Journal of Mathematical Inequalities
Volume 4, Number 2 (2010), 285–299

TRIGONOMETRIC APPROXIMATION IN GENERALIZED LEBESGUE SPACES $L^{p(x)}$

ALI GUVEN AND DANIYAL M. ISRAFILOV

(*Communicated by S. Samko*)

Abstract. The approximation properties of Nörlund (N_n) and Riesz (R_n) means of trigonometric Fourier series are investigated in generalized Lebesgue spaces $L^{p(x)}$. The deviations $\|f-N_n(f)\|_{p(x)}$ and $\|f-R_n(f)\|_{p(x)}$ are estimated by $n^{-\alpha}$ for $f \in Lip(\alpha, p(x))$ $(0<\alpha \leqslant 1)$.

1. Introduction and main results

Let $p:\mathbb{R}\to[1,\infty)$ be a measurable $2\pi-$periodic function, that is $p(x+2\pi)=p(x)$. Denote by $L^{p(x)}=L^{p(x)}([0,2\pi])$ the set of all measurable $2\pi-$periodic functions f such that $m_p(\lambda f)<\infty$ for some $\lambda=\lambda(f)>0$, where

$$m_p(f):=\int_0^{2\pi}|f(x)|^{p(x)}\,dx.$$

$L^{p(x)}$ becomes a Banach space with respect to the norm

$$\|f\|_{p(x)}=\inf\left\{\lambda>0:m_p\left(\frac{f}{\lambda}\right)\leqslant 1\right\}.$$

If $p(x)\equiv p$ is constant $(1\leqslant p<\infty)$, then the space $L^{p(x)}$ is isometrically isomorphic to the Lebesgue space L^p.

If the function p satisfies

$$1<p_-:=\operatorname*{ess\,inf}_{x\in[0,2\pi]}p(x),\quad p_+:=\operatorname*{ess\,sup}_{x\in[0,2\pi]}p(x)<\infty \tag{1}$$

then the function

$$p'(x):=\frac{p(x)}{p(x)-1}$$

is well defined and satisfies (1) itself.

Mathematics subject classification (2010): 41A25, 42A10, 46E30.
Keywords and phrases: Generalized Lebesgue space, Lipschitz class, modulus of continuity, Nörlund mean, Riesz mean.

The space $L^{p(x)}$ consists of all measurable $2\pi-$periodic functions f such that

$$\int_0^{2\pi} |f(x)g(x)|\,dx < \infty$$

for all measurable g with $m_{p'}(g) \leqslant 1$ and

$$\|f\|_{p(x)}^{*} = \sup\left\{\int_0^{2\pi} |f(x)g(x)|\,dx : m_{p'}(g) \leqslant 1\right\}$$

is also a norm on $L^{p(x)}$. It is known that the inequalities

$$\|f\|_{p(x)} \leqslant \|f\|_{p(x)}^{*} \leqslant r_p \|f\|_{p(x)}$$

satisfied for all functions $f \in L^{p(x)}$, where

$$r_p := 1 + \frac{1}{p_-} - \frac{1}{p_+},$$

and hence the norms $\|f\|_{p(x)}$ and $\|f\|_{p(x)}^{*}$ are equivalent. We refer to [13], [9], [10] and [7] for properties above and for more general information about $L^{p(x)}$ spaces.

Denote by M the Hardy-Littlewood maximal operator, defined for $f \in L^1$ by

$$M(f)(x) = \sup_I \frac{1}{|I|}\int_I |f(t)|\,dt, \quad x \in [0, 2\pi],$$

where the supremum is taken over all intervals I with $x \in I$.

The boundedness problem of the operator M on the space $L^{p(x)}$ was studied by many authors ([5], [8], [17], [18], etc.).

In [8] it was proved that if the function p satisfies (1) and the condition

$$|p(x) - p(y)| \leqslant \frac{C}{-\ln|x-y|}, \quad 0 < |x-y| \leqslant \frac{1}{2}, \tag{2}$$

then the maximal operator M is bounded on $L^{p(x)}$, that is,

$$\|M(f)\|_{p(x)} \leqslant c\,\|f\|_{p(x)} \tag{3}$$

for all $f \in L^{p(x)}$, where c is a constant depends only on p.

The set of all measurable $2\pi-$periodic functions $p : \mathbb{R} \to [1,\infty)$ satisfies the conditions (1) and (2) will be denoted by $\mathscr{M}$.

Let $p \in \mathscr{M}$ and $f \in L^{p(x)}$. The modulus of continuity of the function f is defined by

$$\Omega_{p(x)}(f,\delta) = \sup_{|h| \leqslant \delta} \|T_h(f)\|_{p(x)}, \quad \delta > 0, \tag{4}$$

where

$$T_h(f)(x) := \frac{1}{h}\int_0^h |f(x+t) - f(x)|\, dt. \tag{5}$$

The existence of $\Omega_{p(x)}(f,\delta)$ follows from (3), and also the inequality

$$\Omega_{p(x)}(f,\delta) \leqslant c\,\|f\|_{p(x)}$$

satisfied for all $\delta > 0$.

