

A GOOD λ ESTIMATE FOR MULTILINEAR COMMUTATOR OF FRACTIONAL INTEGRAL ON SPACES OF HOMOGENEOUS TYPE

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Abstract. In this paper, a good λ estimate for the multilinear commutator associated to the fractional integral operator on the spaces of homogeneous type is obtained. Under this result, we get the $(L^p(X), L^q(X))$ -boundedness of the multilinear commutator.

1. Introduction

Let T be the Calderón-Zygmund operator, Coifman, Rochberg and Weiss (see [9]) proves that the commutator $[b, T](f) = bT(f) - T(bf)$ (where $b \in BMO(\mathbb{R}^n)$) is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo (see [5]) proves a similar result when T is replaced by the fractional operators. In [13], [16], Janson and Paluszynski study these results for the Triebel-Lizorkin spaces and the case $b \in Lip_\beta(\mathbb{R}^n)$, where $Lip_\beta(\mathbb{R}^n)$ is the homogeneous Lipschitz space. The main purpose of this paper is to find the good λ estimate for the multilinear commutator associated to the fractional integral operator on the spaces of homogeneous type, where $b \in Lip_\beta(X)$ and $b \in BMO(X)$. Under this result, we get $(L^p(X), L^q(X))$ -boundedness of the multilinear commutator.

2. Preliminaries and Theorems

Give a set X , a function $d : X \times X \rightarrow \mathbb{R}^+$ is called a quasi-distance on X if the following conditions are satisfied:

- (i) for every x and y in X , $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,
- (ii) for every x and y in X , $d(x, y) = d(y, x)$,
- (iii) there exists a constant $k \geq 1$ such that

$$d(x, y) \leq k(d(x, z) + d(z, y)) \quad (1)$$

for every x, y and z in X .

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Let μ be a positive measure on the σ -algebra of subsets of X which contains the r -balls $B(x, r) = \{y : d(x, y) < r\}$. We assume that μ satisfies a doubling condition, that is, there exists a constant A such that

$$0 < \mu(B(x, 2r)) \leq A\mu(B(x, r)) < \infty \tag{2}$$

holds for all $x \in X$ and $r > 0$.

A structure (X, d, μ) , with d and μ as above, is called a space of homogeneous type. The constants k and A in (1) and (2) will be called the constants of the space.

Then let us introduce some notations (see [3], [11], [16], [21]). Throughout this paper, B will denote a ball of X , and for a ball B let $f_B = \mu(B)^{-1} \int_B f(x) d\mu(x)$ and the sharp function of f is defined by

$$f^\#(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y) - f_B| d\mu(y).$$

It is well-known that (see [11])

$$f^\#(x) \approx \sup_{B \ni x} \inf_{c \in \mathbb{C}} \frac{1}{\mu(B)} \int_B |f(y) - c| d\mu(y).$$

We say that b belongs to $BMO(X)$ if $b^\#$ belongs to $L^\infty(X)$ and define $\|b\|_{BMO} = \|b^\#\|_{L^\infty}$. It has been known that(see [11])

$$\|b - b_{2^k B}\|_{BMO} \leq Ck \|b\|_{BMO}.$$

For $1 \leq p < \infty$ and $0 \leq \gamma < 1$, let

$$M_{\gamma,p}(f)(x) = \sup_{x \in B} \left(\frac{1}{\mu(B)^{1-p\gamma}} \int_B |f(y)|^p d\mu(y) \right)^{1/p}.$$

If $\gamma = 0$, $M_{p,\gamma}(f) = M_p(f)$ which is the Hardy-Littlewood maximal function when $p = 1$. For $0 < \beta < 1$, the Lipschitz space λ_β is the space of functions f such that

$$\|f\|_{\lambda_\beta} = \sup_{\substack{x, h \in X \\ h \neq 0}} \rho \left(\Delta_h^{[\beta]+1} f(x) \right) / \rho(x+h, x)^\beta < \infty,$$

where Δ_h^k denotes the k -th difference operator (see [16]) and the existence of ρ is guaranteed by the following Lemma 1.

In this paper, we will study some multilinear commutators as follows.

DEFINITION. Suppose b_j ($j = 1, \dots, m$) are the fixed locally integrable functions on X . Let I_γ be the fractional integral operator as

$$I_\gamma(f)(x) = \int_X K_\gamma(x, y) f(y) d\mu(y), \quad 0 < \gamma < 1,$$

where

$$K_\gamma(x,y) = \begin{cases} [\mu(B(x,d(x,y)))]^{\gamma-1} & \text{if } x \neq y \\ \mu(x)^{\gamma-1} & \text{if } x = y \text{ and } \mu(x) > 0. \end{cases}$$

The multilinear commutator of fractional integral operator is defined by

$$I_\gamma^{\vec{b}}(f)(x) = \int_X \prod_{j=1}^m (b_j(x) - b_j(y)) K_\gamma(x,y) f(y) d\mu(y). \tag{3}$$

and $I_x^{\gamma\vec{b}}(f)(x) = \sup_{\varepsilon>0} |I_\varepsilon^{\gamma\vec{b}}(f)(x)|$, where

$$I_\varepsilon^{\gamma\vec{b}}(f)(x) = \int_{\rho(x,y)>\varepsilon} \prod_{j=1}^m (b_j(x) - b_j(y)) K_\gamma(x,y) f(y) d\mu(y)$$

Note that when $b_1 = \dots = b_m$, $I_\gamma^{\vec{b}}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors(see [1–6], [10–12], [17–19]). Our main purpose is to find the good λ estimate for the multilinear commutator $I_x^{\gamma\vec{b}}$, and with this result to find $(L^p(X), L^q(X))$ -boundedness for the multilinear commutator $I_\gamma^{\vec{b}}$.

Given some functions b_j ($j = 1, \dots, m$) and a positive integer m and $1 \leq j \leq m$, we set $\|\vec{b}\|_{Lip_\beta} = \prod_{j=1}^m \|b_j\|_{Lip_\beta}$, $\|\vec{b}\|_{BMO} = \prod_{j=1}^m \|b_j\|_{BMO}$ and denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, \dots, m\}$ of j different elements, $|\sigma| = j$ is the element number of σ . For $\sigma \in C_j^m$, set $\sigma^c = \{1, \dots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \dots, b_m)$ and $\sigma = \{\sigma(1), \dots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \dots b_{\sigma(j)}$, $\|\vec{b}_\sigma\|_{\lambda_\beta} = \|b_{\sigma(1)}\|_{\lambda_\beta} \dots \|b_{\sigma(j)}\|_{\lambda_\beta}$ and $\|\vec{b}_\sigma\|_{BMO} = \|b_{\sigma(1)}\|_{BMO} \dots \|b_{\sigma(j)}\|_{BMO}$.

In what follows, $C > 0$ always denotes a constant that is independent of main parameters involved but whose value may differ from line to line. For any index $p \in [1, \infty]$, we denote by p' its conjugate index, namely, $1/p + 1/p' = 1$.

Now we state our results as following.

THEOREM 1. *Let $0 < \gamma < 1$, $\theta = 1 - \gamma$, $0 < \beta < 1$ and $b_j \in \lambda_\beta$ for $j = 1, \dots, m$.*

(a) *Suppose $1 < r < p < \infty$. Then there exists $\xi_0 > 0$ such that, for any $0 < \xi < \xi_0$ and $\lambda > 0$,*

$$\begin{aligned} &\mu \left(\left\{ x \in X : I_x^{\gamma\vec{b}}(f)(x) > 3\lambda, \|\vec{b}\|_{\lambda_\beta} M_{m\beta+\gamma,p}(f)(x) \leq \xi\lambda \right\} \right) \\ &\leq C\xi^r \mu(\{x \in X : I_x^{\gamma\vec{b}}(f)(x) > \lambda\}) \end{aligned}$$

(b) $I_\gamma^{\vec{b}}$ is bounded from $L^p(X)$ to $L^q(X)$ for $1 < p < 1/(m\beta + \gamma)$ and $1/q = 1/p - (m\beta + \gamma)$.

