}^{-n\theta/2}$ ($0 < \theta < 1$, $0 \leq \delta < 1 - \theta$). Then T^b is bounded from $L^\infty(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$.*

THEOREM 2. *Let $1 < p < \infty$ and $D^\alpha b_j \in BMO(\mathbb{R}^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \dots, l$. Suppose that T is the pseudo-differential operator with symbol $\sigma \in S_{1-\theta, \delta}^{-n\theta/2}$ ($0 < \theta < 1$, $0 \leq \delta < 1 - \theta$). Then T^b is bounded from $B_p(\mathbb{R}^n)$ to $CMO(\mathbb{R}^n)$.*

3. Proofs of Theorems

To prove the theorem, we need the following lemma.

LEMMA 1. (see [3]). *Let b be a function on \mathbb{R}^n and $D^\alpha b \in L^q(\mathbb{R}^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(b; x, y)| \leq C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^\alpha b(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

LEMMA 2. ([1]) *Let T be the pseudo-differential operator with symbol $\sigma \in S_{1-\theta, \delta}^{-n\theta/2}$ ($0 < \theta < 1$, $0 \leq \delta < 1 - \theta$). Then, for every $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$,*

$$\|T(f)\|_{L^p} \leq C\|f\|_{L^p}.$$

LEMMA 3. ([1]) *Let $\sigma \in S_{1-\theta, \delta}^{-n\theta/2}$ ($0 < \theta < 1$, $0 \leq \delta < 1 - \theta$) and K be the kernel of the pseudo-differential operator T with symbol $\sigma \in S_{1-\theta, \delta}^{-n\theta/2}$. Then, for $|x_0 - x| \leq d < 1$ and $k \geq 1$,*

$$\left(\int_{(2^k d)^{1-\theta} \leq |y-x_0| < (2^{k+1} d)^{1-\theta}} |K(x, x-y) - K(x_0, x_0-y)|^2 dy \right)^{1/2} \leq C \frac{|x_0 - x|^{(1-\theta)(m-n/2)}}{(2^k d)^{m(1-\theta)}},$$

provided m is an integer such that $n/2 < m < n/2 + 1/(1-\theta)$.

LEMMA 4. ([1]) Let $\sigma \in S_{\rho, \delta}^0$ ($0 < \rho < 1$) and

$$K(x, w) = \int_{\mathbb{R}^n} e^{2\pi i w \cdot \xi} \sigma(x, \xi) d\xi.$$

Then, for $|w| \geq 1/4$ and any integer $N \geq 1$,

$$|K(x, w)| \leq C_N |w|^{-2N}.$$

Proof of Theorem 1. It is enough to prove that there exists a constant C_Q such that

$$\frac{1}{|Q|} \int_Q |T^b(f)(x) - C_Q| dx \leq C \|f\|_{L^\infty}$$

holds for any cube Q . Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$. We consider the following two cases:

Case 1. $d \leq 1$. In this case, let Q^* be the cube concentric with Q of side length $d^{1-\theta}$. Let $\tilde{Q} = 5\sqrt{n}Q^*$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha b_j)_{\tilde{Q}} x^\alpha$, then $R_{m_j+1}(b_j; x, y) = R_{m_j+1}(\tilde{b}_j; x, y)$ and $D^\alpha \tilde{b}_j = D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f_1 = f \chi_{\tilde{Q}}$ and $f_2 = f \chi_{\mathbb{R}^n \setminus \tilde{Q}}$,

$$\begin{aligned} T^b(f)(x) &= \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m} K(x, x-y) f(y) dy \\ &= \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m} K(x, x-y) f_2(y) dy \\ &\quad + \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m} K(x, x-y) f_1(y) dy \\ &\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{\mathbb{R}^n} \frac{R_{m_1}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \tilde{b}_1(y) K(x, x-y) f_1(y) dy \\ &\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{\mathbb{R}^n} \frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{b}_2(y) K(x, x-y) f_1(y) dy \\ &\quad + \sum_{\substack{|\alpha_1|=m_1, \\ |\alpha_2|=m_2}} \frac{1}{\alpha_1! \alpha_2!} \int_{\mathbb{R}^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y)}{|x-y|^m} K(x, x-y) f_1(y) dy, \end{aligned}$$