The modulus $\Omega_{p(x)}(f,\cdot)$ is nonnegative, continuous function such that

$$\lim_{\delta\to 0}\Omega_{p(x)}(f,\delta) = 0, \quad \Omega_{p(x)}(f_1+f_2,\cdot) \leqslant \Omega_{p(x)}(f_1,\cdot) + \Omega_{p(x)}(f_2,\cdot).$$

In the Lebesgue spaces L^p $(1 < p < \infty)$, the classical modulus of continuity $\omega_p(f,\cdot)$ is defined by

$$\omega_p(f,\delta) = \sup_{|h|\leqslant\delta}\left\|T_h'(f)\right\|_p, \quad \delta > 0, \tag{6}$$

where

$$T_h'(f)(x) := f(x+h) - f(x).$$

It is known that in the Lebesgue spaces L^p the moduli of continuity (4) and (6) are equivalent (see [14]).

We define in the spaces $L^{p(x)}$ the modulus of continuity by using the shift (5), because the space $L^{p(x)}$ is not translation invariant, in general (see, for example [13, Example 2.9]).

Let $p \in \mathscr{M}$ and $0 < \alpha \leqslant 1$. We define the Lipschitz class $Lip(\alpha, p(x))$ as

$$Lip(\alpha, p(x)) = \left\{f \in L^{p(x)} : \Omega_{p(x)}(f,\delta) = O(\delta^\alpha), \delta > 0\right\}.$$

Let $f \in L^1$ has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kx + b_k\sin kx). \tag{7}$$

Denote by $S_n(f)(x)$, $n = 0,1,\ldots$ the nth partial sums of the series (7) at the point x, that is,

$$S_n(f)(x) = \sum_{k=0}^{n} A_k(f)(x),$$

where

$$A_0(f)(x) = \frac{a_0}{2}, \quad A_k(f)(x) = a_k\cos kx + b_k\sin kx, \quad k = 1,2,\ldots.$$

Let $\{p_n\}_0^\infty$ be a sequence of positive real numbers. We consider two means of the series (7) defined by

$$N_n(f)(x) = \frac{1}{P_n}\sum_{m=0}^{n} p_{n-m}S_m(f)(x)$$

and

$$R_n(f)(x) = \frac{1}{P_n} \sum_{m=0}^{n} p_m S_m(f)(x),$$

where $P_n := \sum_{m=0}^{n} p_m, \quad p_{-1} = P_{-1} := 0$. The means $N_n(f)$ and $R_n(f)$ are called the Nörlund and Riesz means of the series (7), respectively. In the case $p_n = 1, n \geqslant 0$, both of $N_n(f)$ and $R_n(f)$ are equal to the Cesàro mean

$$\sigma_n(f)(x) = \frac{1}{n+1} \sum_{m=0}^{n} S_m(f)(x).$$

If we take $p_n = A_n^{\beta-1} \ (\beta > 0)$, where

$$A_0^{\beta} := 1, \quad A_k^{\beta} := \frac{\beta(\beta+1)\ldots(\beta+k)}{k!}, \quad k \geqslant 1,$$

the mean $N_n(f)$ be the generalized Cesàro mean $\sigma_n^{\beta}(f)(x)$, that is

$$N_n(f)(x) = \frac{1}{A_n^{\beta}} \sum_{m=0}^{n} A_{n-m}^{\beta-1} S_m(f)(x).$$

The approximation properties of the Cesàro means σ_n in Lipschitz classes $Lip(\alpha, p)$, $1 \leqslant p < \infty$, $0 < \alpha \leqslant 1$ were investigated by Quade in [19]. The generalizations of Quade's results were studied by Mohapatra and Russell [16], Chandra ([1], [2], [3], [4]) and Leindler [15]. In [1], Chandra obtained estimates for $\|f - N_n(f)\|_p$, where $1 < p < \infty$. Chandra also gave estimates for the difference $\|f - R_n(f)\|_p$, where $f \in Lip(\alpha, p)$, $1 < p < \infty$, $0 < \alpha \leqslant 1$ [2]. In the paper [4], Chandra gave some conditions on the sequence $\{p_n\}_0^{\infty}$ and obtained very satisfactory results about approximation by the means $N_n(f)$ and $R_n(f)$ in $Lip(\alpha, p)$, $1 \leqslant p < \infty$, $0 < \alpha \leqslant 1$. Later, Leindler in [15] weakened the conditions given by Chandra on the sequence $\{p_n\}_0^{\infty}$ and generalized his results. In [11], the analogues of Chandra's results was obtained for weighted Lebesgue spaces.

In the present paper we give $L^{p(x)}$ analogues of the results obtained by Leindler in [15] and Chandra in [4].