THEOREM 2. *Let $0 < \gamma < 1$, $\theta = 1 - \gamma$, and $b_j \in BMO(X)$ for $j = 1, \dots, m$.*

(a) *Suppose $1 < r < p < \infty$. Then there exists $\xi_0 > 0$ such that, for any $0 < \xi < \xi_0$ and $\lambda > 0$,*

$$\begin{aligned} \mu \left(\left\{ x \in X : I_{\star}^{\vec{b}}(f)(x) > 3\lambda, \|\vec{b}\|_{BMO M_{\gamma,p}}(f)(x) \leq \xi \lambda \right\} \right) \\ \leq C \xi^r \mu(\{x \in X : I_{\star}^{\vec{b}}(f)(x) > \lambda\}) \end{aligned}$$

(b) *$I_{\gamma}^{\vec{b}}$ is bounded from $L^p(X)$ to $L^q(X)$ for $1 < p < 1/\gamma$ and $1/q = 1/p - \gamma$.*

3. Proofs of Theorems

To prove the theorem, we need the following lemmas.

LEMMA 1. (see [15]). *Let d be a quasi-distance on a set X . Then there exists a quasi-distance ρ on X , a finite constant C and a number $0 < \alpha < 1$, such that ρ is equivalent to d and, for every x, y and z in X*

$$|\rho(x, y) - \rho(z, y)| \leq C \rho(x, z)^{\alpha} (\rho(x, y) + \rho(z, y))^{1-\alpha}.$$

LEMMA 2. (see [16]). *Let $0 < \beta < 1, 1 \leq p \leq \infty$, then*

$$\begin{aligned} \|b\|_{\lambda_{\beta}} &\approx \sup_B \frac{1}{\mu(B)^{1+\beta}} \int_B |b(x) - b_B| d\mu(x) \\ &\approx \sup_B \frac{1}{\mu(B)^{\beta}} \left(\frac{1}{\mu(B)} \int_B |b(x) - b_B|^p d\mu(x) \right)^{1/p} \\ &\approx \sup_B \inf_c \frac{1}{\mu(B)^{1+\beta}} \int_B |b(x) - c| d\mu(x) \\ &\approx \sup_B \inf_c \frac{1}{\mu(B)^{\beta}} \left(\frac{1}{\mu(B)} \int_B |b(x) - c|^p d\mu(x) \right)^{1/p} \end{aligned}$$

LEMMA 3. (see [5]). *Let $0 \leq v < 1, 1 \leq r < p < 1/v$ and $1/q = 1/p - v$, then*

$$\|M_{v,r}(f)\|_{L^q} \leq C \|f\|_{L^p}.$$

REMARK. In order to prove Theorems 1, it is clear that we may replace the kernel K_{γ} by any kernel Q_{γ} equivalent to K_{γ} , in the sense that $1/cK_{\gamma} \leq Q_{\gamma} \leq cK_{\gamma}$ for some positive constant. We shall use an equivalent kernel having some smoothness properties that fractional kernel K_{γ} does not have.

In fact, notice that the function $K(x, y)$ defined as $\mu(B(x, d(x, y)))$ if $x \neq y$ and $K(x, y) = 0$ is not a quasi-distance because it might not be symmetric. However, it is easy to prove that the function

$$\delta(x, y) = \begin{cases} \frac{1}{2} [\mu(B(x, d(x, y))) + \mu(B(y, d(x, y)))] & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

is a quasi-distance equivalent to $K(x, y)$. Now let ρ be a continuous quasi-distance equivalent to δ (the existence of ρ is guaranteed by lemma 1). Associated to ρ we define the kernel

$$Q_\gamma(x, y) = \begin{cases} \rho(x, y)^{\gamma-1} & \text{if } x \neq y, \\ \mu(x)^{\gamma-1} & \text{if } x = y \text{ and } \mu(x) > 0. \end{cases}$$

and the operator

$$\mathcal{J}_\gamma^{\vec{b}}(f)(x) = \int_X \prod_{j=1}^m (b_j(x) - b_j(y)) Q_\gamma(x, y) f(y) d\mu(y) \tag{4}$$

It is clear that the above operator is equivalent to the one defined in (3). Consequently, we shall work in the proof Theorem 1 and the following Theorems 2 with the operator defined in (4).

Proof of Theorem 1(a). When $\mathbf{m} = \mathbf{1}$, by the Whitney decomposition, $\{x \in X : \mathcal{J}_\star^{\gamma b_1}(f)(x) > \lambda\}$ may be written as a union of balls $\{B_k\}$ with mutually disjoint interiors and with distance from each to $X \setminus \cup_k B_k$ comparable to the diameter of B_k . It suffices to prove the good λ estimate for each B_k . There exists a constant $C = C(n)$ such that for each k , the cube \tilde{B}_k intersects $X \setminus \cup_k B_k$, where \tilde{B}_k denotes the ball with the same center as B_k and with the diam $\tilde{B}_k = C \text{ diam } B_k$. Then, for each k , there exists a point $x_0 = x_0(k) \in \tilde{B}_k$ such that

$$\mathcal{J}_\star^{\gamma b_1}(f)(x_0) \leq \lambda.$$

Now, we fix a ball B_k . Without loss of generality, we may assume there exists a point $z = z(k)$ with

$$\|b_1\|_{\lambda_\beta} M_{\beta+\gamma, p}(f)(z) \leq \xi \lambda.$$

Set $\bar{B}_k = \tilde{\tilde{B}}_k$ and write $f = f_1 + f_2$ for $f_1 = f\chi_{\bar{B}_k}$ and $f_2 = f\chi_{X \setminus \bar{B}_k}$. We turn to the estimates on f_1 and f_2 .

The estimates on f_1 . For $x \in B_k$, $1/r = 1/p + 1/q < 1$

$$\begin{aligned} \|\mathcal{J}_\star^{\gamma b_1}(f_1)\|_{L^r} &\leq C \left(\int_{\rho(x, y) > \varepsilon} |(b_1(x) - b_1(y)) Q_\gamma(x, y) f_1(y)|^r d\mu(y) \right)^{1/r} \\ &\leq C \mu(\bar{B}_k)^{\gamma-1} \left(\int_{\bar{B}_k} |b_1(x) - b_1(y)|^q d\mu(y) \right)^{1/q} \left(\int_{\bar{B}_k} |f(y)|^p d\mu(y) \right)^{1/p} \\ &\leq C \mu(\bar{B}_k)^{\gamma-1} \mu(\bar{B}_k)^{\beta+1/q} \|b_1\|_{\lambda_\beta} \mu(\bar{B}_k)^{1/p-\beta-\gamma} \\ &\quad \times \left(\frac{1}{\mu(\bar{B}_k)^{1-(\beta+\gamma)p}} \int_{\bar{B}_k} |f(y)|^p d\mu(y) \right)^{1/p} \\ &\leq C \mu(\bar{B}_k)^{1/r-1} \|b_1\|_{\lambda_\beta} M_{\beta+\gamma, p}(f)(z). \end{aligned}$$

Let $\eta > 0$, we have

$$\begin{aligned} \mu(\{x \in X : \mathcal{I}_*^{\gamma b_1}(f_1)(x) > \eta\lambda\}) &\leq C(\eta\lambda)^{-r} \|\mathcal{I}_*^{\gamma b_1}(f_1)\|_{L^r}^r \\ &\leq C(\eta\lambda)^{-r} [\|b_1\|_{\lambda_\beta} M_{\beta+\gamma,p}(f)(z)]^r \mu(\bar{B}_k)^{1-1/r} \\ &\leq C(\eta\lambda)^{-r} (\xi\lambda)^r \mu(\bar{B}_k) \\ &\leq C(\xi/\eta)^r \mu(B_k). \end{aligned}$$

The estimates on f_2 . Let $H = H(X)$ be a large positive integer depending only on X . We consider the following two cases:

Case 1. $\text{diam}(\tilde{B}_k) \leq \varepsilon \leq H \text{diam}(\tilde{B}_k)$. Set $V(x, y) = (b_1(x) - b_1(y))Q_\gamma(x, y)f(y)$. Choose $x_0 \in \tilde{B}_k$ such that $x_0 \in X \setminus \bigcup_k B_k$. For $x \in B_k$, following [8], we have

$$\begin{aligned} |\mathcal{I}_\varepsilon^{\gamma b_1}(f_2)(x)| &\leq \left| \int_{\rho(x,y) > \varepsilon} [V(x, y) - V(x_0, y)]f_2(y)d\mu(y) \right| + \int_{R(x)} |V(x_0, y)f(y)|d\mu(y) \\ &\quad + \int_{R(x_0)} |V(x_0, y)f(y)|d\mu(y) + |\mathcal{I}_\varepsilon^{\gamma b_1}(f)(x_0)| \\ &= I + II + III + IV, \end{aligned}$$

where $R(u) = \{y \in X : \text{diam}(\tilde{B}_k) < \rho(u, y) \leq H \text{diam}(\tilde{B}_k)\}$. Now let us treat I, II and III respectively. For I , we write

$$\begin{aligned} |V(x, y) - V(x_0, y)| &= |(b_1(x) - b_1(y))Q_\gamma(x, y)f_2(y) - (b_1(x_0) - b_1(y))Q_\gamma(x_0, y)f_2(y)| \\ &= |b_1(x)Q_\gamma(x, y)f_2(y) - b_1(y)Q_\gamma(x, y)f_2(y) - b_1(x_0)Q_\gamma(x_0, y)f_2(y) \\ &\quad + b_1(y)Q_\gamma(x_0, y)f_2(y) - b_1(x_0)Q_\gamma(x, y)f_2(y) + b_1(x_0)Q_\gamma(x, y)f_2(y)| \\ &\leq |(b_1(x) - b_1(x_0))Q_\gamma(x, y)f_2(y)| \\ &\quad + |(b_1(x_0) - b_1(y))(Q_\gamma(x, y) - Q_\gamma(x_0, y))f_2(y)| \\ &= I_1 + I_2. \end{aligned}$$

For I_1 , by Lemma 2, Hölder’s inequality and the following inequality, for $b \in \lambda_\beta$

$$|b(x) - b_B| \leq \frac{1}{\mu(B)} \int_B \|b\|_{\lambda_\beta} |x - y|^\beta d\mu(y) \leq \|b\|_{\lambda_\beta} \mu(B)^\beta,$$

we have

$$\begin{aligned} \int_{\rho(x,y) > \varepsilon} I_1 d\mu(y) &\leq C \sum_{v=1}^\infty \int_{B_{2^{v+1}\varepsilon}(x_0) \setminus B_{2^v\varepsilon}(x_0)} |b_1(x) - b_1(x_0)| |Q_\gamma(x, y)| |f(y)| d\mu(y) \\ &\leq C \|b_1\|_{\lambda_\beta} \mu(\tilde{B}_k)^\beta \sum_{v=1}^\infty \mu(B_{2^v\varepsilon}(x_0))^{\gamma-1} \\ &\quad \times \left(\int_{B_{2^{v+1}\varepsilon}(x_0)} |f(y)|^p d\mu(y) \right)^{1/p} \mu(B_{2^{v+1}\varepsilon}(x_0))^{1-1/p} \end{aligned}$$

$$\begin{aligned}
 &\leq C \|b_1\|_{\dot{\lambda}_\beta} \mu(\tilde{B}_k)^\beta \sum_{v=1}^\infty \mu(B_{2^{v+1}\varepsilon}(x_0))^{\gamma-1+1-1/p+1/p-\beta-\gamma} \\
 &\quad \times \left(\frac{1}{\mu(B_{2^{v+1}\varepsilon}(x_0))^{1-(\beta+\gamma)p}} \int_{B_{2^{v+1}\varepsilon}(x_0)} |f(y)|^p d\mu(y) \right)^{1/p} \\
 &\leq C \|b_1\|_{\dot{\lambda}_\beta} \mu(\tilde{B}_k)^\beta \sum_{v=1}^\infty \mu(B_{2^{v+1}\varepsilon}(x_0))^{-\beta} M_{\beta+\gamma,p}(f)(z) \\
 &\leq C \|b_1\|_{\dot{\lambda}_\beta} \sum_{v=1}^\infty 2^{-v\beta} M_{\beta+\gamma,p}(f)(z) \\
 &\leq C \|b_1\|_{\dot{\lambda}_\beta} M_{\beta+\gamma,p}(f)(z) \\
 &\leq C \xi \lambda.
 \end{aligned}$$

For I_2 , notice $\theta = 1 - \gamma$, by μ 's doubling condition, Hölder's inequality, Lemma 2 and the inequality $\|a\|^\alpha - \|c\|^\alpha \leq \|a - c\|^\alpha$ for $a, c \in \mathbf{R}$ and $0 < \alpha < 1$, we obtain

$$\begin{aligned}
 &\int_{\rho(x,y) > \varepsilon} I_2 d\mu(y) \\
 &\leq C \sum_{v=1}^\infty \int_{B_{2^{v+1}\varepsilon}(x_0) \setminus B_{2^v\varepsilon}(x_0)} |(b_1(x_0) - b_1(y))(Q_\gamma(x,y) - Q_\gamma(x_0,y))f(y)| d\mu(y) \\
 &\leq C \sum_{v=1}^\infty \int_{B_{2^{v+1}\varepsilon}(x_0)} |b_1(x_0) - b_1(y)| \frac{|\eta(x_0,y)^{1-\gamma} - \eta(x,y)^{1-\gamma}|}{|\eta(x,y)^{1-\gamma}\eta(x_0,y)^{1-\gamma}|} |f(y)| d\mu(y) \\
 &\leq C \sum_{v=1}^\infty \int_{B_{2^{v+1}\varepsilon}(x_0)} |b_1(x_0) - b_1(y)| \frac{|\eta(x_0,x)^{1-\gamma}|}{|\eta(x_0,y)^{2(1-\gamma)}|} |f(y)| d\mu(y) \\
 &\leq C \sum_{v=1}^\infty \int_{B_{2^{v+1}\varepsilon}(x_0)} |b_1(x_0) - b_1(y)| \frac{\eta(x_0,x)^\theta}{\eta(x_0,y)^{1-\gamma+\theta}} |f(y)| d\mu(y) \\
 &\leq C \sum_{v=1}^\infty \frac{\mu(\tilde{B}_k)^\theta}{\mu(B_{2^{v+1}\varepsilon}(x_0))^{1-\gamma+\theta}} \left(\int_{B_{2^{v+1}\varepsilon}(x_0)} |b_1(x_0) - b_1(y)|^{p'} d\mu(y) \right)^{1/p'} \\
 &\quad \times \left(\int_{B_{2^{v+1}\varepsilon}(x_0)} |f(y)|^p d\mu(y) \right)^{1/p} \\
 &\leq C \sum_{v=1}^\infty \mu(\tilde{B}_k)^\theta \mu(B_{2^{v+1}\varepsilon}(x_0))^{-1+\gamma-\theta+\beta+1/p'+1/p-\beta-\gamma} \|b_1\|_{\dot{\lambda}_\beta} M_{\beta+\gamma,p}(f)(z) \\
 &\leq C \sum_{v=1}^\infty 2^{-v\theta} \|b_1\|_{\dot{\lambda}_\beta} M_{\beta+\gamma,p}(f)(z) \\
 &\leq C \|b_1\|_{\dot{\lambda}_\beta} M_{\beta+\gamma,p}(f)(z) \\
 &\leq C \xi \lambda.
 \end{aligned}$$

Therefore $I \leq C \xi \lambda$.

For *II* and *III*, note that, for $y \in R(x)$,

$$\rho(x, y) \leq H \text{diam}(\tilde{B}_k),$$

we get, by Lemma 2 and Hölder’s inequality

$$\begin{aligned} II &\leq C \int_{H\tilde{B}_k} |b_1(x_0) - b_1(y)| |Q_\gamma(x, y)| |f(y)| d\mu(y) \\ &\leq C \mu(H\tilde{B}_k)^{\gamma-1} \left(\int_{H\tilde{B}_k} |b_1(x_0) - b_1(y)|^{p'} d\mu(y) \right)^{1/p'} \left(\int_{H\tilde{B}_k} |f(y)|^p d\mu(y) \right)^{1/p} \\ &\leq C \mu(H\tilde{B}_k)^{\gamma-1+\beta+1/p'+1/p-\beta-\gamma} \|b_1\|_{\lambda_\beta} \left(\frac{1}{\mu(H\tilde{B}_k)^{1-(\beta+\gamma)/p}} \int_{H\tilde{B}_k} |f(y)|^p d\mu(y) \right)^{1/p} \\ &\leq C \|b_1\|_{\lambda_\beta} M_{\beta+\gamma,p}(f)(z) \\ &\leq C \xi \lambda. \end{aligned}$$

Similar *III* $\leq C \xi \lambda$.

Thus $I + II + III \leq C \xi \lambda$.