then

$$\begin{aligned} &\frac{1}{|Q|} \int_Q |T^b(f)(x) - T^{\tilde{b}}(f_2)(x_0)| dx \\ &\leq \frac{1}{|Q|} \int_Q \left| \int_{\mathbb{R}^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m} K(x, x-y) f_1(y) dy \right| dx \\ &\quad + \frac{C}{|Q|} \int_Q \left| \sum_{|\alpha_1|=m_1} \int_{\mathbb{R}^n} \frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \tilde{b}_1(y) K(x, x-y) f_1(y) dy \right| dx \end{aligned}$$

$$\begin{aligned}
& + \frac{C}{|Q|} \int_Q \left| \sum_{|\alpha_2|=m_2} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{b}_2(y) K(x, x-y) f_1(y) dy \right| dx \\
& + \frac{C}{|Q|} \int_Q \left| \sum_{\substack{|\alpha_1|=m_1, \\ |\alpha_2|=m_2}} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y)}{|x-y|^m} K(x, x-y) f_1(y) dy \right| dx \\
& + \frac{1}{|Q|} \int_Q \left| T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0) \right| dx \\
& := I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Now, let us estimate I_1 , I_2 , I_3 , I_4 and I_5 , respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, by Lemma 1, we get

$$R_{m_j}(\tilde{b}_j; x, y) \leq C|x-y|^{m_j} \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO}.$$

Let $\sigma(x, \xi) = \sigma(x, \xi)|\xi|^{n\theta/2}|\xi|^{-n\theta/2} = q(x, \xi)|\xi|^{-n\theta/2}$ and S be the pseudo-differential operator with symbol $q(x, \xi)$. We know $q(x, \xi) \in S_{1-\theta, \delta}^0$. By the Hardy-Littlewood-Sobolev fractional integration theorem and the $L^2(R^n)$ -boundedness of S (see [1], [6], [15]), we obtain, for $1/p = 1/2 - \theta/2$,

$$\begin{aligned}
I_1 & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \frac{1}{|Q|} \int_Q |T(f_1)(x)| dx \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \left(\frac{1}{|Q|} \int_Q |T(f_1)(x)|^p dx \right)^{1/p} \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) |Q|^{-1/p} \left(\int_{R^n} |S(f_1)(x)|^2 dx \right)^{1/2} \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) |Q|^{-1/p} \left(\int_{R^n} |f_1(x)|^2 dx \right)^{1/2} \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \frac{|\tilde{Q}|^{1/2}}{|Q|^{1/p}} \|f\|_{L^\infty} \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty}.
\end{aligned}$$

For I_2 , by Hölder' inequality, we get

$$\begin{aligned}
 I_2 &\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |T(D^{\alpha_1} \tilde{b}_1 f_1)(x)|^p dx \right)^{1/p} \\
 &\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} |\tilde{Q}|^{-1/p} \left(\int_{R^n} |S(D^{\alpha_1} \tilde{b}_1 f_1)(x)|^2 dx \right)^{1/2} \\
 &\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} |\tilde{Q}|^{-1/p} \left(\int_{R^n} |D^{\alpha_1} \tilde{b}_1(x) f_1(x)|^2 dx \right)^{1/2} \\
 &\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \frac{|\tilde{Q}|^{1/2}}{|\tilde{Q}|^{1/p}} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_1} b_1(x) - (D^{\alpha_1} b_1)_{\tilde{Q}}|^2 dx \right)^{1/2} \|f\|_{L^\infty} \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty}.
 \end{aligned}$$

For I_3 , similar to the proof of I_2 , we get

$$I_3 \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty}.$$

Similarly, for I_4 , choose $1 < p, q < \infty$ such that $1/p + 1/q = 1/2$, we obtain, by Hölder' inequality,

$$\begin{aligned}
 I_4 &\leq C \sum_{\substack{|\alpha_1|=m_1, \\ |\alpha_2|=m_2}} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |T(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1)(x)|^p dx \right)^{1/p} \\
 &\leq C \sum_{\substack{|\alpha_1|=m_1, \\ |\alpha_2|=m_2}} |\tilde{Q}|^{-1/p} \left(\int_{R^n} |S(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1)(x)|^2 dx \right)^{1/2} \\
 &\leq C \sum_{\substack{|\alpha_1|=m_1, \\ |\alpha_2|=m_2}} |\tilde{Q}|^{-1/p} \left(\int_{R^n} |D^{\alpha_1} \tilde{b}_1(x) D^{\alpha_2} \tilde{b}_2(x) f_1(x)|^2 dx \right)^{1/2} \\
 &\leq C \frac{|\tilde{Q}|^{1/2}}{|\tilde{Q}|^{1/p}} \sum_{\substack{|\alpha_1|=m_1, \\ |\alpha_2|=m_2}} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_1} \tilde{b}_1(x)|^p dx \right)^{1/p} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_2} \tilde{b}_2(x)|^q dx \right)^{1/q} \|f\|_{L^\infty} \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty}.
 \end{aligned}$$