We shall use the notations

$$\Delta g_n := g_n - g_{n+1}, \quad \Delta_m g(n,m) := g(n,m) - g(n,m+1).$$

A sequence of positive real numbers $\{p_n\}_0^{\infty}$ is called almost monotone decreasing (increasing) if there exists a constant c, depending only on the sequence $\{p_n\}_0^{\infty}$ such that for all $n \geqslant m$ the inequality

$$p_n \leqslant c\, p_m \quad (c\, p_n \geqslant p_m)$$

holds. Such sequences will be denoted by $\{p_n\}_0^{\infty} \in AMDS$ $(\{p_n\}_0^{\infty} \in AMIS)$.

Our main results are the following.

THEOREM 1. *Let $p \in \mathscr{M}$, $0 < \alpha < 1$, $f \in Lip(\alpha, p(x))$ and $\{p_n\}_0^\infty$ be a sequence of positive numbers. If*

$$\{p_n\}_0^\infty \in AMDS,$$

or

$$\{p_n\}_0^\infty \in AMIS \quad and \quad (n+1)\, p_n = O(P_n),$$

then

$$\|f - N_n(f)\|_{p(x)} = O\left(n^{-\alpha}\right).$$

THEOREM 2. *Let $p \in \mathscr{M}$, $f \in Lip\,(1, p(x))$ and $\{p_n\}_0^\infty$ be a sequence of positive numbers. If*

$$\sum_{k=1}^{n-1} k\,|\Delta p_k| = O(P_n),$$

or

$$\sum_{k=0}^{n-1} |\Delta p_k| = O\left(\frac{P_n}{n}\right),$$

then the estimate

$$\|f - N_n(f)\|_{p(x)} = O\left(n^{-1}\right)$$

holds for $n = 1, 2, \ldots$.

THEOREM 3. *Let $p \in \mathscr{M}$, $0 < \alpha \leqslant 1$, $f \in Lip(\alpha, p(x))$ and $\{p_n\}_0^\infty$ be a sequence of positive numbers. If*

$$\sum_{m=0}^{n-1} \left|\Delta\left(\frac{P_m}{m+1}\right)\right| = O\left(\frac{P_n}{n+1}\right), \tag{8}$$

then for $n = 1, 2, \ldots$ the estimate

$$\|f - R_n(f)\|_{p(x)} = O\left(n^{-\alpha}\right)$$

holds.

In the classical Lebesgue spaces L^p, the analogues of Theorem 1 and Theorem 2 were proved by Leindler in [15], and Theorem 3 in L^p spaces was obtained by Chandra [4].

2. Some auxiliary results

In this section c will denote the constants (in general, different in different relations) depend only on quantities that are not important for the questions of interest.

Let $p \in \mathscr{M}$. Denote by $E_n(f)_{p(x)}$ $(n = 0, 1, \ldots)$ the best approximation of $f \in L^{p(x)}$ in Π_n (the set of trigonometric polynomials of degree at most n), that is

$$E_n(f)_{p(x)} = \inf\left\{\|f - t_n\|_{p(x)} : t_n \in \Pi_n\right\}.$$

It follows that, for example from Theorem 1.1 in [6], there exists a trigonometric polynomial $t_n^* \in \Pi_n$ such that

$$E_n(f)_{p(x)} = \|f - t_n^*\|_{p(x)}$$

for $n = 0, 1, \ldots$.

By $W^{p(x)} = W^{p(x)}([0, 2\pi])$ we denote the set of absolutely continuous functions f such that $f' \in L^{p(x)}$.

LEMMA 1. *Let $p \in \mathscr{M}$ and $f \in W^{p(x)}$. Then the estimate*

$$E_n(f)_{p(x)} = O\left(\frac{1}{n}\|f'\|_{p(x)}\right) \tag{9}$$

holds for $n = 1, 2, \ldots$.

Proof. It follows from Theorem 6.1 of [21] that

$$\|f - S_n(f)\|_{p(x)} \to 0, \quad n \to \infty.$$

It is easy to see that

$$A_k\left(\widetilde{f'}\right)(x) = kA_k(f)(x), \quad k = 1, 2, \ldots,$$

where $\widetilde{f'}$ is the conjugate function of f'. By considering the uniform boundedness of $\{S_n\}_0^\infty$ and the boundedness of the conjugation operator in the space $L^{p(x)}$ (see [21]), we get

$$\begin{aligned}
\|f - S_n(f)\|_{p(x)} &= \left\|\sum_{k=n+1}^{\infty} A_k(f)\right\|_{p(x)} = \left\|\sum_{k=n+1}^{\infty} \frac{1}{k} A_k\left(\widetilde{f'}\right)\right\|_{p(x)} \\
&= \left\|\sum_{k=n+1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)\left(S_k\left(\widetilde{f'}\right) - \widetilde{f'}\right) + \frac{1}{n+1}\left(S_n\left(\widetilde{f'}\right) - \widetilde{f'}\right)\right\|_{p(x)} \\
&\leqslant \sum_{k=n+1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)\left\|S_k\left(\widetilde{f'}\right) - \widetilde{f'}\right\|_{p(x)} + \frac{1}{n+1}\left\|S_n\left(\widetilde{f'}\right) - \widetilde{f'}\right\|_{p(x)} \\
&\leqslant c\left\{\sum_{k=n+1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)\right\}\|f'\|_{p(x)} + \frac{1}{n+1}\|f'\|_{p(x)} \\
&\leqslant \frac{c}{n}\|f'\|_{p(x)},
\end{aligned}$$