For *IV*, since $x \notin \cup_k Q_k$, then $|\mathcal{I}_\varepsilon^{\gamma b_1}(f)(x_0)| \leq \lambda$. For $x \in B_j$,

$$\sup_{\varepsilon \approx \text{diam}(\tilde{B}_k)} |\mathcal{I}_\varepsilon^{\gamma b_1}(f_2)(x)| \leq C \xi \lambda + \lambda.$$

Case 2. $\varepsilon > H \text{diam}(\tilde{B}_k)$. Let B_k^ε denote the ball with the same center as B_k and with the $\text{diam } B_k^\varepsilon = \varepsilon$. Similar to the proof of *Case 1*, we get

$$\sup_{\varepsilon > \text{diam}(\tilde{B}_k)} |\mathcal{I}_\varepsilon^{\gamma b_1}(f_2)(x)| \leq C \xi \lambda + \lambda.$$

Thus, we have shown that for $x \in B_k$,

$$\mathcal{I}_*^{\gamma b_1}(f_2)(x) \leq C \xi \lambda + \lambda.$$

Now, choose ξ_0 such that $C \xi_0 < 1$, let $\eta = 1$ and combine the estimates on f_1 with f_2 , we get

$$\begin{aligned} &\mu \left(\left\{ x \in B_k : I_*^{\gamma b_1}(f)(x) > 3\lambda, \|b_1\|_{\lambda_\beta} M_{\beta+\gamma,p}(f)(x) \leq \xi \lambda \right\} \right) \\ &\leq \mu(\{x \in B_k : I_*^{\gamma b_1}(f_1)(x) > 2\lambda - C \xi \lambda\}) + \mu(\{x \in X : I_*^{\gamma b_1}(f_2)(x) > \lambda + C \xi \lambda\}) \\ &\leq \mu(\{x \in B_k : I_*^{\gamma b_1}(f_1)(x) > \lambda\}) \leq \xi^r \mu(B_k). \end{aligned}$$

When $\mathbf{m} > \mathbf{1}$, similar to the case $\mathbf{m} = \mathbf{1}$, there exists a point $x_0 = x_0(k) \in \tilde{B}_k$ such that

$$\mathcal{I}_*^{\gamma b}(f)(x_0) \leq \lambda.$$

Now, we fix a ball B_k . Without loss of generality, we may assume there exists a point $z = z(k)$ with

$$\|\vec{b}\|_{\lambda_\beta} M_{m\beta+\gamma,p}(f)(z) \leq \xi\lambda.$$

Set $\bar{B}_k = \tilde{B}_k$ and write $f = f_1 + f_2$ for $f_1 = f\chi_{\bar{B}_k}$ and $f_2 = f\chi_{X \setminus \bar{B}_k}$. We turn to the estimates on f_1 and f_2 .

The estimates on f_1 . For $x \in B_k$, $1/r = 1/p + 1/q < 1$

$$\begin{aligned} \|\mathcal{I}_*^{\gamma\vec{b}}(f_1)\|_{L^r} &\leq C \left(\int_{\rho(x,y) > \varepsilon} \left| \prod_{j=1}^m (b_j(x) - b_j(y)) Q_\gamma(x,y) f_1(y) \right|^r d\mu(y) \right)^{1/r} \\ &\leq C\mu(\bar{B}_k)^{\gamma-1} \left(\int_{\bar{B}_k} |\prod_{j=1}^m b_j(x) - b_j(y)|^q d\mu(y) \right)^{1/q} \left(\int_{\bar{B}_k} |f(y)|^p d\mu(y) \right)^{1/p} \\ &\leq C\mu(\bar{B}_k)^{\gamma-1} \mu(\bar{B}_k)^{m\beta+1/q} \|\vec{b}\|_{\lambda_\beta} \mu(\bar{B}_k)^{1/p-m\beta-\gamma} \\ &\quad \times \left(\frac{1}{\mu(\bar{B}_k)^{1-(m\beta+\gamma)p}} \int_{\bar{B}_k} |f(y)|^p d\mu(y) \right)^{1/p} \\ &\leq C\mu(\bar{B}_k)^{1/r-1} \|\vec{b}\|_{\lambda_\beta} M_{m\beta+\gamma,p}(f)(z). \end{aligned}$$

Let $\eta > 0$, we have

$$\begin{aligned} \mu(\{x \in X : \mathcal{I}_*^{\gamma\vec{b}}(f_1)(x) > \eta\lambda\}) &\leq C(\eta\lambda)^{-r} \|\mathcal{I}_*^{\gamma\vec{b}}(f_1)\|_{L^r}^r \\ &\leq C(\eta\lambda)^{-r} [\|\vec{b}\|_{\lambda_\beta} M_{m\beta+\gamma,p}(f)(z)]^r \mu(\bar{B}_k)^{1-1/r} \\ &\leq C(\eta\lambda)^{-r} (\xi\lambda)^r \mu(\bar{B}_k) \\ &\leq C(\xi/\eta)^r \mu(B_k). \end{aligned}$$

The estimates on f_2 . Let $H = H(X)$ be a large positive integer depending only on X . We consider the following two cases:

Case 1. $\text{diam}(\tilde{B}_k) \leq \varepsilon \leq H\text{diam}(\tilde{B}_k)$. Set $U(x,y) = \prod_{j=1}^m (b_j(x) - b_j(y)) Q_\gamma(x,y) f(y)$. Choose $x_0 \in \tilde{B}_k$ such that $x_0 \in X \setminus \cup_k B_k$. For $x \in B_k$, following [8], we have

$$\begin{aligned} |\mathcal{I}_\varepsilon^{\gamma\vec{b}}(f_2)(x)| &\leq \left| \int_{\rho(x,y) > \varepsilon} [U(x,y) - U(x_0,y)] f_2(y) d\mu(y) \right| + \int_{R(x)} |U(x_0,y) f(y)| d\mu(y) \\ &\quad + \int_{R(x_0)} |U(x_0,y) f(y)| d\mu(y) + |\mathcal{I}_\varepsilon^{\gamma\vec{b}}(f)(x_0)| \\ &= J + JJ + JJJ + JJJJ, \end{aligned}$$

where $R(u) = \{y \in X : \text{diam}(\tilde{B}_k) < \rho(u,y) \leq H\text{diam}(\tilde{B}_k)\}$. Now let us treat J, JJ and JJJ respectively. For J , we write

$$J \leq |(b_1(x) - b_1(x_0)) \cdots (b_m(x) - b_m(x_0)) \int_{\rho(x,y) > \varepsilon} Q_\gamma(x,y) f_2(y) d\mu(y)|$$

$$\begin{aligned}
 & + \sum_{j=1}^m \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}(x_0))_{\sigma^c} \int_{\rho(x,y) > \varepsilon} (\vec{b}(y) - \vec{b}(x_0))_{\sigma} (Q_{\gamma}(x,y) - Q_{\gamma}(x_0,y)) f_2(y) d\mu(y)| \\
 & = J_1 + J_2.
 \end{aligned}$$

For J_1 , we have

$$\begin{aligned}
 J_1 & \leq \prod_{j=1}^m |b_j(x) - b_j(x_0)| \sum_{v=1}^{\infty} \int_{B_{2^{v+1}\varepsilon}(x_0) \setminus B_{2^v\varepsilon}(x_0)} |Q_{\gamma}(x,y)| |f(y)| d\mu(y) \\
 & \leq C \|\vec{b}\|_{\lambda_{\beta}} \mu(\tilde{B}_k)^{m\beta} \sum_{v=1}^{\infty} \mu(B_{2^v\varepsilon}(x_0))^{\gamma-1} \\
 & \quad \times \left(\int_{B_{2^{v+1}\varepsilon}(x_0)} |f(y)|^p d\mu(y) \right)^{1/p} \mu(B_{2^{v+1}\varepsilon}(x_0))^{1-1/p} \\
 & \leq C \|\vec{b}\|_{\lambda_{\beta}} \mu(\tilde{B}_k)^{m\beta} \sum_{v=1}^{\infty} \mu(B_{2^{v+1}\varepsilon}(x_0))^{\gamma-1+1-1/p+1/p-m\beta-\gamma} \\
 & \quad \times \left(\frac{1}{\mu(B_{2^{v+1}\varepsilon}(x_0))^{1-(m\beta+\gamma)p}} \int_{B_{2^{v+1}\varepsilon}(x_0)} |f(y)|^p d\mu(y) \right)^{1/p} \\
 & \leq C \|\vec{b}\|_{\lambda_{\beta}} \mu(\tilde{B}_k)^{m\beta} \sum_{v=1}^{\infty} \mu(B_{2^{v+1}\varepsilon}(x_0))^{-m\beta} M_{m\beta+\gamma,p}(f)(z) \\
 & \leq C \|\vec{b}\|_{\lambda_{\beta}} \sum_{v=1}^{\infty} 2^{-vm\beta} M_{m\beta+\gamma,p}(f)(z) \\
 & \leq C \|\vec{b}\|_{\lambda_{\beta}} M_{m\beta+\gamma,p}(f)(z) \\
 & \leq C \xi \lambda.
 \end{aligned}$$