For I_5 , we write, for $x \in Q$,

$$\begin{aligned}
 & T^{\tilde{b}}(f_2)(x) - T^{\tilde{b}}(f_2)(x_0) \\
 &= \int_{R^n} \left(\frac{K(x, x-y)}{|x-y|^m} - \frac{K(x_0, x_0-y)}{|x_0-y|^m} \right) \prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y) f_2(y) dy \\
 &+ \int_{R^n} (R_{m_1}(\tilde{b}_1; x, y) - R_{m_1}(\tilde{b}_1; x_0, y)) \frac{R_{m_2}(\tilde{b}_2; x, y)}{|x_0-y|^m} K(x_0, x_0-y) f_2(y) dy \\
 &+ \int_{R^n} (R_{m_2}(\tilde{b}_2; x, y) - R_{m_2}(\tilde{b}_2; x_0, y)) \frac{R_{m_1}(\tilde{b}_1; x_0, y)}{|x_0-y|^m} K(x_0, x_0-y) f_2(y) dy \\
 &- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[\frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} K(x, x-y) \right. \\
 &\quad \left. - \frac{R_{m_2}(\tilde{b}_2; x_0, y)(x_0-y)^{\alpha_1}}{|x_0-y|^m} K(x_0, x_0-y) \right] D^{\alpha_1} \tilde{b}_1(y) f_2(y) dy \\
 &- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[\frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} K(x, x-y) \right. \\
 &\quad \left. - \frac{R_{m_1}(\tilde{b}_1; x_0, y)(x_0-y)^{\alpha_2}}{|x_0-y|^m} K(x_0, x_0-y) \right] D^{\alpha_2} \tilde{b}_2(y) f_2(y) dy \\
 &+ \sum_{\substack{|\alpha_1|=m_1, \\ |\alpha_2|=m_2}} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} K(x, x-y) - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} K(x_0, x_0-y) \right] \\
 &\quad \times D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y) f_2(y) dy \\
 &= I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)}.
 \end{aligned}$$

By Lemma 2 and the following inequality (see [12])

$$|b_{Q_1} - b_{Q_2}| \leq C \log(|Q_2|/|Q_1|) \|b\|_{BMO} \text{ for } Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $y \in Q(x_0, (2^{k+1}d)^{1-\theta}) \setminus Q(x_0, (2^k d)^{1-\theta})$,

$$\begin{aligned}
 |R_{m_j}(\tilde{b}_j; x, y)| &\leq C|x-y|^{m_j} \sum_{|\alpha|=m_j} (\|D^\alpha b_j\|_{BMO} + |(D^\alpha b_j)_{\tilde{Q}(x,y)} - (D^\alpha b_j)_{\tilde{Q}}|) \\
 &\leq Ck|x-y|^{m_j} \sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO}.
 \end{aligned}$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain

$$\begin{aligned}
 |I_5^{(1)}| &\leq \sum_{k=0}^{\infty} k^2 \int_{(2^k d)^{1-\theta} \leq |y-x_0| < (2^{k+1} d)^{1-\theta}} |K(x, x-y) - K(x_0, x_0-y)| \\
 &\quad \times \frac{1}{|x-y|^m} \prod_{j=1}^2 |R_{m_j}(\tilde{b}_j; x, y)| |f(y)| dy
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{k=0}^{\infty} k^2 \int_{(2^k d)^{1-\theta} \leq |y-x_0| < (2^{k+1} d)^{1-\theta}} \left| \frac{1}{|x-y|^m} - \frac{1}{|x_0-y|^m} \right| \\
 & \times |K(x_0, x_0-y)| \prod_{j=1}^2 |R_{m_j}(\tilde{b}_j; x, y)| |f(y)| dy \\
 & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^{\infty} k^2 \left(\int_{|y-x_0| < (2^{k+1} d)^{1-\theta}} |f(y)|^2 dy \right)^{1/2} \\
 & \times \left(\int_{(2^k d)^{1-\theta} \leq |y-x_0| < (2^{k+1} d)^{1-\theta}} |K(x, x-y) - K(x_0, x_0-y)|^2 dy \right)^{1/2} \\
 & + C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^{\infty} k^2 \left(\int_{|y-x_0| < (2^{k+1} d)^{1-\theta}} |f(y)|^2 dy \right)^{1/2} \\
 & \times \left(\int_{(2^k d)^{1-\theta} \leq |y-x_0| < (2^{k+1} d)^{1-\theta}} \frac{|x_0-x|^2}{|x_0-y|^2} |K(x_0, x_0-y)|^2 dy \right)^{1/2},
 \end{aligned}$$