and hence (9) follows. □

LEMMA 2. *If $p \in \mathscr{M}$, the Jackson type inequality*

$$E_n(f)_{p(x)} = O\left(\Omega_{p(x)}\left(f, \frac{1}{n}\right)\right), \quad n = 1, 2, \ldots$$

holds for $f \in L^{p(x)}$.

Proof. Let $f \in L^{p(x)}$. Consider the transform

$$U_\delta(f)(x) := \frac{2}{\delta} \int_{\delta/2}^{\delta} \left(\frac{1}{h} \int_0^h f(x+t)\,dt \right) dh, \quad \delta > 0.$$

It is clear that $U_\delta(f) \in W^{p(x)}$ for each $\delta > 0$ and

$$(U_\delta(f))'(x) = \frac{2}{\delta} \int_{\delta/2}^{\delta} \frac{1}{h} (f(x+h) - f(x))\,dh$$

for almost all x. Since

$$\left|(U_\delta(f))'(x)\right| \leqslant \frac{4}{\delta} \left(\frac{1}{\delta} \int_0^{\delta} |f(x+h) - f(x)|\,dh \right),$$

it follows from definition of $\Omega_{p(x)}(f,\delta)$ that

$$\begin{aligned} \left\|(U_\delta(f))'\right\|_{p(x)} &\leqslant \frac{4}{\delta} \left\| \frac{1}{\delta} \int_0^{\delta} |f(\cdot+h) - f|\,dh \right\|_{p(x)} \\ &\leqslant \frac{4}{\delta} \Omega_{p(x)}(f,\delta). \end{aligned}$$

On the other hand, since

$$U_\delta(f)(x) - f(x) = \frac{2}{\delta} \int_{\delta/2}^{\delta} \left(\frac{1}{h} \int_0^h (f(x+t) - f(x))\,dt \right) dh,$$

we get

$$\begin{aligned} \|U_\delta(f) - f\|_{p(x)} &\leqslant \frac{2}{\delta} \int_{\delta/2}^{\delta} \left\| \frac{1}{h} \int_0^h |f(\cdot+t) - f|\,dt \right\|_{p(x)} dh \\ &\leqslant \sup_{\delta/2 \leqslant h \leqslant \delta} \frac{2}{\delta} \int_{\delta/2}^{\delta} \left\| \frac{1}{h} \int_0^h |f(\cdot+t) - f|\,dt \right\|_{p(x)} dh \\ &\leqslant \frac{2}{\delta} \int_{\delta/2}^{\delta} \left(\sup_{\delta/2 \leqslant h \leqslant \delta} \left\| \frac{1}{h} \int_0^h |f(\cdot+t) - f|\,dt \right\|_{p(x)} dh \right) \end{aligned}$$

$$= \sup_{\delta/2 \leqslant h \leqslant \delta} \left\| \frac{1}{h} \int_0^h |f(\cdot + t) - f| \, dt \right\|_{p(x)} \leqslant \Omega_{p(x)}(f, \delta).$$

Hence, by subadditivity of the best approximation and (9) we obtain

$$\begin{aligned} E_n(f)_{p(x)} &\leqslant E_n\left(f - U_{1/n}(f)\right)_{p(x)} + E_n\left(U_{1/n}(f)\right)_{p(x)} \\ &\leqslant \left\|f - U_{1/n}(f)\right\|_{p(x)} + \frac{c}{n} \left\| \left(U_{1/n}(f)\right)' \right\|_{p(x)} \\ &\leqslant \Omega_{p(x)}\left(f, \frac{1}{n}\right) + \frac{c}{n} 4n \Omega_{p(x)}\left(f, \frac{1}{n}\right), \end{aligned}$$

which finishes the proof. □

LEMMA 3. *Let* $p \in \mathscr{M}$ *and* $0 < \alpha \leqslant 1$. *Then for every* $f \in Lip(\alpha, p(x))$ *the estimate*

$$\|f - S_n(f)\|_{p(x)} = O\left(n^{-\alpha}\right), \quad n = 1, 2, \ldots$$

holds.