For J_2 , let $\tau, \tau' \in \mathbf{N}$ such that $\tau + \tau' = m$, and $\tau' \neq 0$, we get

$$\begin{aligned}
 J_2 & \leq \sum_{j=1}^m \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}(x_0))_{\sigma^c} \int_{B_{2^{v+1}\varepsilon}(x_0) \setminus B_{2^v\varepsilon}(x_0)} (\vec{b}(y) - \vec{b}(x_0))_{\sigma} \\
 & \quad \times (Q_{\gamma}(x,y) - Q_{\gamma}(x_0,y)) f(y) d\mu(y)| \\
 & \leq C \sum_{j=1}^m \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}(x_0))_{\sigma^c}| \sum_{v=1}^{\infty} \frac{\mu(\tilde{B}_k)^{\theta}}{\mu(B_{2^v\varepsilon}(x_0))^{1-\gamma+\theta}} \\
 & \quad \times \int_{B_{2^{v+1}\varepsilon}(x_0)} |(\vec{b}(y) - \vec{b}(x_0))_{\sigma} f(y) d\mu(y)| \\
 & \leq C \sum_{j=1}^m \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}(x_0))_{\sigma^c}| \sum_{v=1}^{\infty} \frac{\mu(\tilde{B}_k)^{\theta}}{\mu(B_{2^v\varepsilon}(x_0))^{1-\gamma+\theta}} \\
 & \quad \times \left(\int_{B_{2^{v+1}\varepsilon}(x_0)} |(\vec{b}(y) - \vec{b}(x_0))_{\sigma}|^{p'} d\mu(y) \right)^{1/p'} \left(\int_{B_{2^{v+1}\varepsilon}(x_0)} |f(y)|^p d\mu(y) \right)^{1/p}
 \end{aligned}$$

$$\begin{aligned}
 &\leq C \sum_{\tau+\tau'=m} \|\vec{b}_{\sigma^c}\|_{\lambda_\beta} \mu(\tilde{B}_k)^{\tau\beta+\theta} \sum_{v=1}^{\infty} \mu(B_{2^{v+1}\varepsilon}(x_0))^{-1+\gamma-\theta+\tau'\beta+1/p'+1/p-m\beta-\gamma} \\
 &\quad \times \|\vec{b}_\sigma\|_{\lambda_\beta} M_{m\beta+\gamma,p}(f)(z) \\
 &\leq C \sum_{\tau+\tau'=m} \|\vec{b}\|_{\lambda_\beta} \mu(\tilde{B}_k)^{\tau\beta+\theta} \sum_{v=1}^{\infty} \mu(B_{2^{v+1}\varepsilon}(x_0))^{-\theta-\tau\beta} M_{m\beta+\gamma,p}(f)(z) \\
 &\leq C \|\vec{b}\|_{\lambda_\beta} \sum_{v=1}^{\infty} 2^{-v(\theta+\tau\beta)} M_{m\beta+\gamma,p}(f)(z) \\
 &\leq C \|\vec{b}\|_{\lambda_\beta} M_{m\beta+\gamma,p}(f)(z) \\
 &\leq C\xi\lambda.
 \end{aligned}$$

Therefore $J \leq C\xi\lambda$. For JJ and JJJ , note that, for $y \in R(x)$,

$$\rho(x, y) \leq H \text{diam}(\tilde{B}_k),$$

we get, by Lemma 2 and Hölder's inequality

$$\begin{aligned}
 JJ &\leq C \int_{H\tilde{B}_k} \left| \prod_{j=1}^m b_j(x_0) - b_j(y) \right| |Q_\gamma(x, y)| |f(y)| d\mu(y) \\
 &\leq C \mu(H\tilde{B}_k)^{\gamma-1} \left(\int_{H\tilde{B}_k} \left| \prod_{j=1}^m b_j(x_0) - b_j(y) \right|^{p'} d\mu(y) \right)^{1/p'} \left(\int_{H\tilde{B}_k} |f(y)|^p d\mu(y) \right)^{1/p} \\
 &\leq C \mu(H\tilde{B}_k)^{\gamma-1+m\beta+1/p'+1/p-m\beta-\gamma} \|\vec{b}\|_{\lambda_\beta} \left(\frac{1}{\mu(H\tilde{B}_k)^{1-(m\beta+\gamma)p}} \int_{H\tilde{B}_k} |f(y)|^p d\mu(y) \right)^{1/p} \\
 &\leq C \|\vec{b}\|_{\lambda_\beta} M_{m\beta+\gamma,p}(f)(z) \\
 &\leq C\xi\lambda.
 \end{aligned}$$

Similar $JJJ \leq C\xi\lambda$.

Thus $J + JJ + JJJ \leq C\xi\lambda$.

For $JJJJ$, since $x \notin \cup_k Q_k$, then $|\mathcal{I}_\varepsilon^{\gamma\vec{b}}(f)(x_0)| \leq \lambda$. For $x \in B_j$,

$$\sup_{\varepsilon \approx \text{diam}(\tilde{B}_k)} |\mathcal{I}_\varepsilon^{\gamma\vec{b}}(f_2)(x)| \leq C\xi\lambda + \lambda.$$

Case 2. $\varepsilon > H \text{diam}(\tilde{B}_k)$. Let B_k^ε denote the ball with the same center as B_k and with the diam $B_k^\varepsilon = \varepsilon$. Similar to the proof of Case 1, we get

$$\sup_{\varepsilon > \text{diam}(\tilde{B}_k)} |\mathcal{I}_\varepsilon^{\gamma\vec{b}}(f_2)(x)| \leq C\xi\lambda + \lambda.$$

Thus, we have shown that for $x \in B_k$,

$$\mathcal{I}_\star^{\gamma\vec{b}}(f_2)(x) \leq C\xi\lambda + \lambda.$$

Now, choose ξ_0 such that $C\xi_0 < 1$, let $\eta = 1$ and combine the estimates on f_1 with f_2 , we get

$$\begin{aligned} & \mu\left(\left\{x \in B_k : I_*^{\vec{b}}(f)(x) > 3\lambda, \|\vec{b}\|_{\lambda_\beta} M_{m\beta+\gamma,p}(f)(x) \leq \xi\lambda\right\}\right) \\ & \leq \mu(\{x \in B_k : I_*^{\vec{b}}(f_1)(x) > 2\lambda - C\xi\lambda\}) + \mu(\{x \in X : I_*^{\vec{b}}(f_2)(x) > \lambda + C\xi\lambda\}) \\ & \leq \mu(\{x \in B_k : I_*^{\vec{b}}(f_1)(x) > \lambda\}) \leq \xi^r \mu(B_k). \end{aligned}$$

This completes the proof of Theorem 1(a). (b) follows from (a) and Lemma 3. \square

Proof of Theorem 2(a). When $\mathbf{m} = \mathbf{1}$, by the Whitney decomposition, $\{x \in X : \mathcal{I}_*^{\gamma b_1}(f)(x) > \lambda\}$ may be written as a union of balls $\{B_k\}$ with mutually disjoint interiors and with distance from each to $X \setminus \cup_k B_k$ comparable to the diameter of B_k . It suffices to prove the good λ estimate for each B_k . There exists a constant $C = C(n)$ such that for each k , the cube \tilde{B}_k intersects $X \setminus \cup_k B_k$, where \tilde{B}_k denotes the ball with the same center as B_k and with the diam $\tilde{B}_k = C \text{ diam } B_k$. Then, for each k , there exists a point $x_0 = x_0(k) \in \tilde{B}_k$ such that

$$\mathcal{I}_*^{\gamma b_1}(f)(x_0) \leq \lambda.$$

Now, we fix a ball B_k . Without loss of generality, we may assume there exists a point $z = z(k)$ with

$$\|b_1\|_{BMO M_{\gamma,p}(f)}(z) \leq \xi\lambda.$$

Set $\bar{B}_k = \tilde{B}_k$ and write $f = f_1 + f_2$ for $f_1 = f\chi_{\bar{B}_k}$ and $f_2 = f\chi_{X \setminus \bar{B}_k}$. We turn to the estimates on f_1 and f_2 .