for the second term above, similar to the proof of Lemma 2.1 in [1], we have

$$\left(\int_{(2^k d)^{1-\theta} \leq |y-x_0| < (2^{k+1} d)^{1-\theta}} \frac{|x_0-x|^2}{|x_0-y|^2} |K(x_0, x_0-y)|^2 dy \right)^{1/2} \leq C \frac{|x_0-x|^{(1-\theta)(m-n/2)}}{(2^k d)^{m(1-\theta)}},$$

thus, by Lemma 3 and recall that $n/2 < m$,

$$\begin{aligned}
 |I_5^{(1)}| & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 \frac{d^{(1-\theta)(m-n/2)}}{(2^k d)^{m(1-\theta)}} (2^k d)^{n(1-\theta)/2} \|f\|_{L^\infty} \\
 & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 2^{k(1-\theta)(n/2-m)} \|f\|_{L^\infty} \\
 & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \|f\|_{L^\infty}.
 \end{aligned}$$

For $I_5^{(2)}$, by the formula (see [3]):

$$R_{m_j}(\tilde{b}_j; x, y) - R_{m_j}(\tilde{b}_j; x_0, y) = \sum_{|\beta| < m} \frac{1}{\beta!} R_{m_j-|\beta|}(D^\beta \tilde{b}_j; x, x_0) (x-y)^\beta$$

and Lemma 1, we have

$$|R_{m_j}(\tilde{b}_j; x, y) - R_{m_j}(\tilde{b}_j; x_0, y)| \leq C \sum_{|\beta| < m_j} \sum_{|\alpha|=m_j} |x-x_0|^{m_j-|\beta|} |x-y|^{|\beta|} \|D^\alpha b_j\|_{BMO},$$

thus

$$\begin{aligned}
 |I_5^{(2)}| &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \\
 &\quad \times \sum_{k=0}^{\infty} \int_{(2^k d)^{1-\theta} \leq |y-x_0| < (2^{k+1} d)^{1-\theta}} k \frac{|x-x_0|}{|x_0-y|} |K(x_0, x_0-y)| |f(y)| dy \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \sum_{k=1}^{\infty} k 2^{k(1-\theta)(n/2-m)} \|f\|_{L^\infty} \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty}.
 \end{aligned}$$

Similarly,

$$|I_5^{(3)}| \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty}.$$

For $I_5^{(4)}$, recall that $|b_{2^k Q} - b_{2Q}| \leq Ck \|b\|_{BMO}$, similar to the proof of $I_5^{(1)}$ and $I_5^{(2)}$, we get

$$\begin{aligned}
 |I_5^{(4)}| &\leq C \sum_{|\alpha_1|=m_1} \int_{\mathbb{R}^n} \left| \frac{(x-y)^{\alpha_1}}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1}}{|x_0-y|^m} \right| |R_{m_2}(\tilde{b}_2; x, y)| |K(x, x-y)| |D^{\alpha_1} \tilde{b}_1(y)| |f_2(y)| dy \\
 &\quad + C \sum_{|\alpha_1|=m_1} \int_{\mathbb{R}^n} |R_{m_2}(\tilde{b}_2; x, y) - R_{m_2}(\tilde{b}_2; x_0, y)| \frac{|(x_0-y)^{\alpha_1}|}{|x_0-y|^m} |K(x, x-y)| \\
 &\quad \times |D^{\alpha_1} \tilde{b}_1(y)| |f_2(y)| dy \\
 &\quad + C \sum_{|\alpha_1|=m_1} \int_{\mathbb{R}^n} |K(x, x-y) - K(x_0, x_0-y)| \frac{|(x_0-y)^{\alpha_1}|}{|x_0-y|^m} |R_{m_2}(\tilde{b}_2; x_0, y)| \\
 &\quad \times |D^{\alpha_1} \tilde{b}_1(y)| |f_2(y)| dy \\
 &\leq C \sum_{|\alpha|=m_2} \|D^\alpha b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \sum_{k=1}^{\infty} k \frac{d^{(1-\theta)(m-n/2)}}{(2^k d)^{m(1-\theta)}} (2^k d)^{n(1-\theta)/2} \|f\|_{L^\infty} \\
 &\quad \times \left(\frac{1}{|Q(x_0, (2^k d)^{1-\theta})|} \int_{Q(x_0, (2^k d)^{1-\theta})} |D^{\alpha_1} b_1(y) - (D^{\alpha_1} b_1)_Q|^2 dy \right)^{1/2} \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \sum_{k=1}^{\infty} k^2 2^{k(1-\theta)(n/2-m)} \|f\|_{L^\infty} \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty}
 \end{aligned}$$