Proof. Let t_n^* $(n = 0, 1, \ldots)$ be the trigonometric polynomial of best approximation to $f \in Lip(\alpha, p(x))$. By Lemma 2

$$\|f - t_n^*\|_{p(x)} = O\left(\Omega_{p(x)}(f, 1/n)\right),$$

and hence

$$\|f - t_n^*\|_{p(x)} = O\left(n^{-\alpha}\right).$$

By the uniform boundedness of the partial sums $S_n(f)$ in the space $L^{p(x)}$ (see [21]), we get

$$\begin{aligned} \|f - S_n(f)\|_{p(x)} &\leqslant \|f - t_n^*\|_{p(x)} + \|t_n^* - S_n(f)\|_{p(x)} \\ &= \|f - t_n^*\|_{p(x)} + \|S_n(t_n^* - f)\|_{p(x)} \\ &= O\left(\|f - t_n^*\|_{p(x)}\right) \\ &= O\left(n^{-\alpha}\right). \qquad \square \end{aligned}$$

LEMMA 4. *Let* $p \in \mathscr{M}$. *If* $f \in Lip(1, p(x))$, *then* f *is absolutely continuous and* $f' \in L^{p(x)}$, *that is* $f \in W^{p(x)}$.

Proof. Let $f \in Lip(1, p(x))$ and $\delta > 0$. Since $p_- \leqslant p(x)$ almost everywhere, by Theorem 2.8 of [13] the space $L^{p(x)}$ is continuously embedded in L^{p_-}. Hence we have

$$\|T_h(f)\|_{p_-} \leqslant c \|T_h(f)\|_{p(x)}$$

for every h with $|h| \leqslant \delta$. This inequality and equivalence of $\omega_{p_-}(f, \cdot)$ and $\Omega_{p_-}(f, \cdot)$ yield

$$\omega_{p_-}(f, \delta) \leqslant c \, \Omega_{p(x)}(f, \delta).$$

Hence, $f \in Lip(1, p(x))$ implies $\omega_{p_-}(f,\delta) = O(\delta)$, and this implies that f is absolutely continuous and $f' \in L^{p_-}$ ([6, pp. 51–54]).

Since the relation

$$\frac{f(x+t)-f(x)}{t} \to f'(x), \quad t \to 0$$

holds almost everywhere, for almost all x we get

$$\frac{2}{\delta}\int_{\delta/2}^{\delta} \frac{|f(x+t)-f(x)|}{t}dt \to |f'(x)|, \quad \delta \to 0^+.$$

By Fatou Lemma, for every measurable function g with $m_{p'}(g) \leqslant 1$,

$$\begin{aligned}
\int_0^{2\pi} |f'(x)|\,|g(x)|\,dx &= \int_0^{2\pi} \left(\lim_{\delta\to 0^+} \frac{2}{\delta}\int_{\delta/2}^{\delta} \frac{|f(x+t)-f(x)|}{t}dt \right) |g(x)|\,dx \\
&\leqslant \liminf_{\delta\to 0^+} \int_0^{2\pi} \left(\frac{2}{\delta}\int_{\delta/2}^{\delta} \frac{|f(x+t)-f(x)|}{t}dt \right) |g(x)|\,dx \\
&\leqslant \liminf_{\delta\to 0^+} \frac{4}{\delta}\int_0^{2\pi} \left(\frac{1}{\delta}\int_0^{\delta} |f(x+t)-f(x)|\,dt \right) |g(x)|\,dx \\
&= \liminf_{\delta\to 0^+} \frac{4}{\delta}\int_0^{2\pi} T_\delta(f)(x)\,|g(x)|\,dx \\
&\leqslant \liminf_{\delta\to 0^+} \frac{4}{\delta}\,\|T_\delta(f)\|_{p(x)} \leqslant \liminf_{\delta\to 0^+} \frac{4}{\delta}\,\Omega_{p(x)}(f,\delta) \\
&= \liminf_{\delta\to 0^+} \frac{4}{\delta}\,O(\delta) = O(1),
\end{aligned}$$

and this means that $f' \in L^{p(x)}$. □

LEMMA 5. *Let $p \in \mathscr{M}$ and $f \in Lip(1, p(x))$. Then for $n = 1, 2, \ldots$ the estimate*

$$\|S_n(f) - \sigma_n(f)\|_{p(x)} = O\left(n^{-1}\right)$$

holds.

Proof. By Lemma 4, $f \in W^{p(x)}$. If f has the Fourier series

$$f(x) \sim \sum_{k=0}^{\infty} A_k(f)(x),$$

then the Fourier series of the conjugate function $\widetilde{f'}$ be

$$\widetilde{f'}(x) \sim \sum_{k=1}^{\infty} kA_k(f)(x).$$

On the other hand,

$$\begin{aligned} S_n(f)(x) - \sigma_n(f)(x) &= \sum_{k=1}^{n} \frac{k}{n+1} A_k(f)(x) \\ &= \frac{1}{n+1} S_n\left(\widetilde{f'}\right)(x). \end{aligned}$$

Hence, by considering the uniform boundedness of the partial sums and the conjugation operator in the space $L^{p(x)}$ (see [21]), we obtain

$$\|S_n(f) - \sigma_n(f)\|_{p(x)} = O\left(n^{-1}\right)$$

for $n = 1, 2, \ldots$. □

The following Lemma was proved in [15].