The estimates on f_1 . For $x \in B_k$, $1/r = 1/p + 1/q < 1$

$$\begin{aligned} \|\mathcal{I}_*^{\gamma b_1}(f_1)\|_{L^r} & \leq C \left(\int_{\rho(x,y) > \varepsilon} |(b_1(x) - b_1(y))\mathcal{Q}_\gamma(x,y)f_1(y)|^r d\mu(y) \right)^{1/r} \\ & \leq C\mu(\bar{B}_k)^{\gamma-1} \left(\int_{\bar{B}_k} |b_1(x) - b_1(y)|^q d\mu(y) \right)^{1/q} \left(\int_{\bar{B}_k} |f(y)|^p d\mu(y) \right)^{1/p} \\ & \leq C\mu(\bar{B}_k)^{\gamma-1} \mu(\bar{B}_k)^{1/q} \|b_1\|_{BMO} \mu(\bar{B}_k)^{1/p-\gamma} \\ & \quad \times \left(\frac{1}{\mu(\bar{B}_k)^{1-\gamma p}} \int_{\bar{B}_k} |f(y)|^p d\mu(y) \right)^{1/p} \\ & \leq C\mu(\bar{B}_k)^{1/r-1} \|b_1\|_{BMO M_{\gamma,p}(f)}(z). \end{aligned}$$

Let $\eta > 0$, we have

$$\begin{aligned} \mu(\{x \in X : \mathcal{I}_*^{\gamma b_1}(f_1)(x) > \eta\lambda\}) & \leq C(\eta\lambda)^{-r} \|\mathcal{I}_*^{\gamma b_1}(f_1)\|_{L^r}^r \\ & \leq C(\eta\lambda)^{-r} [\|b_1\|_{BMO M_{\gamma,p}(f)}(z)]^r \mu(\bar{B}_k)^{1-1/r} \\ & \leq C(\eta\lambda)^{-r} (\xi\lambda)^r \mu(\bar{B}_k) \\ & \leq C(\xi/\eta)^r \mu(B_k). \end{aligned}$$

The estimates on f_2 . Let $H = H(X)$ be a large positive integer depending only on X . We consider the following two cases:

Case 1. $\text{diam}(\tilde{B}_k) \leq \varepsilon \leq H \text{diam}(\tilde{B}_k)$. Set $V(x, y) = (b_1(x) - b_1(y))Q_\gamma(x, y)f(y)$. Choose $x_0 \in \tilde{B}_k$ such that $x_0 \in X \setminus \bigcup_k B_k$. For $x \in B_k$, following [8], we have

$$\begin{aligned} |\mathcal{I}_\varepsilon^{\gamma b_1}(f_2)(x)| &\leq \left| \int_{\rho(x,y) > \varepsilon} [V(x, y) - V(x_0, y)]f_2(y)d\mu(y) \right| + \int_{R(x)} |V(x_0, y)f(y)|d\mu(y) \\ &\quad + \int_{R(x_0)} |V(x_0, y)f(y)|d\mu(y) + |\mathcal{I}_\varepsilon^{\gamma b_1}(f)(x_0)| \\ &= I' + II' + III' + IV', \end{aligned}$$

where $R(u) = \{y \in X : \text{diam}(\tilde{B}_k) < \rho(u, y) \leq H \text{diam}(\tilde{B}_k)\}$. Now let us treat I', II' and III' respectively. For I' , we write

$$\begin{aligned} &|V(x, y) - V(x_0, y)| \\ &= |(b_1(x) - b_1(y))Q_\gamma(x, y)f_2(y) - (b_1(x_0) - b_1(y))Q_\gamma(x_0, y)f_2(y)| \\ &= |b_1(x)Q_\gamma(x, y)f_2(y) - b_1(y)Q_\gamma(x, y)f_2(y) - b_1(x_0)Q_\gamma(x_0, y)f_2(y) \\ &\quad + b_1(y)Q_\gamma(x_0, y)f_2(y) - b_1(x_0)Q_\gamma(x, y)f_2(y) + b_1(x_0)Q_\gamma(x, y)f_2(y)| \\ &\leq |(b_1(x) - b_1(x_0))Q_\gamma(x, y)f_2(y)| + |(b_1(x_0) - b_1(y))(Q_\gamma(x, y) - Q_\gamma(x_0, y))f_2(y)| \\ &= I'_1 + I'_2. \end{aligned}$$

For I'_1 , by Hölder's inequality and the following inequality, for $b \in BMO(X)$

$$|b(x) - b_B| \leq \frac{1}{\mu(B)} \int_B |b(x) - b_B|d\mu(x) \leq \|b\|_{BMO},$$

we have

$$\begin{aligned} \int_{\rho(x,y) > \varepsilon} I_1 d\mu(y) &\leq C \sum_{v=1}^{\infty} \int_{B_{2^{v+1}\varepsilon}(x_0) \setminus B_{2^v\varepsilon}(x_0)} |b_1(x) - b_1(x_0)| |Q_\gamma(x, y)| |f(y)| d\mu(y) \\ &\leq C \|b_1\|_{BMO} \sum_{v=1}^{\infty} \mu(B_{2^v\varepsilon}(x_0))^{\gamma-1} \left(\int_{B_{2^{v+1}\varepsilon}(x_0)} |f(y)|^p d\mu(y) \right)^{1/p} \\ &\quad \times \mu(B_{2^{v+1}\varepsilon}(x_0))^{1-1/p} \\ &\leq C \|b_1\|_{BMO} \sum_{v=1}^{\infty} \mu(B_{2^{v+1}\varepsilon}(x_0))^{\gamma-1+1-1/p+1/p-\gamma} \\ &\quad \times \left(\frac{1}{\mu(B_{2^{v+1}\varepsilon}(x_0))^{1-\gamma p}} \int_{B_{2^{v+1}\varepsilon}(x_0)} |f(y)|^p d\mu(y) \right)^{1/p} \\ &\leq C \|b_1\|_{BMO} M_{\gamma,p}(f)(z) \\ &\leq C \xi^\lambda. \end{aligned}$$

For I'_2 , similar to I_2 , we obtain

$$\begin{aligned}
 \int_{\rho(x,y) > \varepsilon} I_2 d\mu(y) &\leq C \sum_{v=1}^{\infty} \int_{B_{2^{v+1}\varepsilon}(x_0) \setminus B_{2^v\varepsilon}(x_0)} |(b_1(x_0) - b_1(y))(Q_\gamma(x,y) - Q_\gamma(x_0,y))f(y)| d\mu(y) \\
 &\leq C \sum_{v=1}^{\infty} \int_{B_{2^{v+1}\varepsilon}(x_0)} |b_1(x_0) - b_1(y)| \frac{|\eta(x_0,y)^{1-\gamma} - \eta(x,y)^{1-\gamma}|}{|\eta(x,y)^{1-\gamma}\eta(x_0,y)^{1-\gamma}|} |f(y)| d\mu(y) \\
 &\leq C \sum_{v=1}^{\infty} \int_{B_{2^{v+1}\varepsilon}(x_0)} |b_1(x_0) - b_1(y)| \frac{|\eta(x_0,x)^{1-\gamma}|}{|\eta(x_0,y)^{2(1-\gamma)}|} |f(y)| d\mu(y) \\
 &\leq C \sum_{v=1}^{\infty} \int_{B_{2^{v+1}\varepsilon}(x_0)} |b_1(x_0) - b_1(y)| \frac{\eta(x_0,x)^\theta}{\eta(x_0,y)^{1-\gamma+\theta}} |f(y)| d\mu(y) \\
 &\leq C \sum_{v=1}^{\infty} \frac{\mu(\tilde{B}_k)^\theta}{\mu(B_{2^{v+1}\varepsilon}(x_0))^{1-\gamma+\theta}} \left(\int_{B_{2^{v+1}\varepsilon}(x_0)} |b_1(x_0) - b_1(y)|^{p'} d\mu(y) \right)^{1/p'} \\
 &\quad \times \left(\int_{B_{2^{v+1}\varepsilon}(x_0)} |f(y)|^p d\mu(y) \right)^{1/p} \\
 &\leq C \sum_{v=1}^{\infty} \mu(\tilde{B}_k)^\theta \mu(B_{2^{v+1}\varepsilon}(x_0))^{-1+\gamma-\theta+1/p'+1/p-\gamma} \|b_1\|_{\lambda_\beta} M_{\gamma,p}(f)(z) \\
 &\leq C \sum_{v=1}^{\infty} 2^{-v\theta} \|b_1\|_{BMO} M_{\gamma,p}(f)(z) \\
 &\leq C \|b_1\|_{BMO} M_{\gamma,p}(f)(z) \\
 &\leq C \xi \lambda.
 \end{aligned}$$

Therefore $I' \leq C \xi \lambda$.