Similarly,

$$|I_5^{(5)}| \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty}.$$

For $I_5^{(6)}$, taking $1 < r_1, r_2 < \infty$ such that $1/r_1 + 1/r_2 = 1/2$, similar to the proof of $I_5^{(1)}$, we get

$$\begin{aligned} |I_5^{(6)}| &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{R^n} \left| \frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} \right| \\ &\quad \times |K(x, x-y)| |D^{\alpha_1} \tilde{b}_1(y)| |D^{\alpha_2} \tilde{b}_2(y)| |f_2(y)| dy \\ &\quad + C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{R^n} |K(x, x-y) - K(x_0, x_0-y)| \frac{|(x_0-y)^{\alpha_1+\alpha_2}|}{|x_0-y|^m} \\ &\quad \times |D^{\alpha_1} \tilde{b}_1(y)| |D^{\alpha_2} \tilde{b}_2(y)| |f_2(y)| dy \\ &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \sum_{k=1}^\infty 2^{k(1-\theta)(n/2-m)} \|f\|_{L^\infty} \\ &\quad \times \prod_{j=1}^2 \left(\frac{1}{|Q(x_0, (2^k d)^{1-\theta})|} \int_{Q(x_0, (2^k d)^{1-\theta})} |D^{\alpha_j} b_j(y) - (D^{\alpha_j} b_j)_{\tilde{Q}}|^{r_j} dy \right)^{1/r_j} \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty}. \end{aligned}$$

Thus

$$I_5 \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{L^\infty}.$$

Case 2. $d > 1$. In this case, let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!} (D^\alpha b_j)_{\tilde{Q}} x^\alpha$,

then $R_{m_j+1}(b_j; x, y) = R_{m_j+1}(\tilde{b}_j; x, y)$ and $D^\alpha \tilde{b}_j = D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. Write, for $f = f\chi_{\tilde{Q}} + f\chi_{R^n \setminus \tilde{Q}} = f_1 + f_2$,

$$\begin{aligned} &\frac{1}{|Q|} \int_Q |T^b(f)(x)| dx \\ &\leq \frac{1}{|Q|} \int_Q \left| \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m} K(x, x-y) f_1(y) dy \right| dx \\ &\quad + \frac{C}{|Q|} \int_Q \left| \sum_{|\alpha_1|=m_1} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \tilde{b}_1(y) K(x, x-y) f_1(y) dy \right| dx \\ &\quad + \frac{C}{|Q|} \int_Q \left| \sum_{|\alpha_2|=m_2} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{b}_2(y) K(x, x-y) f_1(y) dy \right| dx \end{aligned}$$

$$\begin{aligned}
& + \frac{C}{|\tilde{Q}|} \int_{\tilde{Q}} \left| \sum_{\substack{|\alpha_1|=m_1, \\ |\alpha_2|=m_2}} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y)}{|x-y|^m} K(x, x-y) f_1(y) dy \right| dx \\
& + \frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |T^{\tilde{b}}(f_2)(x)| dx \\
& := J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

Similar to the proof of I_1 , I_2 , I_3 and I_4 , by using L^r ($1 < r < \infty$)-boundedness of T (see Lemma 2), we get