LEMMA 6. *Let* $\{p_n\}_0^\infty$ *be a sequence of positive numbers. If* $\{p_n\}_0^\infty \in AMDS$, *or* $\{p_n\}_0^\infty \in AMIS$ *and* $(n+1)p_n = O(P_n)$, *then*

$$\sum_{m=1}^{n} m^{-\alpha} p_{n-m} = O\left(n^{-\alpha} P_n\right)$$

for $0 < \alpha < 1$.

3. Proofs of the main results

Proof of Theorem 1. Since

$$f(x) = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} f(x),$$

we have

$$f(x) - N_n(f)(x) = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} \{f(x) - S_m(f)(x)\}.$$

By Lemma 3 and Lemma 6 we obtain

$$\begin{aligned} \|f - N_n(f)\|_{p(x)} &\leqslant \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} \|f - S_m(f)\|_{p(x)} \\ &= \frac{1}{P_n} \sum_{m=1}^{n} p_{n-m} O\left(m^{-\alpha}\right) + \frac{p_n}{P_n} \|f - S_0(f)\|_{p(x)} \\ &= \frac{1}{P_n} O\left(n^{-\alpha} P_n\right) + O\left(\frac{1}{n+1}\right) \\ &= O\left(n^{-\alpha}\right). \qquad \square \end{aligned}$$

Proof of Theorem 2. It is clear that

$$N_n(f)(x) = \frac{1}{P_n}\sum_{m=0}^{n} P_{n-m}A_m(f)(x).$$

By Abel transform,

$$\begin{aligned} S_n(f)(x) - N_n(f)(x) &= \frac{1}{P_n}\sum_{m=1}^{n}(P_n - P_{n-m})A_m(f)(x) \\ &= \frac{1}{P_n}\sum_{m=1}^{n}\Delta_m\left(\frac{P_n - P_{n-m}}{m}\right)\left(\sum_{k=1}^{m}kA_k(f)(x)\right) + \frac{1}{n+1}\sum_{k=1}^{n}kA_k(f)(x), \end{aligned}$$

and hence

$$\begin{aligned} \|S_n(f) - N_n(f)\|_{p(x)} \leqslant{}& \frac{1}{P_n}\sum_{m=1}^{n}\left|\Delta_m\left(\frac{P_n - P_{n-m}}{m}\right)\right|\left\|\sum_{k=1}^{m}kA_k(f)\right\|_{p(x)} \\ &+ \frac{1}{n+1}\left\|\sum_{k=1}^{n}kA_k(f)\right\|_{p(x)}. \end{aligned}$$

Since

$$S_n(f)(x) - \sigma_n(f)(x) = \frac{1}{n+1}\sum_{k=1}^{n}kA_k(f)(x),$$

by Lemma 5 we get

$$\frac{1}{n+1}\left\|\sum_{k=1}^{n}kA_k(f)\right\|_{p(x)} = \|S_n(f) - \sigma_n(f)\|_{p(x)} = O\left(n^{-1}\right).$$

Hence,

$$\|S_n(f) - N_n(f)\|_{p(x)} = O\left(\frac{1}{P_n}\right)\sum_{m=1}^{n}\left|\Delta_m\left(\frac{P_n - P_{n-m}}{m}\right)\right| + O\left(n^{-1}\right). \tag{10}$$

Suppose the condition

$$\sum_{k=1}^{n-1}k|\Delta p_k| = O(P_n)$$

holds. This implies that (see [15])

$$\sum_{m=1}^{n}\left|\Delta_m\left(\frac{P_n - P_{n-m}}{m}\right)\right| = O\left(\frac{P_n}{n}\right),$$

and hence by (10) we have

$$\|S_n(f) - N_n(f)\|_{p(x)} = O\left(n^{-1}\right).$$

This relation and Lemma 3 yield

$$\|f-N_n(f)\|_{p(x)} = O\left(n^{-1}\right).$$

Now let

$$\sum_{k=0}^{n-1} |\Delta p_k| = O\left(\frac{P_n}{n}\right). \tag{11}$$

A simple calculation yields

$$\Delta_m\left(\frac{P_n-P_{n-m}}{m}\right) = \frac{1}{m(m+1)}\left(\sum_{k=n-m}^{n} p_k-(m+1)\,p_{n-m}\right),$$

and by induction one can easily get

$$\left|\sum_{k=n-m}^{n} p_k-(m+1)\,p_{n-m}\right| \leqslant \sum_{k=1}^{m} k\,|p_{n-k+1}-p_{n-k}|.$$