For II' and III' , note that, for $y \in R(x)$,

$$\rho(x,y) \leq H \text{diam}(\tilde{B}_k),$$

we get, by Hölder's inequality

$$\begin{aligned}
 II' &\leq C \int_{H\tilde{B}_k} |b_1(x_0) - b_1(y)| |Q_\gamma(x,y)| |f(y)| d\mu(y) \\
 &\leq C \mu(H\tilde{B}_k)^{\gamma-1} \left(\int_{H\tilde{B}_k} |b_1(x_0) - b_1(y)|^{p'} d\mu(y) \right)^{1/p'} \left(\int_{H\tilde{B}_k} |f(y)|^p d\mu(y) \right)^{1/p} \\
 &\leq C \mu(H\tilde{B}_k)^{\gamma-1+1/p'+1/p-\gamma} \|b_1\|_{BMO} \left(\frac{1}{\mu(H\tilde{B}_k)^{1-\gamma p}} \int_{H\tilde{B}_k} |f(y)|^p d\mu(y) \right)^{1/p} \\
 &\leq C \|b_1\|_{BMO} M_{\gamma,p}(f)(z) \\
 &\leq C \xi \lambda.
 \end{aligned}$$

Similar $III' \leq C \xi \lambda$.

Thus $I' + II' + III' \leq C \xi \lambda$.

For IV' , since $x \notin \bigcup_k Q_k$, then $|\mathcal{I}_\varepsilon^{\gamma b_1}(f)(x_0)| \leq \lambda$. For $x \in B_j$,

$$\sup_{\varepsilon \approx \text{diam}(\tilde{B}_k)} |\mathcal{I}_\varepsilon^{\gamma b_1}(f_2)(x)| \leq C\xi\lambda + \lambda.$$

Case 2. $\varepsilon > H\text{diam}(\tilde{B}_k)$. Let B_k^ε denote the ball with the same center as B_k and with the diam $B_k^\varepsilon = \varepsilon$. Similar to the proof of Case 1, we get

$$\sup_{\varepsilon > \text{diam}(\tilde{B}_k)} |\mathcal{I}_\varepsilon^{\gamma b_1}(f_2)(x)| \leq C\xi\lambda + \lambda.$$

Thus, we have shown that for $x \in B_k$,

$$\mathcal{I}_*^{\gamma b_1}(f_2)(x) \leq C\xi\lambda + \lambda.$$

Now, choose ξ_0 such that $C\xi_0 < 1$, let $\eta = 1$ and combine the estimates on f_1 with f_2 , we get

$$\begin{aligned} \mu\left(\left\{x \in B_k : I_*^{\gamma b_1}(f)(x) > 3\lambda, \|b_1\|_{BMO} M_{\gamma,p}(f)(x) \leq \xi\lambda\right\}\right) \\ \leq \mu(\{x \in B_k : I_*^{\gamma b_1}(f_1)(x) > 2\lambda - C\xi\lambda\}) + \mu(\{x \in X : I_*^{\gamma b_1}(f_2)(x) > \lambda + C\xi\lambda\}) \\ \leq \mu(\{x \in B_k : I_*^{\gamma b_1}(f_1)(x) > \lambda\}) \leq \xi^r \mu(B_k). \end{aligned}$$

When $\mathbf{m} > \mathbf{1}$, similar to the case $\mathbf{m} = \mathbf{1}$, there exists a point $x_0 = x_0(k) \in \tilde{B}_k$ such that

$$\mathcal{I}_*^{\gamma \vec{b}}(f)(x_0) \leq \lambda.$$

Now, we fix a ball B_k . Without loss of generality, we may assume there exists a point $z = z(k)$ with

$$\|\vec{b}\|_{BMO} M_{\gamma,p}(f)(z) \leq \xi\lambda.$$

Set $\bar{B}_k = \tilde{\tilde{B}}_k$ and write $f = f_1 + f_2$ for $f_1 = f\chi_{\bar{B}_k}$ and $f_2 = f\chi_{X \setminus \bar{B}_k}$. We turn to the estimates on f_1 and f_2 .

The estimates on f_1 . For $x \in B_k$, $1/r = 1/p + 1/q_j < 1$, $j = 1, \dots, m$

$$\begin{aligned} \|\mathcal{I}_*^{\gamma \vec{b}}(f_1)\|_{L^r} &\leq C \left(\int_{\rho(x,y) > \varepsilon} \left| \prod_{j=1}^m (b_j(x) - b_j(y)) \mathcal{Q}_\gamma(x,y) f_1(y) \right|^r d\mu(y) \right)^{1/r} \\ &\leq C\mu(\bar{B}_k)^{\gamma-1} \left(\int_{\bar{B}_k} \left| \prod_{j=1}^m b_j(x) - b_j(y) \right|^{q_j} d\mu(y) \right)^{1/q_j} \left(\int_{\bar{B}_k} |f(y)|^p d\mu(y) \right)^{1/p} \\ &\leq C\mu(\bar{B}_k)^{\gamma-1} \mu(\bar{B}_k)^{\sum_{j=1}^m 1/q_j} \|\vec{b}\|_{BMO} \mu(\bar{B}_k)^{1/p-\gamma} \\ &\quad \times \left(\frac{1}{\mu(\bar{B}_k)^{1-\gamma p}} \int_{\bar{B}_k} |f(y)|^p d\mu(y) \right)^{1/p} \end{aligned}$$

$$\leq C\mu(\bar{B}_k)^{1/r-1} \|\vec{b}\|_{BMO M_{\gamma,p}}(f)(z).$$

Let $\eta > 0$, we have

$$\begin{aligned} \mu(\{x \in X : \mathcal{I}_\star^{\vec{b}}(f_1)(x) > \eta\lambda\}) &\leq C(\eta\lambda)^{-r} \|\mathcal{I}_\star^{\vec{b}}(f_1)\|_{L^r}^r \\ &\leq C(\eta\lambda)^{-r} [\|\vec{b}\|_{BMO M_{\gamma,p}}(f)(z)]^r \mu(\bar{B}_k)^{1-1/r} \\ &\leq C(\eta\lambda)^{-r} (\xi\lambda)^r \mu(\bar{B}_k) \\ &\leq C(\xi/\eta)^r \mu(B_k). \end{aligned}$$

The estimates on f_2 . Let $H = H(X)$ be a large positive integer depending only on X . We consider the following two cases:

Case I. $\text{diam}(\tilde{B}_k) \leq \varepsilon \leq H \text{diam}(\tilde{B}_k)$. Set $U(x, y) = \prod_{j=1}^m (b_j(x) - b_j(y)) Q_\gamma(x, y) f(y)$. Choose $x_0 \in \tilde{B}_k$ such that $x_0 \in X \setminus \cup_k B_k$. For $x \in B_k$, following [8], we have

$$\begin{aligned} |\mathcal{I}_\varepsilon^{\vec{b}}(f_2)(x)| &\leq \left| \int_{\rho(x,y) > \varepsilon} [U(x, y) - U(x_0, y)] f_2(y) d\mu(y) \right| + \int_{R(x)} |U(x_0, y) f(y)| d\mu(y) \\ &\quad + \int_{R(x_0)} |U(x_0, y) f(y)| d\mu(y) + |\mathcal{I}_\varepsilon^{\vec{b}}(f)(x_0)| \\ &= J' + JJ' + JJJ' + JJJJ', \end{aligned}$$

where $R(u) = \{y \in X : \text{diam}(\tilde{B}_k) < \rho(u, y) \leq H \text{diam}(\tilde{B}_k)\}$. Now let us treat J', JJ' and JJJ' respectively. For J' , we write

$$\begin{aligned} J' &\leq |(b_1(x) - b_1(x_0)) \cdots (b_m(x) - b_m(x_0)) \int_{\rho(x,y) > \varepsilon} Q_\gamma(x, y) f_2(y) d\mu(y)| \\ &\quad + \sum_{j=1}^m \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}(x_0))_{\sigma^c} \int_{\rho(x,y) > \varepsilon} (\vec{b}(y) - \vec{b}(x_0))_{\sigma} (Q_\gamma(x, y) - Q_\gamma(x_0, y)) f_2(y) d\mu(y)| \\ &= J'_1 + J'_2. \end{aligned}$$