$$\begin{aligned}
J_1 & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \left(\frac{1}{|\tilde{Q}|} \int_{R^n} |T(f_1)(x)|^r dx \right)^{1/r} \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) |\tilde{Q}|^{-1/r} \left(\int_{R^n} |f_1(x)|^r dx \right)^{1/r} \\
& \leq C \sum_{|\alpha|=m_2} \|D^{\alpha} b_2\|_{BMO} \|f\|_{L^\infty}; \\
J_2 & \leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|\tilde{Q}|} \int_{R^n} |T(D^{\alpha_1} \tilde{b}_1 f_1)(x)|^r dx \right)^{1/r} \\
& \leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} |\tilde{Q}|^{-1/r} \left(\int_{R^n} |D^{\alpha_1} \tilde{b}_1(x) f_1(x)|^r dx \right)^{1/r} \\
& \leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_1} b_1(x) - (D^{\alpha_1} b_1)_{\tilde{Q}}|^r dx \right)^{1/r} \|f\|_{L^\infty} \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{BMO} \right) \|f\|_{L^\infty}; \\
J_3 & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{BMO} \right) \|f\|_{L^\infty}; \\
J_4 & \leq C \sum_{\substack{|\alpha_1|=m_1, \\ |\alpha_2|=m_2}} \left(\frac{1}{|\tilde{Q}|} \int_{R^n} |T(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{b}_2 f_1)(x)|^r dx \right)^{1/r} \\
& \leq C \sum_{\substack{|\alpha_1|=m_1, \\ |\alpha_2|=m_2}} |\tilde{Q}|^{-1/r} \left(\int_{R^n} |D^{\alpha_1} \tilde{b}_1(x) D^{\alpha_2} \tilde{b}_2(x) f_1(x)|^r dx \right)^{1/r} \\
& \leq C \sum_{\substack{|\alpha_1|=m_1, \\ |\alpha_2|=m_2}} \prod_{j=1}^2 \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_j} \tilde{b}_j(x)|^{p_j} dx \right)^{1/p_j} \|f\|_{L^\infty}
\end{aligned}$$

$$\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \|f\|_{L^\infty}.$$

For J_5 , we write, for $x \in Q$,

$$\begin{aligned} T^{\tilde{b}}(f_2)(x) &= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m} K(x, x-y) f_2(y) dy \\ &\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} K(x, x-y) D^{\alpha_1} \tilde{b}_1(y) f_2(y) dy \\ &\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} K(x, x-y) D^{\alpha_2} \tilde{b}_2(y) f_2(y) dy \\ &\quad + \sum_{\substack{|\alpha_1|=m_1, \\ |\alpha_2|=m_2}} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} K(x, x-y) D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y) f_2(y) dy, \end{aligned}$$

similar to the proof of I_5 and by using lemma 4, we get

$$\begin{aligned} |T^{\tilde{b}}(f_2)(x)| &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \sum_{k=0}^{\infty} k^2 \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |x-y|^{-2n} |f(y)| dy \\ &\quad + C \sum_{|\alpha|=m_2} \|D^\alpha b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \sum_{k=0}^{\infty} k \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |x-y|^{-2n} |D^{\alpha_1} \tilde{b}_1(y)| |f(y)| dy \\ &\quad + C \sum_{|\alpha|=m_1} \|D^\alpha b_1\|_{BMO} \sum_{|\alpha_2|=m_2} \sum_{k=0}^{\infty} k \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |x-y|^{-2n} |D^{\alpha_2} \tilde{b}_2(y)| |f(y)| dy \\ &\quad + C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} |x-y|^{-2n} |D^{\alpha_1} \tilde{b}_1(y)| |D^{\alpha_2} \tilde{b}_2(y)| |f(y)| dy \\ &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) d^{-n} \|f\|_{L^\infty} \sum_{k=1}^{\infty} k^2 2^{-kn} \\ &\quad + C \sum_{|\alpha|=m_2} \|D^\alpha b_2\|_{BMO} d^{-n} \|f\|_{L^\infty} \sum_{k=1}^{\infty} k 2^{-kn} \\ &\quad \times \sum_{|\alpha_1|=m_1} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |(D^{\alpha_1} b_1(y) - (D^{\alpha_1} b_1)_{\tilde{Q}})| dy \right) \\ &\quad + C \sum_{|\alpha|=m_1} \|D^\alpha b_1\|_{BMO} d^{-n} \|f\|_{L^\infty} \sum_{k=1}^{\infty} k 2^{-kn} \\ &\quad \times \sum_{|\alpha_2|=m_2} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |(D^{\alpha_2} b_2(y) - (D^{\alpha_2} b_2)_{\tilde{Q}})| dy \right) \\ &\quad + C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} d^{-n} \|f\|_{L^\infty} \sum_{k=1}^{\infty} 2^{-kn} \end{aligned}$$

$$\begin{aligned} & \times \prod_{j=1}^2 \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_j} b_j(y) - (D^{\alpha_j} b_j)_{\tilde{Q}}|^{p_j} dy \right)^{1/p_j} \\ & \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{BMO} \right) \|f\|_{L^\infty}, \end{aligned}$$

thus

$$|J_5| \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{BMO} \right) \|f\|_{L^\infty}.$$