Thus,

$$\begin{aligned}
\sum_{m=1}^{n}\left|\Delta_m\left(\frac{P_n-P_{n-m}}{m}\right)\right| &\leqslant \sum_{m=1}^{n}\frac{1}{m(m+1)}\left(\sum_{k=1}^{m} k\,|p_{n-k+1}-p_{n-k}|\right)\\
&= \sum_{k=1}^{n} k\,|p_{n-k+1}-p_{n-k}|\left(\sum_{m=k}^{n}\frac{1}{m(m+1)}\right)\\
&\leqslant \sum_{k=1}^{n} k\,|p_{n-k+1}-p_{n-k}|\left(\sum_{m=k}^{\infty}\frac{1}{m(m+1)}\right)\\
&= \sum_{k=1}^{n} |p_{n-k+1}-p_{n-k}| = \sum_{k=0}^{n-1}|\Delta p_k|.
\end{aligned}$$

Combining this, the assumption (11) and (10) we get

$$\|S_n(f)-N_n(f)\|_{p(x)} = O\left(n^{-1}\right),$$

and considering Lemma 3 again we obtain the desired result. □

Proof of Theorem 3. Let $0<\alpha<1$. By definition of $R_n(f)(x)$,

$$f(x)-R_n(f)(x) = \frac{1}{P_n}\sum_{m=0}^{n} p_m\left\{f(x)-S_m(f)(x)\right\}.$$

From Lemma 3, we get

$$\begin{aligned}
\|f-R_n(f)\|_{p(x)} &\leqslant \frac{1}{P_n}\sum_{m=0}^{n} p_m\,\|f-S_m(f)\|_{p(x)} \\
&= O\left(\frac{1}{P_n}\right)\sum_{m=1}^{n} p_m m^{-\alpha} + \frac{p_0}{P_n}\,\|f-S_0(f)\|_{p(x)}\\
&= O\left(\frac{1}{P_n}\right)\sum_{m=1}^{n} p_m m^{-\alpha}.
\end{aligned} \tag{12}$$

By Abel transform,

$$\begin{aligned}\sum_{m=1}^{n} p_m m^{-\alpha} &= \sum_{m=1}^{n-1} P_m \left\{m^{-\alpha} - (m+1)^{-\alpha}\right\} + n^{-\alpha} P_n \\ &\leqslant \sum_{m=1}^{n-1} m^{-\alpha} \frac{P_m}{m+1} + n^{-\alpha} P_n,\end{aligned}$$

and

$$\begin{aligned}\sum_{m=1}^{n-1} m^{-\alpha} \frac{P_m}{m+1} &= \sum_{m=1}^{n-1} \Delta\left(\frac{P_m}{m+1}\right)\left(\sum_{k=1}^{m} k^{-\alpha}\right) + \frac{P_n}{n+1} \sum_{m=1}^{n-1} m^{-\alpha} \\ &= O\left(n^{-\alpha} P_n\right)\end{aligned}$$

by condition (8). This yields

$$\sum_{m=1}^{n} p_m m^{-\alpha} = O\left(n^{-\alpha} P_n\right)$$

and from this and (12) we get

$$\|f - R_n(f)\|_{p(x)} = O\left(n^{-\alpha}\right).$$

Let's consider the case $\alpha = 1$.

By Abel transform,

$$\begin{aligned}R_n(f)(x) &= \frac{1}{P_n} \sum_{m=0}^{n-1} \left\{P_m \left(S_m(f)(x) - S_{m+1}(f)(x)\right) + P_n S_n(f)(x)\right\} \\ &= \frac{1}{P_n} \sum_{m=0}^{n-1} P_m \left(-A_{m+1}(f)(x)\right) + S_n(f)(x),\end{aligned}$$

and hence

$$R_n(f)(x) - S_n(f)(x) = -\frac{1}{P_n} \sum_{m=0}^{n-1} P_m A_{m+1}(f)(x).$$

Using Abel transform again yields

$$\begin{aligned}\sum_{m=0}^{n-1} P_m A_{m+1}(f)(x) &= \sum_{m=0}^{n-1} \frac{P_m}{m+1} (m+1) A_{m+1}(f)(x) \\ &= \sum_{m=0}^{n-1} \Delta\left(\frac{P_m}{m+1}\right)\left(\sum_{k=0}^{m} (k+1) A_{k+1}(f)(x)\right) \\ &\quad + \frac{P_n}{n+1} \sum_{k=0}^{n-1} (k+1) A_{k+1}(f)(x).\end{aligned}$$

Thus, by considering Lemma 5 and (8) we obtain

$$\begin{aligned}\left\|\sum_{m=0}^{n-1} P_m A_{m+1}(f)\right\|_{p(x)} &\leqslant \sum_{m=0}^{n-1}\left|\Delta\left(\frac{P_m}{m+1}\right)\right|\left\|\sum_{k=0}^{m}(k+1)A_{k+1}(f)\right\|_{p(x)} \\ &\quad + \frac{P_n}{n+1}\left\|\sum_{k=0}^{n-1}(k+1)A_{k+1}(f)\right\|_{p(x)} \\ &= \sum_{m=0}^{n-1}\left|\Delta\left(\frac{P_m}{m+1}\right)\right|(m+2)\left\|S_{m+1}(f)-\sigma_{m+1}(f)\right\|_{p(x)} \\ &\quad + P_n\left\|S_n(f)-\sigma_n(f)\right\|_{p(x)} \\ &= O(1)\sum_{m=0}^{n-1}\left|\Delta\left(\frac{P_m}{m+1}\right)\right| + O\left(\frac{P_n}{n}\right).\end{aligned}$$