For J'_1 , we have

$$\begin{aligned} J'_1 &\leq \prod_{j=1}^m |b_j(x) - b_j(x_0)| \sum_{v=1}^{\infty} \int_{B_{2^v+1\varepsilon}(x_0) \setminus B_{2^v\varepsilon}(x_0)} |Q_\gamma(x, y)| |f(y)| d\mu(y) \\ &\leq C \|\vec{b}\|_{BMO} \sum_{v=1}^{\infty} \mu(B_{2^v\varepsilon}(x_0))^{\gamma-1} \left(\int_{B_{2^v+1\varepsilon}(x_0)} |f(y)|^p d\mu(y) \right)^{1/p} \mu(B_{2^v+1\varepsilon}(x_0))^{1-1/p} \\ &\leq C \|\vec{b}\|_{BMO} \sum_{v=1}^{\infty} \mu(B_{2^v+1\varepsilon}(x_0))^{\gamma-1+1-1/p+1/p-\gamma} \\ &\quad \times \left(\frac{1}{\mu(B_{2^v+1\varepsilon}(x_0))^{1-\gamma p}} \int_{B_{2^v+1\varepsilon}(x_0)} |f(y)|^p d\mu(y) \right)^{1/p} \\ &\leq C \|\vec{b}\|_{BMO M_{\gamma,p}}(f)(z) \\ &\leq C \xi \lambda. \end{aligned}$$

For J'_2 , we get

$$\begin{aligned}
 J'_2 &\leq \sum_{j=1}^m \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}(x_0))_{\sigma^c}| \int_{B_{2^{v+1}\varepsilon}(x_0) \setminus B_{2^v\varepsilon}(x_0)} (\vec{b}(y) - \vec{b}(x_0))_{\sigma} \\
 &\quad \times (Q_{\gamma}(x, y) - Q_{\gamma}(x_0, y)) f(y) d\mu(y) | \\
 &\leq C \sum_{j=1}^m \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}(x_0))_{\sigma^c}| \sum_{v=1}^{\infty} \frac{\mu(\tilde{B}_k)^{\theta}}{\mu(B_{2^v\varepsilon}(x_0))^{1-\gamma+\theta}} \\
 &\quad \times \int_{B_{2^{v+1}\varepsilon}(x_0)} |(\vec{b}(y) - \vec{b}(x_0))_{\sigma} f(y)| d\mu(y) | \\
 &\leq C \sum_{j=1}^m \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}(x_0))_{\sigma^c}| \sum_{v=1}^{\infty} \frac{\mu(\tilde{B}_k)^{\theta}}{\mu(B_{2^v\varepsilon}(x_0))^{1-\gamma+\theta}} \\
 &\quad \times \left(\int_{B_{2^{v+1}\varepsilon}(x_0)} |(\vec{b}(y) - \vec{b}(x_0))_{\sigma}|^{p'} d\mu(y) \right)^{1/p'} \left(\int_{B_{2^{v+1}\varepsilon}(x_0)} |f(y)|^p d\mu(y) \right)^{1/p} \\
 &\leq C \|\vec{b}_{\sigma^c}\|_{BMO} \mu(\tilde{B}_k)^{\theta} \sum_{v=1}^{\infty} \mu(B_{2^{v+1}\varepsilon}(x_0))^{-1+\gamma-\theta+1/p'+1/p-\gamma} \|\vec{b}_{\sigma}\|_{BMO} M_{\gamma,p}(f)(z) \\
 &\leq C \|\vec{b}\|_{BMO} \mu(\tilde{B}_k)^{\theta} \sum_{v=1}^{\infty} \mu(B_{2^{v+1}\varepsilon}(x_0))^{-\theta} M_{\gamma,p}(f)(z) \\
 &\leq C \|\vec{b}\|_{BMO} \sum_{v=1}^{\infty} 2^{-v\theta} M_{\gamma,p}(f)(z) \\
 &\leq C \|\vec{b}\|_{BMO} M_{\gamma,p}(f)(z) \\
 &\leq C\xi\lambda.
 \end{aligned}$$

Therefore $J' \leq C\xi\lambda$. For JJ' and JJJ' , note that, for $y \in R(x)$,

$$\rho(x, y) \leq H \text{diam}(\tilde{B}_k),$$

we get, by Hölder's inequality, for $1/p'_1 + \dots + 1/p'_m + 1/p = 1$

$$\begin{aligned}
 JJ' &\leq C \int_{H\tilde{B}_k} \left| \prod_{j=1}^m b_j(x_0) - b_j(y) \right| |Q_{\gamma}(x, y)| |f(y)| d\mu(y) \\
 &\leq C \mu(H\tilde{B}_k)^{\gamma-1} \left(\int_{H\tilde{B}_k} \left| \prod_{j=1}^m b_j(x_0) - b_j(y) \right|^{p'_j} d\mu(y) \right)^{1/p'_j} \left(\int_{H\tilde{B}_k} |f(y)|^p d\mu(y) \right)^{1/p} \\
 &\leq C \mu(H\tilde{B}_k)^{\gamma-1+1/p'_1+\dots+1/p'_m+1/p-\gamma} \|\vec{b}\|_{BMO} \left(\frac{1}{\mu(H\tilde{B}_k)^{1-\gamma p}} |f(y)|^p d\mu(y) \right)^{1/p} \\
 &\leq C \|\vec{b}\|_{BMO} M_{\gamma,p}(f)(z) \\
 &\leq C\xi\lambda.
 \end{aligned}$$

Similar $JJJ' \leq C\xi\lambda$.

Thus $J' + JJ' + JJJ' \leq C\xi\lambda$.

For $JJJJ'$, since $x \notin \cup_k Q_k$, then $|\mathcal{I}_\varepsilon^{\gamma\vec{b}}(f)(x_0)| \leq \lambda$. For $x \in B_j$,

$$\sup_{\varepsilon \approx \text{diam}(\tilde{B}_k)} |\mathcal{I}_\varepsilon^{\gamma\vec{b}}(f_2)(x)| \leq C\xi\lambda + \lambda.$$

Case 2. $\varepsilon > H\text{diam}(\tilde{B}_k)$. Let B_k^ε denote the ball with the same center as B_k and with the diam $B_k^\varepsilon = \varepsilon$. Similar to the proof of Case 1, we get

$$\sup_{\varepsilon > \text{diam}(\tilde{B}_k)} |\mathcal{I}_\varepsilon^{\gamma\vec{b}}(f_2)(x)| \leq C\xi\lambda + \lambda.$$

Thus, we have shown that for $x \in B_k$,

$$\mathcal{I}_*^{\gamma\vec{b}}(f_2)(x) \leq C\xi\lambda + \lambda.$$

Now, choose ξ_0 such that $C\xi_0 < 1$, let $\eta = 1$ and combine the estimates on f_1 with f_2 , we get

$$\begin{aligned} &\mu\left(\left\{x \in B_k : I_*^{\gamma\vec{b}}(f)(x) > 3\lambda, \|\vec{b}\|_{BMO_{\gamma,p}}(f)(x) \leq \xi\lambda\right\}\right) \\ &\leq \mu(\{x \in B_k : I_*^{\gamma\vec{b}}(f_1)(x) > 2\lambda - C\xi\lambda\}) + \mu(\{x \in X : I_*^{\gamma\vec{b}}(f_2)(x) > \lambda + C\xi\lambda\}) \\ &\leq \mu(\{x \in B_k : I_*^{\gamma\vec{b}}(f_1)(x) > \lambda\}) \leq \xi^r \mu(B_k). \end{aligned}$$

This completes the proof of Theorem 2(a). (b) follows from (a) and Lemma 3. \square

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