This completes the proof of Theorem 1. \square

Proof of Theorem 2. It is enough to prove that

$$\frac{1}{|Q|} \int_Q |T^b(f)(x)| dx \leq C \|f\|_{B_p}$$

holds for any cube $Q = Q(0, d)$ with $d > 1$. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(0, d)$ with $d > 1$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!} (D^{\alpha} b_j)_{\tilde{Q}} x^{\alpha}$, then $R_{m_j+1}(b_j; x, y) = R_{m_j+1}(\tilde{b}_j; x, y)$ and $D^{\alpha} \tilde{b}_j = D^{\alpha} b_j - (D^{\alpha} b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. Write, for $f_1 = f \chi_{\tilde{Q}}$ and $f_2 = f \chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned} & \frac{1}{|Q|} \int_Q |T^b(f)(x)| dx \\ & \leq \frac{1}{|Q|} \int_Q \left| \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m} K(x, x-y) f_1(y) dy \right| dx \\ & \quad + \frac{C}{|Q|} \int_Q \left| \sum_{|\alpha_1|=m_1} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \tilde{b}_1(y) K(x, x-y) f_1(y) dy \right| dx \\ & \quad + \frac{C}{|Q|} \int_Q \left| \sum_{|\alpha_2|=m_2} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{b}_2(y) K(x, x-y) f_1(y) dy \right| dx \\ & \quad + \frac{C}{|Q|} \int_Q \left| \sum_{\substack{|\alpha_1|=m_1, \\ |\alpha_2|=m_2}} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y)}{|x-y|^m} K(x, x-y) f_1(y) dy \right| dx \\ & \quad + \frac{1}{|Q|} \int_Q |T^{\tilde{b}}(f_2)(x)| dx \\ & := L_1 + L_2 + L_3 + L_4 + L_5. \end{aligned}$$

Similar to **Case 2** in proof of Theorem 1, we get, for $1 < r < p$,

$$\begin{aligned}
 L_1 + L_2 + L_3 + L_4 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \left(\frac{1}{|Q|} \int_Q |T(f_1)(x)|^p dx \right)^{1/p} \\
 &\quad + C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \tilde{b}_1 f_1)(x)|^r dx \right)^{1/r} \\
 &\quad + C \sum_{\substack{|\alpha_1|=m_1, \\ |\alpha_2|=m_2}} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \tilde{b}_1 D^{\alpha_2} \tilde{A}_2 f_1)(x)|^r dx \right)^{1/r} \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) |\tilde{Q}|^{-1/p} \|f \chi_{\tilde{Q}}\|_{L^p} \\
 &\quad + C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} b_2\|_{BMO} |Q|^{-1/r} \sum_{|\alpha_1|=m_1} \left(\int_{R^n} |D^{\alpha_1} \tilde{b}_1(x) f_1(x)|^r dx \right)^{1/r} \\
 &\quad + C \sum_{\substack{|\alpha_1|=m_1, \\ |\alpha_2|=m_2}} |Q|^{-1/r} \left(\int_{R^n} |D^{\alpha_1} \tilde{b}_1(x) D^{\alpha_2} \tilde{b}_2(x) f_1(x)|^r dx \right)^{1/r} \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} b_j\|_{BMO} \right) \|f\|_{B_p}.
 \end{aligned}$$

For L_5 , similar to the proof of Theorem 1, we get, for $x \in Q$ and $1 < q_1, q_2 < \infty$ with $1/q_1 + 1/q_2 + 1/p = 1$,

$$\begin{aligned}
 |T^{\tilde{b}}(f_2)(x)| &\leq \left| \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m} K(x, x-y) f_2(y) dy \right| \\
 &\quad + C \left| \sum_{|\alpha_1|=m_1} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} K(x, x-y) D^{\alpha_1} \tilde{b}_1(y) f_2(y) dy \right| \\
 &\quad + C \left| \sum_{|\alpha_2|=m_2} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} K(x, x-y) D^{\alpha_2} \tilde{b}_2(y) f_2(y) dy \right| \\
 &\quad + C \left| \sum_{\substack{|\alpha_1|=m_1, \\ |\alpha_2|=m_2}} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} K(x, x-y) D^{\alpha_1} \tilde{b}_1(y) D^{\alpha_2} \tilde{b}_2(y) f_2(y) dy \right| \\
 &\leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^{\alpha} b_j\|_{BMO} \right) d^{-n} \sum_{k=1}^{\infty} k^2 2^{-kn} |2^k \tilde{Q}|^{-1/p} \|f \chi_{2^k \tilde{Q}}\|_{L^p} \\
 &\quad + C \sum_{|\alpha|=m_2} \|D^{\alpha} b_2\|_{BMO} d^{-n} \sum_{k=1}^{\infty} k^2 2^{-kn} |2^k \tilde{Q}|^{-1/p} \|f \chi_{2^k \tilde{Q}}\|_{L^p}
 \end{aligned}$$