This gives

$$\begin{aligned}\left\|R_n(f)-S_n(f)\right\|_{p(x)} &= \frac{1}{P_n}\left\|\sum_{m=0}^{n-1} P_m A_{m+1}(f)\right\|_{p(x)} \\ &= \frac{1}{P_n} O\left(\frac{P_n}{n}\right) = O\left(\frac{1}{n}\right).\end{aligned}$$

Combining this estimate with Lemma 3 yields

$$\left\|f-R_n(f)\right\|_{p(x)} = O\left(n^{-1}\right). \qquad \square$$

REFERENCES

[1] P. CHANDRA, *Approximation by Nörlund operators*, Mat. Vestnik 38 (1986), 263–269.

[2] P. CHANDRA, *Functions of classes L_p and $Lip(\alpha, p)$ and their Riesz means*, Riv. Mat. Univ. Parma (4) 12 (1986), 275–282.

[3] P. CHANDRA, *A note on degree of approximation by Nörlund and Riesz operators*, Mat. Vestnik 42 (1990), 9–10.

[4] P. CHANDRA, *Trigonometric approximation of functions in L_p-norm*, J. Math. Anal. Appl. 275 (2002), 13–26.

[5] D. CRUZ-URIBE, A. FIORENZA, C. J. NEUGEBAUER, *The maximal function on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 223–238, and 29 (2004), 247–249.

[6] R. A. DEVORE, G. G. LORENTZ, *Constructive Approximation*, Springer-Verlag (1993).

[7] L. DIENING, M. RUZICKA, *Calderon-Zygmund operators on generalized Lebesgue spaces $L^{p(x)}$ and problems related to fluid dynamics*, J. Reine Angew. Math. 563 (2003), 197–220.

[8] L. DIENING, *Maximal function on generalized Lebesgue spaces $L^{p(x)}$*, Math. Inequal. Appl. 7 (2004), 245–253.

[9] D. E. EDMUNDS, J. LANG, A. NEKVINDA, *On $L^{p(x)}$ norms*, Proc. R. Soc. Lond. A 455 (1999), 219–225.

[10] X. FAN, D. ZHAO, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. 263 (2001), 424–446.

[11] A. GUVEN, *Trigonometric approximation of functions in weighted L^p spaces*, Sarajevo J. Math 5 (17) (2009), 99–108.

[12] D. M. ISRAFILOV, V. KOKILASHVILI, S. SAMKO, *Approximation in weighted Lebesgue and smirnov Spaces with variable exponents*, Proc. A. Razmadze Math. Inst. 143 (2007), 25–35.

[13] O. KOVACIK, J. RAKOSNIK, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. 41 (1991), 592–618.

[14] N. X. KY, *Moduli of Mean Smoothness and Approximation with A_p-weights*, Annales Univ. Sci. Budapest 40 (1997), 37–48.

[15] L. LEINDLER, *Trigonometric approximation in L_p-norm*, J. Math. Anal. Appl. 302 (2005), 129–136.

[16] R. N. MOHAPATRA, D. C. RUSSELL, *Some direct and inverse theorems in approximation of functions*, J. Austral. Math. Soc. (Ser. A) 34 (1983), 143–154.

[17] A. NEKVINDA, *Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R})$*, Math. Inequal. Appl. 7 (2004), 255–265.

[18] L. PICK, M. RUZICKA, *An example of a space $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded*, Expo. Math. 19 (2001), 369–371.

[19] E. S. QUADE, *Trigonometric approximation in the mean*, Duke Math. J. 3 (1937), 529–542.

[20] I. I. SHARAPUDINOV, *Uniform boundedness in L^p $(p = p(x))$ of some families of convolution operators*, Math. Notes 59 (1996), 205–212.

[21] I. I. SHARAPUDINOV, *Some problems in approximation theory in the spaces $L^{p(x)}$*, (Russian), Analysis Mathematica 33 (2007), 135–153.

[22] A. ZYGMUND, *Trigonometric Series, Vol I*, Cambridge Univ. Press, 2nd edition, (1959).

(Received February 16, 2009)

Ali Guven
Department of Mathematics
Faculty of Art and Science
Balikesir University
10145, Balikesir
Turkey
e-mail: ag_guven@yahoo.com

Daniyal M. Israfilov
Department of Mathematics
Faculty of Art and Science
Balikesir University
10145, Balikesir
Turkey
e-mail: mdaniyal@balikesir.edu.tr