$$\begin{aligned}
& \times \sum_{|\alpha_1|=m_1} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_1} \tilde{b}_1(y)|^{p'} dy \right)^{1/p'} \\
& + C \sum_{|\alpha|=m_1} \|D^\alpha b_1\|_{BMO} d^{-n} \sum_{k=1}^{\infty} k 2^{-kn} |2^k \tilde{Q}|^{-1/p} \|f \chi_{2^k \tilde{Q}}\|_{L^p} \\
& \times \sum_{|\alpha_2|=m_2} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_2} \tilde{b}_2(y)|^{p'} dy \right)^{1/p'} \\
& + C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} d^{-n} \sum_{k=1}^{\infty} 2^{-kn} |2^k \tilde{Q}|^{-1/p} \|f \chi_{2^k \tilde{Q}}\|_{L^p} \\
& \times \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_1} \tilde{b}_1(y)|^{q_1} dy \right)^{1/q_1} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_2} \tilde{b}_2(y)|^{q_2} dy \right)^{1/q_2} \\
& \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \|f\|_{B_p},
\end{aligned}$$

thus

$$L_5 \leq C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha b_j\|_{BMO} \right) \|f\|_{B_p}.$$

This finishes the proof of Theorem 2. \square

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REFERENCES

- [1] S. CHANILLO AND A. TORCHINSKY, *Sharp function and weighted L^p estimates for a class of pseudo-differential operators*, Ark. Math., **24** (1986), 1–25.
- [2] W. G. CHEN AND G. E. HU, *Weak type (H^1, L^1) estimate for multilinear singular integral operator*, Adv. in Math.(China), **30** (2001), 63–69.
- [3] J. COHEN, *A sharp estimate for a multilinear singular integral on R^n* , Indiana Univ. Math. J., **30** (1981), 693–702.
- [4] J. COHEN AND J. GOSSELIN, *On multilinear singular integral operators on R^n* , Studia Math., **72** (1982), 199–223.
- [5] J. COHEN AND J. GOSSELIN, *A BMO estimate for multilinear singular integral operators*, Illinois J. Math., **30** (1986), 445–465.
- [6] R. COIFMAN AND Y. MEYER, *Au delà des opérateurs pseudo-différentiels*, Astérisque, **57** (1978).
- [7] R. COIFMAN AND Y. MEYER, *Wavelets, Calderón-Zygmund and multilinear operators*, Cambridge Studies in Advanced Math., **48**, Cambridge University Press, Cambridge, 1997.
- [8] C. FEFFERMAN, *L^p bounds for pseudo-differential operators*, Israel J. Math., **14** (1973), 413–417.
- [9] J. GARCIA-CUERVA AND J. L. RUBIO DE FRANCIA, *Weighted norm inequalities and related topics*, North-Holland Math., **16**, Amsterdam, 1985.
- [10] E. HARBOURE, C. SEGOVIA AND J. L. TORREA, *Boundedness of commutators of fractional and singular integrals for the extreme values of p* , Illinois J. Math., **41** (1997), 676–700.

- [11] N. MILLER, *Weighted Sobolev spaces and pseudo-differential operators with smooth symbols*, Trans. Amer. Math. Soc., **269** (1982), 91–109.
- [12] E. M. STEIN, *Harmonic Analysis: real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton NJ, 1993.
- [13] C. PÉREZ AND R. TRUJILLO-GONZALEZ, *Sharp weighted estimates for multilinear commutators*, J. London Math. Soc., **65** (2002), 672–692.
- [14] M. SAIDANI, A. LAHMAR-BENBERNOU AND S. GALA, *Pseudo-differential operators and commutators in multiplier spaces*, African Diaspora J. of Math., **6** (2008), 31–53.
- [15] M. E. TAYLOR, *Pseudo-differential operators and nonlinear PDE*, Birkhauser, Boston, 1991.

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